Let \( p \) be a prime number, \( F \) the field of \( p \) elements, \( M \) the semigroup of all 2 \( \times \) 2 matrices over \( F \), \( G \) the group \( \text{GL}(2, p) \) of invertible elements of \( M \), and \( S \) the normal subgroup \( \text{SL}(2, p) \) of \( G \) consisting of the matrices of determinant one. The aim of this thesis is to study the representations of \( M, G \), and \( S \) over \( F \).

A certain construction is of great help. Let \( V \) be the commutative polynomial algebra in two indeterminants, \( x \) and \( y \) say, over \( F \). For each positive integer \( m \), the homogeneous polynomials of degree \( m \) form a subspace \( V_m \), of dimension \( m \), in \( V \), and \( V = \bigoplus_{m=1}^{\infty} V_m \). Each \( V_m \) may be regarded as an \( FM \)-module by considering the elements of \( M \) as homogeneous linear substitutions, so an element \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of \( M \) applied to a monomial \( x^iy^j \) yields \( (ax+by)^i(cx+dy)^j \).

The one dimensional \( FM \)-modules, other than \( V_1 \), are the tensor powers \( D_1^n \) (\( n = 1, 2, \ldots, p-1 \)) of the module \( D \) which affords the determinant representation (so on this each element of \( M \) acts as the scalar which is its determinant); we put \( D^0 = V_1 \). We show that \( M \) has precisely \( p^2 \) (isomorphism classes of) irreducible modules over \( F \);
namely the $V_m \otimes D^n$ with $1 \leq m \leq p$, $0 \leq n \leq p-1$. These modules, after restriction, yield all the irreducibles of $G$ and $S$ as well, but to keep them pairwise nonisomorphic the upper end of the range of $n$ has to be brought down to $p-2$ and $0$ respectively. For each of $M, G, S$, the principal indecomposable modules are described in sufficient detail to reveal their complete submodule structure. For $G$ and $S$, all principal indecomposables occur as direct summands in the restrictions of the $V_m$, while the same will hold for some, but definitely not all, principal indecomposables in the case of $M$. For $G$ and $S$, direct decompositions of restrictions of the $V_m$ have at most one nonprojective summand each; this is also false for $M$. For $G$ and $S$, the nonprojective indecomposable direct summands of the $V_m$ form periodic sequences with period $p(p-1)$. In the (repeated) initial segment of length $p(p-1)$ of this sequence, every $p$th term is 0 while the others are nonzero and pairwise nonisomorphic. In the case of $S$, every nonprojective indecomposable is isomorphic to one and only one of these, while in the case of $G$, every nonprojective indecomposable is isomorphic to one and only one of these tensored with a $D^n$. The nonprojective indecomposables for $G$ and $S$ can therefore be described in sufficient detail to reveal a great deal of their submodule structure. Each $G$-module can be made into an $M$-module by making all elements of $M$ outside $G$ annihilate it: call such $M$-modules singular. Now $M$ has infinitely many isomorphism types of nonsingular indecomposables, but only finitely many of them occur as direct summands of the $V_m$, and we do not attempt to classify even those which do.

To complete this summary of the results in this thesis, it remains to indicate what we know about the structure of the $V_m$. As far as the action of $G$ or $S$ is concerned, each $V_m$ is a direct summand of $V_{m+p(p-1)}$ with a complement which depends only on the residue class of $m \mod p^2 - 1$. This makes it possible to describe all $V_m$ by dealing, as we do, with the first few. By contrast, with respect to the action of $M$, the $V_m$ do not fit such an arithmetic pattern: for instance, $V_2$ is not a
direct summand of any other \( V_m \), and is not even embeddable in \( V_m \) unless \( m - 1 \) is a power of \( p \).