A CONTRIBUTION TO THE THEORY OF CHROMATIC POLYNOMIALS

W. T. TUTTE

SUMMARY

Two polynomials \( \theta(G, n) \) and \( \phi(G, n) \) connected with the colourings of a graph \( G \) or of associated maps are discussed. A result believed to be new is proved for the lesser-known polynomial \( \phi(G, n) \). Attention is called to some unsolved problems concerning \( \phi(G, n) \) which are natural generalizations of the Four Colour Problem from planar graphs to general graphs. A polynomial \( \chi(G, x, y) \) in two variables \( x \) and \( y \), which can be regarded as generalizing both \( \theta(G, n) \) and \( \phi(G, n) \) is studied. For a connected graph \( \chi(G, x, y) \) is defined in terms of the "spanning" trees of \( G \) (which include every vertex) and in terms of a fixed enumeration of the edges. The invariance of \( \chi(G, x, y) \) under a change of this enumeration is apparently a new result about spanning trees. It is observed that the theory of spanning trees now links the theory of graph-colourings to that of electrical networks.

1. Introduction. A graph \( G \) consists of a set \( V(G) \) of elements called vertices together with a set \( E(G) \) of elements called edges, the two sets having no common element. With each edge there are associated either one or two vertices called its ends. An edge of \( G \) is a loop or link according as the number of its ends is 1 or 2. For convenience we sometimes say that a link has two distinct ends and a loop two equal ends.

We restrict ourselves to finite graphs, that is graphs for which \( V(G) \) and \( E(G) \) are both finite.

If \( V(G) = 0 \) we must have \( E(G) = 0 \) also.

A graph \( H \) is a subgraph of \( G \) if \( V(H) \subseteq V(G) \), \( E(H) \subseteq E(G) \) and each edge of \( H \) has the same ends in \( H \) as in \( G \). The subgraph \( H \) of \( G \) is a spanning subgraph of \( G \) if \( V(H) = V(G) \). The subgraph of \( G \) for which \( V(H) \) is a given subset \( W \) of \( V(G) \) and \( E(H) \) is the set of all edges of \( G \) having no end outside \( W \), will be denoted by \( G[W] \).

A sequence \( (a_0, A_1, a_1, A_2, a_2, \ldots, A_n, a_n) \), in which the terms are alternately vertices \( a_i \) and edges \( A_j \) of \( G \) is a path from \( a_0 \) to \( a_n \) in \( G \) if it satisfies the following conditions.

(i) If \( 1 \leq i \leq n \) the ends of \( A_i \) are \( a_{i-1} \) and \( a_i \).
(ii) If \( 1 \leq i \leq n \) then \( a_{i-1} = a_i \) if and only if \( A_i \) is a loop.

It is not required that the terms of the sequence shall be all distinct. If they are distinct the path is simple. If the sequence has more than one term and its terms are distinct except that \( a_0 = a_n \) then the path is circular.

If \( x \) and \( y \) are elements of \( V(G) \) we say \( x \) and \( y \) are connected in \( G \) if there is a path from \( x \) to \( y \) in \( G \). The relation of connection in \( G \) is clearly an equivalence

Received October 1, 1952.
CHROMATIC POLYNOMIALS

relation. Hence if \( V(G) \) is non-null it can be partitioned into disjoint non-null subsets \( V_1, \ldots, V_k \) such that two vertices of \( G \) are connected in \( G \) if and only if they belong to the same set \( V_i \). The subgraphs \( G[V_i] \) of \( G \) are the components of \( G \). Together they include all the edges and vertices of \( G \), and no two of them have an edge or vertex in common. We denote the number of components of \( G \) by \( p_0(G) \). The graph \( G \) is connected if \( p_0(G) = 0 \) or \( 1 \). The first case arises only when \( V(G) = 0 \) and \( E(G) = 0 \). Clearly each component of a graph is connected.

A connected graph in which there is no circular path is a tree.

We write \( a_0(G) \) and \( a_1(G) \) for the numbers of elements of \( V(G) \) and \( E(G) \) respectively.

Let \( Q_n \) be a finite set of \( n > 0 \) elements. Let \( f \) be a mapping of \( V(G) \) into \( Q_n \). We call \( f \) an \( n \)-colouring of \( G \) if each edge of \( G \) has two ends \( x \) and \( y \) such that \( f(x) \neq f(y) \). We denote the number of \( n \)-colourings of \( G \), defined in terms of \( Q_n \), by \( P(G, n) \). If \( V(G) = 0 \) we take this number to be 1. We observe that \( P(G, n) = 0 \) if \( G \) has a loop.

\( P(G, n) \) is not altered by replacing \( Q_n \) by another set of \( n \) elements. We find it convenient to take \( Q_n \) as the ring of residue classes mod \( n \).

The function \( P(G, n) \) was studied by Hassler Whitney (6; 7). He showed that when \( G \) is loopless, \( P(G, n) \) is a polynomial in \( n \) of degree \( a_0(G) \). For planar graphs \( G \) the polynomial has been studied in great detail by Birkhoff and Lewis (1), who associated it with the dual map of \( G \). Following them we call \( P(G, n) \) the chromatic polynomial of \( G \).

The following explicit formula for \( P(G, n) \) is due to Hassler Whitney.

\[
P(G, n) = \sum_S (-1)^{a_1(S)} n^{p_0(S)}.
\]

The summation is over all spanning subgraphs \( S \) of \( G \). We shall find another explicit formula in terms of the spanning trees of \( G \) valid when \( G \) is connected. A spanning tree is a spanning subgraph which is a tree.

At this stage it is convenient to apply some of the concepts of elementary combinatorial topology. We orient \( G \) by distinguishing one end of each edge \( A \) as the positive end \( p(A) \) and one as the negative end \( q(A) \). The positive and negative ends coincide if \( A \) is a loop but not if \( A \) is a link. If \( a \in V(G) \) and \( A \in E(G) \) we write \( \eta(A, a) = 0 \) if \( A \) is a loop or if \( a \) is not an end of \( A \). Otherwise we write \( \eta(A, a) = 1 \) or \(-1\) according as \( a \) is the positive or the negative end of \( A \). A mapping \( f \) of \( V(G) \) or \( E(G) \) into \( Q_n \) is a 0-chain or 1-chain respectively on \( G \) over \( Q_n \).

If \( V(G) \) is null we consider that there is just one 0-chain on \( G \) over \( Q_n \). Similarly if \( E(G) \) is null there is just one 1-chain on \( G \) over \( Q_n \).

If \( h \) is a 0-chain on \( G \) over \( Q_n \) its coboundary \( \partial h \) is the 1-chain on \( G \) over \( Q_n \) satisfying

\[
(\partial h)(A) = \sum_a \eta(A, a) h(a)
\]

for each \( A \in E(G) \). This may be rewritten as
\[(2a) \quad (\partial h)(A) = h(p(A)) - h(q(A)).\]

If \(g\) is a 1-chain on \(G\) over \(Q_n\) its boundary \(\partial g\) is the 0-chain on \(G\) over \(Q_n\) satisfying
\[(3) \quad (\partial g)(a) = \sum_A \eta(A, a)g(A)\]
for each \(a \in V(G)\). We call \(g\) a 1-cycle on \(G\) over \(Q_n\) if \(\partial g = 0\), that is \((\partial g)(a) = 0\) for each \(a\).

2. Colour-coboundaries and colour-cycles. A colour-coboundary or colour-cycle on \(G\) over \(Q_n\) is a 1-chain \(g\) on \(G\) over \(Q_n\) which is a coboundary or a 1-cycle respectively and which satisfies \(g(A) \neq 0\) for each \(A \in E(G)\).

We denote the numbers of colour-coboundaries and colour-cycles on \(G\) over \(Q_n\) by \(\theta(G, n)\) and \(\phi(G, n)\) respectively. These numbers are independent of the orientation of \(G\), by \((2a)\) and \((3)\). We consider that \(\theta(G, n) = \phi(G, n) = 1\) if \(G\) has no edge.

The colour-coboundaries on \(G\) over \(Q_n\) are the coboundaries of the \(n\)-colourings of \(G\), by \((2a)\). Another consequence of \((2a)\) is that \(\partial h_1 = \partial h_2\) for 0-chains \(h_1\) and \(h_2\) on \(G\) over \(Q_n\) if and only if \(h_1(a) - h_2(a)\) is constant in each component of \(G\). Accordingly
\[(4) \quad \theta(G, n) = n^{-p_{-n}(G)}P(G, n).\]

It follows that \(\theta(G, n) = 0\) if \(G\) has a loop. The function \(\phi(G, n)\) need not vanish if \(G\) has a loop. Indeed if \(g\) is a 1-chain on \(G\) over \(Q_n\) and \(A\) is a loop of \(G\) then the 0-chain \(\partial g\) is independent of \(g(A)\), by \((3)\). Hence if \(G_0\) is the graph obtained from \(G\) by suppressing the loops, say \(l(G)\) in number, we have
\[(5) \quad \phi(G_0, n) = (n - 1)^{-l(G)}\phi(G, n).\]

However \(\phi(G, n)\) does vanish if \(G\) has an isthmus. An edge \(A\) of \(G\) with ends \(x\) and \(y\) is called an isthmus of \(G\) if each path from \(x\) to \(y\) has \(A\) as a term. Thus an isthmus is necessarily a link. If \(G_A^\prime\) is the graph obtained from \(G\) by suppressing \(A\) we may say that \(A\) is an isthmus of \(G\) provided \(x\) and \(y\) belong to different components of \(G_A^\prime\). An equivalent definition is that \(A\) is an isthmus of \(G\) provided that it is a term of no circular path in \(G\). For if \(A\) is a term of such a circular path then \(x\) and \(y\) are clearly connected in \(G_A^\prime\). And if a path from \(x\) to \(y\) exists in \(G_A^\prime\) the path of this kind with fewest terms is simple and can be extended to form a circular path in \(G\) having \(A\) as a term.\(^1\)

We observe that a tree may be defined as a connected graph in which each edge is an isthmus.

The proof that \(\phi(G, n)\) vanishes when \(G\) has an isthmus \(A\) is as follows. Let \(H\) be the component of \(G_A^\prime\) having the end \(x\) of \(A\) in \(G\) as a vertex. Let \(g\) be any 1-cycle on \(G\) over \(Q_n\). Then
\[\sum_B \eta(B, b)g(B) = 0\]

\(^1\)Our term “isthmus” applies to each of the two kinds of edge for which König uses the terms “Brücke” and “Endkante” (3, pp. 3, 179).
for each $b \in V(H)$, where $B$ runs through $E(G)$, by (3). Summing this over all the vertices of $H$ we obtain $\eta(A, x)g(A) = 0$. Hence $g(A) = 0$. Accordingly no 1-cycle on $G$ over $Q_n$ is a colour-cycle.

The connection between the function $\phi(G, n)$ and the ordinary theory of map-colourings is best seen by considering two dual graphs $G$ and $G^*$ on the sphere. It may be shown—though we do not prove it here—that $\phi(G^*, n) = \theta(G, n)$. Accordingly each of the following unproved propositions is equivalent to the famous Four Colour Conjecture.

(i) $\theta(G, 4) > 0$ if $G$ is a planar graph without a loop.
(ii) $\phi(G, 4) > 0$ if $G$ is a planar graph without an isthmus.

I wish to draw attention to some unsolved problems related to (ii) but having to do with general graphs. They are the problems of proving or disproving the following conjectures.

**Conjecture I:** There exists a positive integer $m$ such that $\phi(G, n) > 0$ whenever $n \geq m$ and $G$ has no isthmus.

**Conjecture II:** $\phi(G, n) > 0$ whenever $n \geq 5$ and $G$ has no isthmus.

Conjecture II is a stronger version of Conjecture I. We cannot replace 5 by a smaller integer because it can be shown that the Petersen graph (3, p. 194) satisfies $\phi(G, 4) = 0$.

We prove $\phi(G, 4) = 0$ for the Petersen graph as follows. If $\phi(G, 4) > 0$ then for any orientation of $G$ we can find a colour-cycle $g$ on $G$ over $Q_n$. Let $[m]$ denote the residue class of an integer $m$ modulo 4. If $a$ is any vertex of $G$ the three residue classes $\eta(A, a)g(A)$ corresponding to the edges $A$ having $a$ as an end are non-zero and sum to zero. Their values must be either $[1]$, $[1]$, and $[2]$ or $[-1]$, $[-1]$ and $[2]$. In the first case we call $a$ a positive vertex, in the second a negative vertex. The edges $A$ such that $g(A) = [1]$ or $[-1]$ are therefore the edges of some disjoint circular paths no two of which have a common term and which together involve all the vertices. Each of these paths has an even number of edges since positive and negative vertices must alternate in it. It follows that the edges of $G$ can be arranged in three disjoint classes so that each vertex is an end of one member of each class. But it is well known that this is not true for the Petersen graph (4).

We may perhaps regard the following theorem as a very short first step towards a verification of Conjecture I.

**Theorem:** If $\phi(G, n) > 0$ then $\phi(G, n + 1) > 0$.

**Proof.** In the preceding combinatorial definitions we may replace $Q_n$ by the ring of ordinary integers, obtaining integral 0-chains, 1-chains, 1-cycles, etc.

If $\phi(G, n) > 0$ there exists a colour-cycle $g$ on $G$ over $Q_n$. It follows that there is an integral 1-cycle $g'$ on $G$ such that $g'(A) \in g(A)$ and $|g'(A)| < n$ for each $A \in E(G)$. This is a consequence of Theorem IV of (5). It is true that that theorem is stated only for the case in which $G$ is a simplicial 1-complex, that
is a graph without loops and in which no two links have the same two ends, but its proof is valid with only trivial modifications in the general case. Now for each \( A \in E(G) \) we have \( g'(A) \equiv 0 \mod (n + 1) \). Replacing each integer \( g'(A) \) by its residue class \( \mod (n + 1) \) we obtain a colour-cycle on \( G \) over \( \mathbb{Q}_{n+1} \). The theorem follows.

The methods now available for the computation of \( \theta(G, n) \) and \( \phi(G, n) \) are laborious. They depend on some recursion formulae which we exhibit below.

If \( A \) is an edge of \( G \) not a loop we define \( G_A'' \) as the graph obtained from \( G \) by suppressing \( A \) and then identifying the ends of \( A \) in \( G \) to form a single vertex \( t \).

By examining the relationships between the colour-coboundaries, and between the colour-cycles, of the three graphs \( G, G_A' \) and \( G_A'' \), where \( A \) is any edge of \( G \) not a loop or isthmus, we obtain the identities

\[
\theta(G, n) = \theta(G_A', n) - \theta(G_A'', n),
\]

\[
\phi(G, n) = \phi(G_A'', n) - \phi(G_A', n).
\]

For a disconnected graph \( G \) with components \( G_1, \ldots, G_k \) we evidently have

\[
\theta(G, n) = \sum_{i=1}^{k} \theta(G_i, n),
\]

\[
\phi(G, n) = \sum_{i=1}^{k} \phi(G_i, n).
\]

**Lemma:** If \( J \) is a graph in which every edge is an isthmus then every 1-chain on \( J \) over \( \mathbb{Q}_n \) is a coboundary on \( J \).

**Proof.** If \( \alpha_1(J) = 0 \) this is trivial. Suppose it is true whenever \( \alpha_1(J) \) is less than some positive integer \( q \). Consider the case \( \alpha_1(J) = q \).

Let \( h \) be any 1-chain on \( J \) over \( \mathbb{Q}_n \). Let \( h_A \) be the 1-chain on \( J_A' \) over \( \mathbb{Q}_n \) such that \( h_A(B) = h(B) \) for each \( B \in E(J) - \{A\} \). By the inductive hypothesis \( h_A \) is the coboundary of a 0-chain \( f \) on \( G_A' \). Let \( x \) be the positive and \( y \) the negative end of \( A \) in \( J \). Let \( J_0 \) be the component of \( J_A' \) of which \( x \) (but not \( y \)) is a vertex. If we replace \( f(a) \) by \( f(a) + s \) for each \( a \in V(J_0) \), where \( s \) is an element of \( \mathbb{Q}_n \), we shall not alter the coboundary of \( f \) in \( J_A' \). We may therefore suppose that \( f(x) - f(y) = h(A) \). Then \( h \) is the coboundary of \( f \) in \( J \).

The Lemma follows by induction.

Now consider a graph for which each edge is either a loop or an isthmus. Suppose such a graph \( H \) has \( l(H) \) loops and \( i(H) \) isthmuses. We have, as a consequence of the Lemma,

\[
\theta(H, n) = \begin{cases} 0 & \text{if } l(H) > 0, \\ (n - 1)^{(i(H))} & \text{if } l(H) = 0. \end{cases}
\]

As a consequence of (5) we have also

\[
\phi(H, n) = \begin{cases} 0 & \text{if } i(H) > 0, \\ (n - 1)^{(i(H))} & \text{if } i(H) = 0. \end{cases}
\]
3. The dichromate of a graph. We now define a function \( \chi(G, x, y) \) of two variables \( x \) and \( y \), which may be regarded as generalizing both \( \theta(G, n) \) and \( \phi(G, n) \). We call it the dichromate of \( G \).

If \( G \) has no edge we write
\[
\chi(G, x, y) = 1.
\]

If \( G \) has an edge and is connected we proceed as follows. First we enumerate the edges of \( G \) as \( A_1, \ldots, A_m \).

Consider any spanning tree \( T \) of \( G \). Suppose \( A_j \) is an edge of \( T \). Then \( T_{A_j} \) has two components, \( C \) and \( D \) say. Each has one end of \( A_j \) as a vertex. We say \( A_j \) is internally active in \( T \) if each edge \( A_k \) of \( G \) other than \( A_j \) which has one end a vertex of \( C \) and one end a vertex of \( D \) satisfies \( k < j \).

Now suppose \( A_j \) is not an edge of \( T \). Denote its ends by \( a \) and \( b \). (They may not be distinct.) There is a simple path \( P \) in \( T \) from \( a \) to \( b \). There is only one such path. For suppose there are two distinct simple paths \( P_1 \) and \( P_2 \) in \( T \) from \( a \) to \( b \). Then we may suppose some edge \( A_k \) of \( T \) appears in \( P_1 \) but not in \( P_2 \). Let its ends be \( c \) and \( d \), \( c \) preceding \( A_k \) in \( P_1 \). Then in \( T_{A_k} \) there are paths from \( d \) to \( b \), from \( a \) to \( b \) and from \( a \) to \( c \). Hence \( c \) and \( d \) are vertices of the same component of \( T_{A_k} \). This is impossible since \( A_k \) is an isthmus of \( T \). We say \( A_j \) is externally active in \( T \) if each \( A_k \) which is a term of \( P \) satisfies \( k < j \).

If \( A_j \) is an edge of \( G \) we write \( \lambda(T, A_j) = 1 \) or \( 0 \) according as \( A_j \) is or is not internally active in \( T \). We write also \( \mu(T, A_j) = 1 \) or \( 0 \) according as \( A_j \) is or is not externally active in \( T \). We call \( \lambda(T, A_j) \) and \( \mu(T, A_j) \) the internal and external activities respectively of \( A_j \) in \( T \). We denote by \( r(T) \) and \( s(T) \) the numbers of edges of \( G \) which are internally and externally active respectively in \( T \).

We define the dichromate of \( G \) by the formula
\[
\chi(G, x, y) = \sum_T x^{r(T)} y^{s(T)},
\]
the summation being over all the spanning trees of \( G \).

We note that at least one spanning tree of \( G \) exists. This is proved by König (3, p. 60). (Our "spanning tree" is König's Gerüst). Hence the polynomial on the right of (13) is not identically zero.

To make the above definition significant we must show that \( \chi(G, x, y) \), as defined by (13), is independent of the particular enumeration of the edges of \( G \) which is used.

To prove this we study the effect of interchanging the symbols \( A_i \) and \( A_{i+1} \) between the two corresponding edges. With respect to the new enumeration let \( \lambda'(T, A_j) \) and \( \mu'(T, A_j) \) denote the internal and external activities respectively in the spanning tree \( T \) of \( G \), of the edge initially denoted by \( A_j \). For each \( T \) let the interchange of the two symbols replace \( r(T) \) and \( s(T) \) by \( r'(T) \) and \( s'(T) \) respectively.

The following argument is stated in terms of the initial enumeration.
First we observe that the interchange leaves \( \lambda(T, A_j) \) and \( \mu(T, A_j) \) unaltered if \( A_j \) is not \( A_i \) or \( A_{i+1} \). Hence

\[
(14) \quad r'(T) = r(T) - \lambda(T, A_i) - \lambda(T, A_{i+1}) + \lambda'(T, A_i) + \lambda'(T, A_{i+1}),
\]

\[
(15) \quad s'(T) = s(T) - \mu(T, A_i) - \mu(T, A_{i+1}) + \mu'(T, A_i) + \mu'(T, A_{i+1}).
\]

We partition the set of all spanning trees of \( G \) into three disjoint classes \( X, Y, \) and \( Z \) as follows. \( T \in X \) if \( A_i \) and \( A_{i+1} \) are both edges of \( T \), \( T \in Y \) if neither \( A_i \) nor \( A_{i+1} \) is an edge of \( T \), and \( T \in Z \) if one but not both of \( A_i \) and \( A_{i+1} \) is an edge of \( T \).

If \( T \in X \) or \( T \in Y \) it is clear that the internal and external activities in \( T \) of \( A_i \) and \( A_{i+1} \) are not altered by the interchange. Hence \( r'(T) = r(T) \) and \( s'(T) = s(T) \) in these cases, by (14) and (15).

If \( T \in Z \) let \( A_j \) be the member of the pair \( \{A_i, A_{i+1}\} \) which is an edge of \( T \) and let \( A_k \) be the other member. Let the ends of \( A_j \) be \( a \) and \( b \). Let \( C \) and \( D \) be the two components of \( T_A' \), having \( a \) and \( b \) respectively as a vertex. Let \( c \) and \( d \) be the ends, not necessarily distinct, of \( A_k \). We partition the set \( Z \) into two disjoint subclasses \( Z_1 \) and \( Z_2 \) by the following rule: \( T \in Z_1 \) if \( c \) and \( d \) are vertices of the same component of \( T_A' \), and \( T \in Z_2 \) otherwise.

If \( T \in Z_1 \) the simple path \( P \) from \( c \) to \( d \) in \( T \) does not have \( A_j \) as a term. Accordingly the internal and external activities of \( A_j \) and \( A_k \) in \( T \) are not affected by the interchange. So \( r'(T) = r(T) \) and \( s'(T) = s(T) \) in this case also.

Suppose \( T \in Z_2 \). Then we may suppose that \( c \) is a vertex of \( C \) and \( d \) is a vertex of \( D \). Let \( \sigma(T) \) be the spanning subgraph of \( G \) obtained by suppressing the edge \( A_j \) and adjoining the edge \( A_k \). Clearly \( \sigma(T) \) is connected. We show that it is a spanning tree of \( G \). For otherwise there is a circular path \( P \) in \( \sigma(T) \). This has \( A_k \) as a term since it is not a path in \( T \). This implies that there is a simple path from \( c \) to \( d \) in \( \sigma(T)_A \), that is in \( T_A \), which is false. Now clearly \( \sigma(T) \in Z_2 \) and \( \sigma(\sigma(T)) = T \). We note that \( A_j \) and \( A_k \) must be redefined in terms of \( \sigma(T) \) before the operation \( \sigma \) is repeated.

We deduce that \( Z_2 \) can be partitioned into disjoint pairs of the form \( \{T, \sigma(T)\} \) such that \( A_i \) is an edge of \( T \). In what follows we take \( T \) to be the first member of such a pair.

Suppose first that some edge \( A_w \) of \( G \) distinct from \( A_i \) and \( A_{i+1} \) is internally active in \( T \) but not in \( \sigma(T) \).

Without loss of generality we may suppose \( A_w \) is an edge of the tree \( C \). Let \( C_1 \) and \( C_2 \) be the two components of \( C_{A_w} \), \( a \) being a vertex of \( C_2 \). Let \( A_w \) have ends \( \alpha \in V(C_1) \) and \( \beta \in V(C_2) \). If \( c \in V(C_2) \) then since \( \lambda(\sigma(T), A_w) = 0 \) there is an edge \( A_\gamma \) of \( G \) such that \( v > w \) and which has one end in \( V(C_1) \) and one end in \( V(C_2) \) or \( V(D) \). But then \( A_w \) cannot be internally active in \( T \), contrary to its definition. We deduce that \( c \in V(C_1) \). Now since \( \lambda(\sigma(T), A_w) = 0 \) it follows that there is an edge \( A_\gamma \) of \( G \) having one end \( \gamma \) in \( V(C_2) \) and one end \( \delta \) in \( V(D) \) or \( V(C_1) \), and which satisfies \( v > w \). Actually \( \delta \in V(D) \) since \( \lambda(T, A_w) = 1 \). This state of affairs is represented in Figure 1.
Since $\lambda(T, A_w) = 1$ we have $v > w > i + 1$. Hence

(16) $\lambda(T, A_i) = \lambda'(T, A_i) = 0, \lambda(\sigma(T), A_{i+1}) = \lambda'(\sigma(T), A_{i+1}) = 0$.

There is a circular path $J$ from $a$ to $a$ in $G$ which has $A_i, A_{i+1}$ and $A_w$ as terms. Apart from these terms it is made up of three simple paths, one from $b$ to $d$ in $D$, one from $c$ to $a$ in $C_1$ and one from $\beta$ to $a$ in $C_2$. It follows that the simple paths from $c$ to $d$ in $T$ and from $a$ to $b$ in $\sigma(T)$ each have $A_w$ as a term. Hence

(17) $\mu(T, A_{i+1}) = \mu'(T, A_{i+1}) = 0, \mu(\sigma(T), A_i) = \mu'(\sigma(T), A_i) = 0$.

Suppose next that some edge $A_w$ of $G$ distinct from $A_i$ and $A_{i+1}$ is externally active in $T$ but not in $\sigma(T)$. Let its ends be $\alpha$ and $\beta$. They are not vertices of the same tree $C$ or $D$; otherwise the simple paths from $\alpha$ to $\beta$ in $T$ and $\sigma(T)$ would be identical and this would imply $\mu(T, A_w) = \mu(\sigma(T), A_w)$. Hence we may suppose $\alpha \in V(C)$ and $\beta \in V(D)$. (See Fig. 2.)
Let $P_1$ and $P_2$ be the simple paths in $T$ and $\sigma(T)$ respectively from $\alpha$ to $\beta$. Then $A_1$ is a term of $P_1$ and $A_{i+1}$ is a term of $P_2$. Since $\mu(T, A_w) = 1$ we have $w > i$ and therefore $w > i + 1$. Hence formula (16) holds in this case also.

In $P_1$ let $\alpha'$ be the last vertex of $G$ preceding $A_i$ which is a term of $P_1$, and let $\beta'$ be the first vertex of $G$ succeeding $\alpha'$ which is a term of $P_2$. Clearly $\alpha' \in V(C)$ and $\beta' \in V(D)$. Let $R_1$ and $R_2$ be the subsequences of $P_1$ and $P_2$ respectively extending from $\alpha$ to $\alpha'$. There is a circular path $J$ in $G$ formed by taking first the terms of $R_1$ and then the terms of $R_2$ in reverse order. It has $A_i$ and $A_{i+1}$ as terms, for they are terms of $P_1$ and $P_2$ respectively. The subsequences of $P_1$ and $P_2$ extending from $\alpha$ to $\alpha'$ are identical, since $C$ is a tree. Similarly the subsequences of $P_1$ and $P_2$ extending from $\alpha'$ to $\beta'$ are identical.

Since $\mu(T, A_w) = 1$ and $\mu(\sigma(T), A_w) = 0$ there must be an edge $A_* \in G$ which is a term of $J$ and satisfies $v > w$. Then the simple paths from $c$ to $d$ in $T$ and from $a$ to $b$ in $\sigma(T)$ each have $A_*$ as a term. Hence formula (17) still holds.

Next we consider the case in which some edge of $G$ is internally or externally active in $\sigma(T)$ but not in $T$. We first go over to the new enumeration by interchanging the symbols $A_i$ and $A_{i+1}$. This interchanges $T$ and $\sigma(T)$. The foregoing argument shows that (16) and (17) are true in the new enumeration. They are relations between the two enumerations. To state them in terms of the old enumeration we have merely to interchange the symbols $X$ and $X'$, $\mu$ and $\mu'$, $A_i$ and $A_{i+1}$ and finally $P$ and $\sigma(T)$. But the sets of equations are invariant under this operation.

In all these three cases (16) and (17) are true. Hence by (14) and (15) we have $r'(T) = r(T) \quad s'(\sigma(T)) = r'(\sigma(T)) \quad s'(T) = s(T) \quad s'(\sigma(T)) = s(\sigma(T)) \quad (18)$

We now consider the remaining case, in which $\lambda(T, A_w) = \lambda(\sigma(T), A_w)$ and $\mu(T, A_w) = \mu(\sigma(T), A_w)$ for each edge $A_w$ of $G$ other than $A_i$ and $A_{i+1}$. If there is an edge $A_*$ of $G$ satisfying $v > i + 1$ and having ends in both $V(C)$ and $V(D)$, then $\lambda(T, A_i) = \lambda'(T, A_i) = \lambda(\sigma(T), A_{i+1}) = \lambda'(\sigma(T), A_{i+1}) = 0$. If there is no such edge $A_*$ we have instead $\lambda(T, A_i) = \lambda(\sigma(T), A_{i+1}) = 0$ and $\lambda(\sigma(T), A_{i+1}) = \lambda'(T, A_i) = 1$.

There is a circular path $J$ from $a$ to $a$ in $G$ having $A_i$ and $A_{i+1}$ as terms and otherwise consisting of a simple path from $a$ to $c$ in $C$ and another from $d$ to $b$ in $D$. If $J$ has a term $A_*$ such that $v > i + 1$, then $\mu(T, A_{i+1}) = \mu'(T, A_{i+1}) = \mu(\sigma(T), A_i) = \mu'(\sigma(T), A_i) = 0$. If $J$ has no such term we have instead $\mu(T, A_{i+1}) = \mu'(\sigma(T), A_i) = 1$ and $\mu(\sigma(T), A_i) = \mu'(T, A_{i+1}) = 0$.

It follows that $r'(T) = r(\sigma(T)) \quad s'(\sigma(T)) = r(T) \quad s'(T) = s(\sigma(T))$ and $s'(\sigma(T)) = s(\sigma(T))$.

The foregoing analysis shows that the sum on the right of (13) is not affected by the interchange of the symbols $A_i$ and $A_{i+1}$. All that happens is that the contributions to the sum of certain pairs of trees are interchanged. But any permutation of the symbols $A_1, \ldots, A_m$ can be effected by a finite number of interchanges of consecutive symbols. Hence the function $\chi(G, x, y)$ defined by (13) is independent of the particular enumeration of edges employed.
We extend the definition of the dichromate to graphs which are not connected as follows. If the components of $G$ are $G_1, \ldots, G_k$, then

\[ \chi(G, x, y) = \prod_{i=1}^{k} \chi(G_i, x, y). \]

This is consistent with (12) in the case of an edgeless graph.

We note some general properties of the dichromate.

(i) $\chi(G, x, y)$ is a polynomial of degree $\alpha_0(G) - \rho_0(G)$ in $x$ and of degree $\alpha_1(G) - \alpha_0(G) + \rho_0(G)$ in $y$.

Proof. By a simple induction we find that a graph $S$ in which each edge is an isthmus satisfies $\alpha_1(S) = \alpha_0(S) - \rho_0(S)$. A connected graph $G$ has at least one spanning tree, and each such tree $T$ satisfies $\alpha_1(T) = \alpha_0(G) - \rho_0(G)$.

If $G$ is connected and has an edge the theorem follows from (13). For the contribution to the sum on the right of (13) of any spanning tree $T$ is of degree at most $\alpha_1(T)$ in $x$ and at most $\alpha_1(G) - \alpha_1(T)$ in $y$. By choosing a suitable enumeration of the edges of $G$ we can arrange that either of these values is attained. The proposition follows in this case.

We extend it to all $G$ by applying (12) and (18).

(ii) If $A$ is an edge of $G$ not a loop or an isthmus, then

\[ \chi(G, x, y) = \chi(G_A', x, y) + \chi(G_A'', x, y). \]

Proof. This proof depends on the observation that for a connected $G$ the spanning trees of $G_A'$ are those spanning trees of $G$ which do not have $A$ as an edge, while the spanning trees of $G_A''$ are the graphs $T_A''$ such that $T$ is a spanning tree of $G$ having $A$ as an edge. We enumerate the edges of $G$ so that $A = A_1$. We obtain corresponding enumerations for $G_A'$ and $G_A''$ by rejecting $A_1$ and then reducing each suffix by 1. With these enumerations each tree not having $A$ as an edge makes the same contribution to $\chi(G_A', x, y)$ as to $\chi(G, x, y)$, and any other tree $T$ makes the same contribution to $\chi(G, x, y)$ as does $T_A''$ to $\chi(G_A'', x, y)$.

The proposition follows for a connected $G$.

If $G$ is not connected let $G_1$ be its component having $A$ as an edge. Then $G_A'$ and $G_A''$ have the same components as $G$ except that $G_1$ is replaced by $(G_1)_A'$ and $(G_1)_A''$ respectively. Since (19) is true for $G_1$ it follows from (18) that it is true also for $G$.

(iii) Let $H$ be a graph having $l(H)$ loops, $i(H)$ isthmuses, and no other edge. Then

\[ \chi(H, x, y) = x^{l(H)} y^{i(H)}. \]

Proof. If $H$ is connected form $H_0$ from it by suppressing the loops. Clearly $H_0$ is the only spanning tree of $H$. So (20) follows from (12) and (13). Using (18) we readily extend the formula to the general case.

Formulae (19) and (20) provide a method for computing the dichromate.
of a given graph $G$. If $G$ has an edge $A$ which is not a loop or an isthmus then (19) expresses the dichromate in terms of the dichromates of simpler graphs. Otherwise (20) gives the dichromate directly.

Such computations may sometimes be shortened by using the following theorem.

(iv) If $G$ consists of two connected graphs $H_1$ and $H_2$ having just one vertex $b$ in common, then

$$\chi(G, x, y) = \chi(H_1, x, y) \chi(H_2, x, y).$$

To prove this we observe that a subgraph of $G$ is a spanning tree of $G$ if and only if it is the union of a spanning tree of $H_1$ and a spanning tree of $H_2$. We then apply (13).

Comparing (19) and (20) with (6) and (10), or with (7) and (11), we arrive at inductive proofs of the following formulae.

$$\theta(G, n) = (-1)^{p(G)_0 - p(G)_1} \chi(G, 1 - n, 0),$$

$$\phi(G, n) = (-1)^{q(G)_0 - q(G)_1 + p(G)_1} \chi(G, 0, 1 - n).$$

These formulae justify our description of $\chi(G, x, y)$ as generalizing both $\theta(G, n)$ and $\phi(G, n)$.

The result that for a connected graph $G$ the sum on the right of (13) is invariant under a change of enumeration of the edges is an interesting theorem about the spanning trees of $G$. As one of its corollaries we have:

For each enumeration of the edges of $G$ there exist spanning trees $T_1$ and $T_2$ of $G$ such that each edge of $T_1$ is internally active in $T_1$ and each edge of $G$ not an edge of $T_1$ is externally active in $T_2$.

The number $C(G)$ of spanning trees of a graph $G$ is important in the theory of electrical networks in which the conductance of each wire is unity. A summary of this theory is given in (2). So the theory of spanning trees provides a link between the theory of graph-colourings and the theory of electrical networks. The dichromate can be regarded as a generalization of $C(G)$, for we have

$$C(G) = \chi(G, 1, 1).$$

$C(G)$ has a simple expression as a determinant, and its properties are well known. Perhaps some of them will suggest new properties of the dichromate and hence of the chromatic polynomials.

References


*University of Toronto.*