ON THE NORMALISER OF A GROUP
IN THE CAYLEY REPRESENTATION

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Let $G$ be a $p$-group of order $n$ and embed $G$ into $S_n$ by the Cayley representation. If $X$ is a group such that $G < X \leq S_n$ and $C_X(G) = G$, then it is proved that $G$ is properly contained in $N_X(G)$.

1.

Let $R$ be the Cayley representation (that is, the right regular representation) of a group $G$ given by $R(g) = (x^g)$ for all $g \in G$ and $x \in G$. Under the mapping $R$, the group $G$ is embedded into a subgroup $R(G)$ of the symmetric group $S_n$ where $n$ is the cardinality of $G$. We identify $G$ with $R(G)$. It is not hard to see that the centraliser of $G$ in $S_n$ consists of precisely the elements of the form $(x/gx)$.

Suppose that the group $G$ is non-abelian. If $X$ is a group containing a permutation of the form $(x/gx)$ for some $g \in G \setminus Z(G)$ such that the property

(*)

$G < X \leq S_n$

holds then it follows that $N_X(G)$ contains $G$ properly. However, it is

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easy to see that any element of \( S_n \) which normalises \( G \) is not always a permutation of the form \( (x_{gx}) \). For example, take \( G = S_3 \) and embed it into \( S_6 \) by the Cayley representation. If \( \alpha = (12) \), \( \beta = (13) \), \( \gamma = (23) \) are elements of \( S_3 \) and \( x = (\alpha\gamma\beta) \), then one can check that \( x \) lies outside \( S_3 \) (in its embedding) and \( x^{-1}S_3x = S_3 \) but \( x \) does not centralise \( S_3 \).

When the group \( G \) is abelian, the permutations \( (x_{gx}) \) all lie in \( G \) and so \( G \) is self-centralising in \( S_n \). (This can also be seen by using the fact that \( G \) is transitive and applying Wielandt [3], Theorem 4.4.) So when \( G \) is abelian one cannot obtain by the above method a group \( X \) satisfying (*) such that \( N_X(G) \) contains \( G \) properly.

However, we have

**Theorem 1.** Let \( G \) be a finite p-group and \( X \) be such that

\[
(*) \quad G \leq X \leq S_n
\]

where \( n = |G| \) and \( G \) is embedded in \( S_n \) by the Cayley representation. Assume that the centraliser of \( G \) in \( X \) is \( G \) itself (such a situation will happen when for example, \( G \) is abelian). Then \( G \) is properly contained in \( N_X(G) \).

When \( G \) is an elementary abelian p-group then \( N_X(G) \) is clearly the group of all "affine transformations" on \( G \) regarded as a vector space over \( \text{GF}(p) \). So from Theorem 1 we derive

**Corollary 2.** If \( G \) is an elementary abelian p-group then there is no subgroup of \( S_n \) containing \( G \) which fails to intersect \( N_{S_n}(T) \setminus T \).

It is also interesting to study the problem in the case when \( G \) is an infinite group. A celebrated theorem of Higman, Neumann and Neumann [2] states that if \( G \) is a group then there is a group \( H \) containing \( G \) properly such that any two elements of \( H \) of the same order are conjugate. The proof involves first embedding \( G \) into an uncountable group and then
use the Cayley representation inductively.

2.

Proof of Theorem 1. Suppose that \( X \) is a group satisfying (\(*\)) such that \( N_X(G) = G \). Then \( G \) must be the Sylow \( p \)-group of \( X \) because if \( G \) is contained properly in a Sylow \( p \)-subgroup of \( X \) then there would be an element of \( X \setminus G \) that normalises \( G \) which contradicts the above assumption. By Burnside transfer theorem (see for example, Hall [1], Theorem 14.3) \( X \) has a normal \( p \)-complement \( H \) say. Now \( X \) operates transitively on the set \( G \). Let \( X_0 \) be the stabiliser of some "point" of the set \( G \). Then we have

\[
\bigcap_{x \in X} x^{-1}X_0x = \text{identity}
\]

since the permutation action map \( X \to \text{Perm}(G) \) is injective. Further we have \( |X_0| = |X|/|G| = |H| \). But considering the composite of group homomorphisms,

\[
X_0 \to X \to X/H \cong G,
\]

where the first homomorphism is the natural embedding, we see that the image of \( X_0 \) must be the identity because \( |X_0| \) and \( |G| \) are co-prime. Thus \( X_0 = H \) which contradicts (1). This completes the proof of Theorem 1.

REMARK. If \( G \) is any group (abelian or non-abelian) and \( X \) is a group satisfying the property (\(*\)) then it is not hard to see that a minimal such \( X \) with the property \( N_X(G) = G \) must be of the form \( X = GU \) where \( U \) is a perfect group.

References


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