DIMENSION AND LOWER CENTRAL SUBGROUPS
OF METABELIAN $p$-GROUPS

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To the memory of the late Takehiko Miyata

§ 1. Introduction

It is a well-known result due to Sjogren [9] that if $G$ is a finitely generated $p$-group then, for all $n \leq p - 1$, the $(n + 2)$-th dimension subgroup $D_{n+2}(G)$ of $G$ coincides with $\gamma_{n+2}(G)$, the $(n + 2)$-th term of the lower central series of $G$. This was earlier proved by Moran [5] for $n \leq p - 2$. For $p = 2$, Sjogren’s result is the best possible as Rips [8] has exhibited a finite 2-group $G$ for which $D_4(G) \nsubseteq \gamma_4(G)$ (see also Tahara [10, 11]). In this note we prove that if $G$ is a finitely generated metabelian $p$-group then, for all $n \leq p$, $D_{n+2}(G) \subseteq \gamma_{n+2}(G)$. It follows, in particular, that, for $p$ odd, $D_{n+2}(G) = \gamma_{n+2}(G)$ for all $n \leq p$ and all metabelian $p$-groups $G$.

§ 2. Notation and preliminaries

While the central idea of the proof of our main result stems from Gupta [1], with a slight repetition, it is equally convenient to give a self-contained proof using a less cumbersome notation.

Let $\mathfrak{i} = ZF(F - 1)$ denote the augmentation ideal of the integral group ring $ZF$ of a free group $F$ freely generated by $x_1, x_2, \ldots, x_m, m \geq 2$. For a fixed prime $p$, let $(p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_m})$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m > 0$ be an $m$-tuple of $p$-powers, and let $S = \langle x_1^{p^\alpha_1}, x_2^{p^\alpha_2}, \ldots, x_m^{p^\alpha_m}, F' \rangle$ be the normal subgroup of $F$ so that $F/S$ is abelian. Set $\bar{s} = ZF(S - 1)$, the ideal of $ZF$ generated by all elements $s - 1, s \in S$. For $1 \leq n \leq p$, we shall need to investigate the structure of the subgroup $D_{n+2}(\bar{s}) = F \cap (1 + \bar{s} + \bar{s}^{p+2})$ of $F$ which consists of all elements $w \in F$ such that $w - 1 \in \bar{s} + \bar{s}^{p+2}$. It is clear that $[F', S]\gamma_{n+2}(F) \subseteq D_{n+2}(\bar{s})$.

Let $w \in D_{n+2}(\bar{s})$ be an arbitrary element. Then $w - 1 \in \mathfrak{i}$ and it

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follows that \( w \in F' \). Thus, modulo \( F'' \), using the Jacobi identity, we may write \( w \) as
\[
(1)\quad w \equiv w_1 w_2 \cdots w_{m-1},
\]
where
\[
(2)\quad w_i = \prod_{j=i+1}^m [x_i, x_j]^{d_{ij}}
\]
and \( d_{ij} = d_{ij}(x_1, x_{i+1}, \ldots, x_m) \in \mathbb{Z}F \). For \( i = 1, 2, \ldots, m \), define homomorphisms \( \theta_i : \mathbb{Z}F \to \mathbb{Z}F \) by \( x_k \mapsto 1 \) if \( k \leq i \), \( x_k \mapsto x_k \) if \( k > i \). Since the ideals \( \mathfrak{j}, \mathfrak{s} \) are invariant under \( \theta_i \)'s, it follows, using \( \theta_1, \theta_2, \ldots, \theta_{m-2} \) in succession, that if \( w - 1 \in \mathfrak{j} + \mathfrak{s} + \mathfrak{i}^{n+2} \) then \( w_i - 1 \in \mathfrak{j} + \mathfrak{s} + \mathfrak{i}^{n+2} \) for each \( i \). For each \( k = 1, 2, \ldots, m \), define
\[
(3)\quad t(x_k) = 1 + x_k + \cdots + x_k^{p^{a_k}-1}.
\]
Then
\[
(4)\quad t(x_k) = \sum_{i=1}^{p^{a_k}} \left( \sum_{j=1}^{m} \binom{p^a - 1}{j} \right) (x_k - 1)^{j-1} \\
\equiv p^{a_k} + \left( \sum_{j=1}^{m} \binom{p^a - 1}{j} \right) (x_k - 1)^{p-1} \mod (\mathfrak{s} + \mathfrak{i}^p).
\]
We can now prove,

**Lemma 2.1.** Let \( w_i \) be as in (2) with \( w_i - 1 \in \mathfrak{j} + \mathfrak{s} + \mathfrak{i}^{n+2} \) and \( n \leq p \). Then, modulo \( \mathfrak{s} + \mathfrak{i}^p \), \( d_{ij} \equiv t(x_i) a_{ij} \equiv t(x_i) b_{ij} \), where \( t(x_i) \), \( t(x_j) \) are given by (3), \( a_{ij} \in Z \) and \( b_{ij} \in \mathbb{Z}F \). Moreover, if \( \alpha_i = \alpha_j \), then \( b_{ij} \in Z \).

**Proof.** Expansion of \( w_i - 1 \) shows
\[
(5)\quad \sum_{j=0}^{m} ((x_i - 1)(x_j - 1) - (x_j - 1)(x_i - 1)) d_{ij} \in \mathfrak{j} + \mathfrak{s} + \mathfrak{i}^{n+2}.
\]
Since \( \mathfrak{j} \) is a free right \( \mathbb{Z}F \)-module on \( x_1, x_2, \ldots, x_m \), it follows from (5) that, for all \( j = i + 1, \ldots, m \),
\[
(x_i - 1)(x_j - 1) d_{ij} \in \mathfrak{j} + \mathfrak{s} + \mathfrak{i}^{n+2},
\]
which yields
\[
(6)\quad (x_i - 1) d_{ij} \in \mathfrak{s} + \mathfrak{i}^{n+1}
\]
and, in turn,
\[
(7)\quad d_{ij} \in t(x_i) \mathbb{Z}F + \mathfrak{s} + \mathfrak{i}^p,
\]
where \( t(x_i) \) is given by (3). Since \( n \leq p \), (4) induces that, for \( k \geq i \),
\[
t(x_i)(x_k - 1) \equiv p^{e_i - a_k} p^{a_k}(x_k - 1) \equiv 0 \mod (\mathfrak{s} + \mathfrak{i}^p).
\]
Thus (7) implies \( d_{ij} \equiv \)
(x_i - 1) \sum_{j=i+1}^{m} (x_j - 1) d_{ij} \in \bar{s} + \bar{t}^{n+1}.

and, as before,

\sum_{j=i+1}^{m} (x_j - 1) d_{ij} \in \bar{s} + \bar{t}^{n+1}.

Using the homomorphisms \( \theta_{i+1}, \ldots, \theta_{m-1} \) in turn, gives

\begin{equation}
(x_j - 1) d_{ij} \in \bar{s} + \bar{t}^{n+1}
\end{equation}

for all \( j = i + 1, \ldots, m \), since \( d_{ij} \equiv t(x_i) a_{ij} \mod (\bar{s} + \bar{t}^{n}) \) with \( a_{ij} \in \mathbb{Z} \). Thus

\begin{equation}
d_{ij} \in t(x_i)ZF + \bar{s} + \bar{t}^{n},
\end{equation}

and if \( \alpha_i = \alpha_j \) then, as before, \( d_{ij} \equiv t(x_i) b_{ij} \mod (\bar{s} + \bar{t}^{n}) \) with \( b_{ij} \in \mathbb{Z} \). This completes the proof of the lemma.

Now, let \( \frac{\partial}{\partial x_k} d \) be a free partial derivative of \( d \in ZF \) with respect to \( x_k \). Then we prove,

**Lemma 2.2.** \( \frac{\partial}{\partial x_k} d_{ij} \in p^{\alpha_i}ZF + \bar{s} + \bar{t}^{n-1}, i < k, \) and

\[ \frac{\partial}{\partial x_i} d_{ij} \in \begin{cases} p^{\alpha_i}ZF + \bar{s} + \bar{t}^{n-1} & \text{if } \alpha_i = \alpha_j \\ p^{\alpha_i}ZF + p^{\alpha_i-1}(x_i - 1)p^{\alpha_i-1}ZF + \bar{s} + \bar{t}^{n-1} & \text{if } \alpha_i > \alpha_j. \end{cases} \]

**Proof.** We have

\[ \frac{\partial}{\partial x_k} (\bar{s}) \subseteq \bar{s} + p^{\alpha_i}ZF; \quad \frac{\partial}{\partial x_k} (\bar{t}^{n}) \subseteq \bar{t}^{n-1}. \]

Thus since \( d_{ij} \equiv t(x_i) a_{ij} \mod (\bar{s} + \bar{t}^{n}) \) with \( a_{ij} \in \mathbb{Z} \), it follows that

\[ \frac{\partial}{\partial x_k} d_{ij} \equiv 0 \mod (p^{\alpha_i}ZF + \bar{s} + \bar{t}^{n-1}). \]

By (4) and \( d_{ij} \equiv t(x_i) a_{ij} \mod (\bar{s} + \bar{t}^{n}) \), we have

\[ \frac{\partial}{\partial x_i} d_{ij} \equiv a_{ij} \left( \frac{p^{\alpha_i}}{p} \right) (p - 1)(x_i - 1)p^{\alpha_i-2} \mod (p^{\alpha_i}ZF + \bar{s} + \bar{t}^{n-1}). \]

Since \( p^{\alpha_i-1} \) divides \( \left( \frac{p^{\alpha_i}}{p} \right) \), \( \frac{\partial}{\partial x_i} d_{ij} \equiv 0 \mod (p^{\alpha_i-1}(x_i - 1)p^{\alpha_i-1}ZF + p^{\alpha_i}ZF + \bar{s} + \bar{t}^{n-1}). \) If \( \alpha_i = \alpha_j \), then \( b_{ij} \in \mathbb{Z} \), and we may differentiate \( d_{ij} \equiv t(x_i) b_{ij} \) with
respect to $x_i$ to obtain the desired result.

Next, we need to expand $[x_i, x_j]^{d_{ij}} - 1$ modulo $(n^{2\delta} + n^{\nu + 2})$. We first observe,

$$
[x_i, x_j]^{d_{ij}} - 1 = x_i \cdot x_j - 1
$$

Thus we have,

$$
[x_i, x_j]^{d_{ij}} - 1 \equiv ([x_i, x_j] - 1)d_{ij} - \sum_{k=1}^{m} (x_k - 1)([x_i, x_j]^{d_{ij}} - 1).
$$

Now, modulo $(n^{2\delta} + n^{\nu + 2})$

$$
([x_i, x_j] - 1)d_{ij} \equiv (x_i - 1)x_j - (x_j - 1)(x_i - 1)d_{ij}
$$

Finally, using (6) and (8), we have, for any $x_a$, mod $[F', S]_F$,

$$
[[x_i, x_j]^{d_{ij}}, x_a] \equiv [x_i, x_j, x_a]^{d_{ij}}
$$

Thus we have,
LEMMA 2.4 (Gupta [2]). \([D_{n+3}(\text{fg}), F] \subseteq [F', S] r_{n+3}(F)\) for all \(n \geq 0\).

This completes our preliminary discussions.

§ 3. The main theorem

Let \(G\) be a finitely generated metabelian \(p\)-group. Then \(G\) admits a presentation of the form

\[
G = F/R = \langle x_1, x_2, \ldots, x_m; x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_m^{\alpha_m}, \gamma_1, \gamma_2, \ldots, F' \rangle,
\]

where \(\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m > 0\) (see for instance [4], page 149). Let \(S\) be the normal subgroup of \(F\) generated by \(x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_m^{\alpha_m}\) and \(F'\), then it follows that \(S' \subseteq R \subseteq S\). In terms of the free group rings, the dimension subgroup \(D_{n+2}(G) = D_{n+2}(x)/R\), where \(\tau = ZF(R - 1)\) and \(D_{n+2}(\tau) = F(1 + r + \cdots + \gamma^{n+2})\). Then \(Rr_{n+2}(F) \subseteq D_{n+2}(\tau)\). If \(z \in D_{n+2}(\tau)\), then \(z - 1 \in \tau + \gamma^{n+2}\) implies that \(zr - 1 \in \tau + \gamma^{n+2}\) for some \(r \in R\). It follows that \(D_{n+2}(G) = r_{n+2}(G)\) if and only if \(D_{n+2}(\tau) = F(1 + \tau + \gamma^{n+2}) \subseteq Rr_{n+2}(F)\). We now prove our main result.

THEOREM 3.1. \(D_{n+2}(\tau) \subseteq Rr_{n+2}(F)\) for all \(n \leq p\).

Proof. Let \(w \in D_{n+2}(\tau)\). Then \(w - 1 \in \tau + \gamma^{n+2} \subseteq \delta + \gamma^{n+2}\), and by Lemma 2.1,

\[
w \equiv \prod_{1 \leq i < j \leq m} x_i x_j d_{ij}^{\delta_{ij}} \mod F',
\]

where \(d_{ij} \equiv t(x_i)a_{ij} \equiv t(x_i)b_{ij} \mod (\delta + \gamma)\). Now, \(w - 1 \in \tau + \gamma^{n+2}\) implies \(w - 1 \in \tau + \gamma^{n+2} \subseteq \delta + \gamma^{n+2}\). Then it follows by Lemma 2.3, that

\[
w - 1 = \sum_{k=1}^m (x_k - 1)(y_k u_k^{-1} - 1) \equiv 0 \mod (\tau + \gamma^{n+2}),
\]

where

\[
y_k = \prod_{i < k} x_i \prod_{k < j} x_j d_{ij}^{\delta_{ij}} , \quad u_k = \prod_{i \leq k} (x_i, x_j) d_{ij}^{\delta_{ij} / \gamma^{n+2}}.
\]

From (10) it follows that for each \(k = 1, 2, \ldots, m\),

\[
y_k u_k^{-1} - 1 \in \tau + \gamma^{n+2} + \gamma^{n+1},
\]

which yields, in turn, using \(\tau \subseteq \delta\),

\[
y_k u_k^{-1} r_k = 1 \in \delta + \gamma^{n+1}
\]

with some \(r_k \in R\), and by Lemma 2.4, for all \(k = 1, 2, \ldots, m\),
which reduces to

\[ [x_k, y_k u_k^{-1} r_k] \in R \gamma_{n+2}(F) \]

and hence

(11) \[ [x_k, u_k^{-1}] [x_k, y_k] \in R \gamma_{n+2}(F) \]

Next, \([x_k, u_k^{-1}] \equiv [x_k, u_k]^{-1} \mod R \gamma_{n+2}(F)\), and \([x_k, u_k]\) is a product of commutators of the form

\[ [x_k, [x_i, x_j]]^{x_k(\beta /\alpha_0 \beta) \delta_{ij}}, \quad 1 \leq i \leq k, \quad 1 \leq i < j \leq m \]

By Lemma 2.2, for either \(i < k\) or \(i = k\) and \(\alpha_i = \alpha_j\),

\[ [x_k, [x_i, x_j]]^{x_k(\beta /\alpha_0 \beta) \delta_{ij}} \equiv [x_k, [x_i, x_j]]^{x_k v_0 u_0} \text{ for some } v \in ZF, \]

\[ \equiv [x_k^{\alpha_k}, [x_i, x_j]]^{x_k^{-1}} \]

\[ \equiv 1 \mod [F', S] \gamma_{n+2}(F) \]

If \(i = k\) and \(\alpha_i > \alpha_j\), then by Lemma 2.2, for some \(v, w \in ZF,

\[ [x_k, [x_i, x_j]]^{x_k(\beta /\alpha_0 \beta) \delta_{ij}} \equiv [x_k, [x_i, x_j]]^{x_k p^{v-1} p^{\alpha_k-1} p^{\beta_0-1} p^{\alpha_j-1}} \]

\[ \equiv [x_k^p, x_i, \ldots, x_j]^{p \alpha_k-1 - \alpha_j} \text{ mod } [F', S] \gamma_{n+2}(F) \]

\[ \equiv 1 \mod R \gamma_{n+2}(F) \]

Thus (11) is reduced to \([x_k, y_k] \in R \gamma_{n+2}(F)\). However,

\[ [x_k, y_k] \equiv \prod_{i < k} [x_i^{\alpha_i q_i}, x_i] \prod_{k < j} [x_k, x_j^{\beta_j q_j}] \]

\[ \equiv \prod_{i < k} [x_i, x_k]^{\delta_{ik}} \prod_{k < j} [x_k, x_j]^{\delta_{kj}} \text{ mod } [F', S] \gamma_{n+2}(F) \]

Thus

\[ w^2 \equiv \prod_{k=1}^{m} [x_k, y_k] \equiv 1 \mod R \gamma_{n+2}(F) \]

This completes the proof of our main theorem.

As a corollary we obtain,

**Theorem 3.2.** Let \(G\) be a finitely generated metabelian \(p\)-group. Then
(a) \( D_{n+2}(G) = \gamma_{n+2}(G) \) for all \( n \leq p - 1 \),
(b) if \( p = 2 \), \( D_2(G) \subseteq \gamma_2(G) \),
(c) if \( p \) is odd, \( D_{p+2}(G) = \gamma_{p+2}(G) \).

For \( p = 3 \), part (a) of Theorem 3.2 was first proved by Passi [6]; part (b) is due to Losey [3]. We refer the reader to Passi [7] for a general background on the dimension subgroup problem.

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