Annals of Actuarial Science (2023), **17**, pp. 385–414
doi:10.1017/S1748499523000015

ORIGINAL RESEARCH PAPER

An attribution analysis of investment risk sharing in collective defined contribution schemes

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(Received 10 May 2022; revised 17 January 2023; accepted 22 January 2023; first published online 28 February 2023)

Abstract

A quantification of the financial implications of the design of a funded, collective defined contribution (CDC) pension scheme is presented and illustrated. It is done through an attribution analysis, which allows the importance of various elements of CDC scheme design to be determined. The model of a CDC scheme analysed is based lightly on the first CDC scheme set to be approved in the UK. In the CDC scheme analysed, contributions are fixed and the initial benefit accrued by each contribution is fixed. Once accrued, benefits are subsequently adjusted annually in response to changes in assumptions and returns. An attribution of the benefit payments shows that this design gives higher benefits to the first generations and lower benefits to the last generations, for a scheme which starts with no members. The contributions paid also affect the balance of benefits paid between generations. Too high a contribution is to the advantage of the first generations. Too low a contribution is in the interests of the later generations. The conclusion, within the simple model considered, is that a constant benefit accrual is an important design choice. Its financial consequences across all generations should be carefully analysed, if it is intended to be implemented. Additionally, contributions should be reviewed regularly in such a CDC scheme, to ensure that cross-subsidies are not borne excessively by particular generations.

Keywords: Scheme design; Royal Mail; Lump sum; Annuity; Retirement; Over-lapping generations

1. Introduction

What are the financial implications of the design choices in a collective defined contribution (CDC) scheme? By proposing a framework which allows the design choices to be analysed, the paper assists in the understanding of CDC plans. This is particularly important in the UK since, with the approval of its first CDC scheme approaching at the time of writing, there is no established UK view of what is the standard design of a CDC plan. Moreover, the legal and regulatory rules around UK CDC schemes do not prescribe the CDC plan design. Thus, it is important for UK pension professionals to understand the implications of scheme design choices.

There are several design choices in a CDC plan whose financial implications are explained in this paper through the framework. Should the rate of benefit accrual be constant, as is common in defined benefit (DB) schemes? Should the benefits accrued by a contribution in a CDC plan be the accumulation of that contribution? If so, then what investment returns should be used to accumulate – the prevailing, predicted returns at the time the contribution was made, or a set of returns fixed for all members at all times? Does it matter for investment risk sharing which one is used?

The attribution analysis proposed in the paper explains the financial impact of these design choices. It starts with the benefit which would have been achieved if the contributions made to the CDC plan had been invested in an individual defined contribution (IDC) pension plan. What is left over is the increase in benefits due to investment risk-sharing. In other words, it is the additional gain due to being in a CDC pension plan, over the benefits accrued in an IDC plan.

The increase in benefits due to investment risk sharing is attributed to three design choices:

- The benefits accrued by each contribution being calculated by reference to the predicted investment returns, with the predictions being the ones prevailing at the time the contribution is made;
- The benefits accrued by each contribution being calculated by reference to a fixed set of predicted investment returns. These are chosen to be the investment returns predicted to be achieved at the time the first generation joins the scheme. Such a set-up corresponds to compound or age-related benefit accrual; and
- The benefits accrued by each contribution being fixed, no matter when the contributions are made. This is analogous to the common situation in most UK DB schemes and in the proposed Royal Mail CDC plan.

Due to the complexities implied by investment risk-sharing, the attribution is explored for two types of benefit payments, one being a single lump sum at retirement (section 3) and the other a fixed-term, but not level, annuity in retirement (section 4). While lump sum pension schemes are not typical in the UK, it turns out to be a helpful model to understand investment risk sharing and to motivate what elements are part of the attribution. Longevity risk sharing is excluded to make the model as simple as possible and to avoid conflating it with investment risk sharing.

The annuity model studied in the paper is closest to the UK's first proposed CDC scheme, for the employees of Royal Mail, a postal service company. The model differs in two key ways from the proposed Royal Mail CDC plan. The first is the absence of longevity risk. The second is the annual change in accrued benefits due to investment risk sharing.

In the CDC models in the paper, the annual changes to accrued benefits are a series of one-off annual increases, which are called benefit increases in this paper. If everything turned out as first predicted (i.e. predictions of future returns did not change and were correct), the benefit increases would be zero. The same benefit increase is applied to each member's accrued pension. If the benefit increase is calculated to be 5% today, then all members' accrued benefits are increased by 5% today. The value of each member's benefits (as opposed to the annual amount of the benefits) changes also by 5% today.

In contrast, the proposed Royal Mail scheme calculates an annual pension increase. If the pension increase is calculated to be 2% per annum today, then a member who is 40 years from retirement has a projected benefit at retirement increased with 40 years' worth of 2% per annum. A member who is 1 year from retirement has a projected benefit at retirement which has only one increase of 2% per annum applied. While each of the member's accrued benefit (rather than the projected benefit) goes up by 2%, the change in the underlying value is different (due to the projected benefits at retirement varying with the time until retirement). The change in the value of accrued benefits is much greater for the younger member, in the Royal Mail CDC plan. Pension increases are expected and funded for via the contributions. If everything turned out as first predicted, the pension increases would be whatever they were first expected to be.

This paper does not calculate a pension increase because the goal is to understand investment risk sharing better and algebraically if possible. It is most certainly not an implied criticism of the pension increase approach. Having to calculate a pension increase *à la* Royal Mail means that the annual pension increase must be solved for implicitly. In contrast, the benefit increase is solved for explicitly and is amenable to an algebraic consideration, at least for the first couple of generations before the expressions become too tortured. Indeed, the algebraic consideration has led to the proposed attribution analysis. These development considerations aside, the proposed attribution is applicable to the pension increase approach.

While this paper aims to answer the financial implications of scheme design choice, this is not the whole story. Helping members to understand what benefits they have accrued, and what they can accrue in the future, is extremely important. The paper does not address the latter aspect. It is simply noted that, a design choice that is undesirable from a financial point-of-view, may be important from a member communication, engagement and planning standpoint.

The first CDC scheme studied in the paper, which pays a lump sum at retirement in exchange for a single lump sum contribution upon joining the scheme, is presented and analysed in section 3. The second CDC scheme, which pays a fixed-term, non-level annuity during retirement in exchange for a series of regular contributions before retirement, is studied in Section 4, with its mathematical description relegated to Appendix A. The paper concludes in Section 5. The Wilkie Model used in the stochastic simulations is detailed in Appendix B.

2. Background

In 2014, under the banner of Freedom and Choice, the UK Government announced – unexpectedly and to the shocked splutterings of pension industry professionals – the relaxation of restrictions around withdrawing money from pension pots (Austin, 2022). Beforehand, all-but-the-largest pension pots had to be used to buy a life annuity. Now every pot could be used flexibly: retirees could withdraw as much as they liked, when they liked. By releasing the restrictions around pensions, the UK Government hoped for a plethora of pension innovation from industry. However, this has yet to materialise (Field *et al*., 2018a, section 2).

To some in the UK pensions arena, CDC plans are the superior choice, providing a less volatile and higher expected income to the alternatives, while still compatible with the Freedom and Choice agenda of the UK Government (Field *et al*., 2018b, section 2, Para 10-13, 20). Others view them as the pension version of a Ponzi scheme, with hapless members paying for the poor returns experienced by their predecessors and younger members disadvantaged (Ralfe, 2014, 2021). With the mention of with-profit contracts precipitating a sharp intake of breath and a swift closing down of the conversation from many UK industry professionals, due to a mortgage endowment mis-selling scandal, the idea of any kind of investment risk sharing has become anathema to some. Many others are stuck in the middle, unsure of whom to believe. Ideological stances aside, the authors believe that the lack of consensus is partially due to a lack of objective evidence. The mechanism of longevity risk sharing, the factors affecting it, its trade-offs and subordinate risks are generally well understood. However, the same cannot be confidently said for investment risk-sharing, in the UK pension scheme context. Up to now, there has been simply no pressing need to understand it well.

Through the resolution of an industrial dispute, Royal Mail has been the public and high-profile vanguard of the development of CDC pensions in the UK. The scheme proposed by Royal Mail, one of the UK's largest employers, looks superficially like a UK career-averaged defined benefit (DB) scheme. The Pensions Regulator, the regulator of work-based pension schemes in the UK, has said that it is open to other types of CDC pension schemes being brought to it for approval. This means that there is an open question of what could and should other CDC pension plan designs look like in the UK. At the same time, it is likely that any whole-of-life schemes will be variations of the proposed Royal Mail one, given the number of major UK actuarial consultancies involved and fairly small number of consultants involved in its long development.

The proposed attribution analysis shows how the different elements in the CDC scheme design contribute to the benefits paid to members. In turn, this enables a greater understanding of which scheme design elements matter most. For example, the first identified challenge by the UK's Department of Work & Pensions, given in a 2022 event organised by The RSA on 'The road ahead for CDC pensions in the UK', was "How will age-related accrual/contributions work in practice to minimize cross-subsidy between generations?". The simple model in this paper suggests that, for a scheme which begins with no members (which is different to the Royal Mail membership),

https://doi.org/10.1017/S1748499523000015 Published online by Cambridge University Press

contributions should be reviewed regularly to avoid burdening later generations with excessive costs. If earlier pensions paid out were too high for the investment returns earned, then the cost of this is shouldered by later generations but not equally. The later that a generation joins the scheme, the larger is the cost borne. To share the cost more equally, contributions should be adjusted to avoid the last generations bearing too much of the cost. This becomes more important if there are fewer members in the later-joining generations.

In the academic literature, the consensus across several papers is that CDC plans are welfare-improving, in the sense that they maximise the utility of each generation, compared to an optimal individual alternative; for example, Cui *et al.* (2011) shows this for a CDC plan paying an annuity benefit in retirement, Gollier (2008) for a lump sum retirement benefit. However, Chen *et al.* (2021) find that the CDC plan which they study is not necessarily welfare-improving. It is only welfare-improving if two conditions hold: the financial market returns are volatile and the participants are risk-averse.

Earlier works on intergenerational risk-sharing include Allen & Gale (1997), Ball & Mankiw (2007), Bovenberg *et al.* (2007). Cui *et al.* (2011) shows that a CDC plan in which both contributions and benefits can be adjusted (in a mechanistic way) in response to the funding level, performs better than one in which only the contributions or only the benefits can be adjusted. Beetsma *et al.* (2012) find that mandatory participation is often necessary to sustain a funded pension system.

As these papers and others such as Bonenkamp & Westerhout (2014) show, CDC plans enable a higher level of investment risk to be taken by their members, which results in higher expected benefits for the members compared to an individual DC plan.

The proposed Royal Mail CDC plan has itself been studied and found to be superior to the considered alternatives, using the replacement ratio as the criterion. Owadally *et al.* (2022) find that it gives a higher replacement ratio in retirement (i.e. the ratio of retirement income to the pre-retirement salary) per unit standard deviation of replacement ratio, compared to the DC and life annuity alternatives. Importantly, a member's replacement ratio as a function of their age is less volatile in the CDC plan, allowing members to modify their pension savings in a timely way. Similar results on the proposed Royal Mail CDC plan are illustrated in industry technical reports, for example Wesbroom *et al.* (2020).

Chen *et al.* (2017) suggest that younger members should not exit a collective, funded scheme before retirement, using an option-pricing approach, if individuals understand the value of their scheme benefit. Picking the results which apply to CDC plans in which the benefit indexation policy is used to amortize surpluses and losses, as is done in this paper, Chen *et al.* (2017) discover that fewer people exit a CDC scheme when the period over which surpluses and deficits are recovered (via benefit indexation) is shortened. In their set-up, the benefit indexation applies the same increase to each pension, as is also done in this paper. However, in the CDC plans considered in this paper, recovery of surpluses and deficits is immediate. Another conclusion of their work is that age-independent contributions are likely to reduce the chance of members wishing to transfer their benefits out of the scheme.

Zhu *et al.* (2021) consider a general model of a plan, which encompasses DC, CDC and DB plans. Using a social planning perspective, they optimise various objective functions by controlling the investment strategy, and sometimes also the benefit and contribution strategies. Using the model in Cui *et al.* (2011), one of their main findings is that the optimal choice of the control variables is highly sensitive to the constant risk-free rate, which is used to discount future benefits and contributions in their model.

3. Scheme with Lump-Sum Benefits and Lump-Sum Contributions

In the lump sum CDC scheme, each member pays a single lump sum upon entry and receives at retirement a lump sum benefit. There are no other cashflows for each member.

3.1. Description of the lump sum benefit schemes

The lump sum CDC scheme begins with $N(0) > 0$ members who all join at time 0. There are no members before time 0. More generations join at later times. Generation g consists of $N(g) > 0$ members, who all join at time g, for $g = 0, 1, 2, \ldots, M - 1$, for some constant integer $M > 1$. Each member of generation g pays a single contribution of amount $C(g)$ at the time of entry g to the scheme in exchange for a lump sum benefit paid in T years' time, for constant integer $T > 1$. This means that the generations overlap. Once a member in generation g is paid a lump sum benefit at time $g + T$, they exit the scheme and are no longer a member of it. No one dies so that the focus is entirely on investment risk sharing.

No new members join on and after some known future time $M > 0$. Thus from time M onwards, the end phase, the scheme is declining in terms of its membership since members leave and none join. The last generation joins at time $M - 1$ and exits the scheme at time $M + T - 1$ with a lump sum benefit equal to the value of the residual assets in the scheme. After this last payment, the scheme ceases to exist.

All members contribute an amount of money upon joining the scheme, with the amount potentially varying over time. No one dies or leaves the scheme early; the only exit from the scheme is at the retirement date.

Upon entry, each generation has an initial target benefit. This is subsequently varied every year, with exactly T possible changes occurring. The last benefit change happens at the time of retirement.

3.2. Two benefit structures

Two possible approaches to calculating the initial target benefit are presented here. Contributions are the same amount for all generations, in the two approaches. These approaches result in two different schemes which are called:

- Fair lump sum scheme. Contributions are the same constant amount C for all members. Each generation has their own initial target benefit which is calculated as the accumulation of their contribution at the prevailing predicted set of investment return. Fairness refers to financial fairness in respect of each member. More specifically, the discounted value of each member's initial target benefit equals their contribution, discounting using the returns prevailing at the time the contribution is made.
- Unfair lump sum scheme. The contributions are the same constant amount C for all members as in the fair lump sum scheme, so that both schemes are comparable. The initial target benefit for each member is the same constant value $\bar{b}$ for all members. This scheme is called unfair because each member's contribution is not, in general, equal to the discounted value of their initial target benefit, if discounting using the returns prevailing at the time the contribution is made.

The constant contribution C is calculated as the discounted value of the first generation's initial target benefit in the unfair scheme. The only difference in the two schemes is in which set of predicted returns are used to calculate the initial target benefit. In the fair scheme, the prevailing predicted returns are used to calculate the initial target benefit. Since these predictions change over time, the initial target benefit varies in the fair scheme according to the time at which a member joins the scheme.

In the unfair scheme, the same set of predicted returns are used at all times, regardless of what are the current predictions of investment returns. It is only for the calculation of the initial target benefit that the fixed set of predictions are used.

After 1 year, the initial target benefit is adjusted and is called a target benefit from then on. The adjustment is applied across all benefits which have been accrued for at least 1 year and is called a benefit increase (although, with a negative benefit increase denoting a cut to the benefits). The benefit increase is calculated so that the total assets of the scheme equal the total discounted value of the adjusted benefits. At a fixed time, the same benefit increase is applied to all benefits which have been accrued for at least 1 year.

If actual returns are accurately predicted then one may think that each generation should be paid their initial target benefit at retirement. However, as discussed below, this is true only in the fair scheme, in which the initial target benefit is the accumulation of the contribution at the current predictions of investment returns.

For the unfair scheme, members will not be paid their initial target benefit at retirement when investment returns have been perfectly predicted. This is entirely due to a fixed set of predicted returns being used to relate the initial target benefit to the contributions. The only exception to the latter result is when both the predictions of returns are the same constant value at all times and at all maturities and investment returns turn out to be as predicted. In that case, both the fair and unfair scheme coincide since they both use the same set of predicted returns to calculate the initial target benefit. The implication is that making a simplifying assumption that returns and their predictions are the same constant value will not elicit the consequences of calculating the initial target benefit differently.

A third approach, of fixing the initial target benefit and setting the contribution to be the fair discounted value of the initial target benefit, was also analysed. This approach is not presented here to keep the paper shorter and the motivation is more of curiosity rather than applicability since most members would not be happy if contributions varied all the time. The results in the latter approach are analogous to the results in the fair scheme; the similarity arises from the contribution fairly funding the initial target benefit.

3.2.1. Target benefit and return notation

Let $B^{(g)}(g+T)$ represent the actual payment made to generation g upon their retirement at time $g+T$. Let $B^{(g)}(r)$ denote the target benefit, i.e. the benefit predicted at time $r \in \{g, g+1, \ldots, g+T-1\}$ to be paid to generation g at their retirement time of $g+T$. The value $B^{(g)}(g)$ is the initial target benefit.

The values $\{B^{(g)}(r); r \in \{g, g+1, \ldots, g+T-1\}\}$ are effectively the predictions of $B^{(g)}(g+T)$, based on what has occurred and what is predicted to occur. Differences between these values and $B^{(g)}(g+T)$ are due to investment risk sharing. There are no funded pension increases.

Invested assets earn a random return of $R(k) > -1$ over the time period $(k-1, k]$, for $k = 1, 2, \ldots$. For integer $\ell \geq 1$ and integer $k \leq \ell - 1$, let $i(\ell, k)$ denote the investment return predicted to be achieved over the time period $(\ell - 1, \ell]$, conditional on being at integer time $k \leq \ell - 1$. For example, $i(1, 0)$ represents the prediction at time 0 of the random return $R(1)$ over the first year. $i(2, 0)$ represents the prediction at time 0 of the random return $R(2)$ to be earned over the second year, and $i(2, 1)$ represents the updated prediction at time 1 of $R(2)$.

For each fixed ℓ and k, $i(\ell, k)$ is a constant to keep the mathematics as simple as possible.

3.2.2. Fair lump sum scheme: varying initial target benefit

Fix the amount of contributions paid by each member of each generation to be a constant $C > 0$. For consistency for generation $g = 0$, who have no one with whom to share risk in their first year, fix their initial target benefit $B^{(0)}(0) := \bar{b} > 0$ and calculate the contribution paid by each generation as

$$C = \bar{b} \prod_{\ell=1}^{T} \frac{1}{1 + i(\ell, 0)}. \tag{1}$$

https://doi.org/10.1017/S1748499523000015 Published online by Cambridge University Press

The initial target benefit for each member of generation $g = 0, 1, 2, \ldots, M-1$ is calculated as the accumulated value of that member's contribution using the prevailing predictions of the investment returns at the time each contribution is made, i.e.

$$B^{(g)}(g) = C \prod_{\ell=g+1}^{g+T} \left(1 + i(\ell, g)\right).$$

This means that, unless returns are predicted to be constant, the initial target benefit varies between generations. For example, if a generation expects to experience higher investment returns than earlier generations, their initial target benefit will be higher.

3.2.3. Unfair lump sum scheme: constant initial target benefit

For the unfair scheme, each member pays a contribution of amount C upon entry, with C calculated from equation (1). In return, each member of generation g has the same initial target benefit

$$B^{(g)}(g) = \bar{b}, \quad \text{for } g = 0, 1, 2, \ldots, M-1.$$

With this choice, it is only the first generation whose contribution is guaranteed to be equal to the discounted value of the initial target benefit. However, in the economic scenario in which predicted returns are all equal to the same constant value at all times and at all maturities, then this is true for all generations. In that scenario, the fair and unfair schemes coincide.

3.3. Evolution of the lump sum benefit scheme

3.3.1. Starting phase of the lump sum benefit scheme

In the starting phase of the lump sum benefit scheme, generations join the scheme but no one has retired yet. The initial asset value is $A(0) = C(0) \cdot N(0)$. The asset value at time $k \in \{1, 2, \ldots, T-1\}$ is the accumulation of the assets since the last time period, i.e.

$$A\left(k_{-}\right) = A\left(k-1\right)\,(1 + R(k)).$$

Before time T, the time that the first generation ($g = 0$) leaves the scheme, no benefits are paid out.

The target benefit $B^{(g)}(k)$ of each member of generation g at time $k \in \{1, 2, \ldots, M+T-1\}$ is calculated for each generation who was in the scheme at time $k-1$. Mathematically, set

$$B^{(g)}(k) = (1 + \delta(k)) B^{(g)}(k-1), \quad \text{for } g \in \{0, 1, \ldots, M+T-1\},$$

in which $\delta(k) > -1$ is the benefit increase awarded at time k, whose calculation is shown below. For the first benefit increase, $\delta(1)$, only generation 0 gets an increase, for the second benefit increase, $\delta(2)$, generations 0 and 1 get an increase, and so on.

The benefit increases account for actual returns not turning out as expected, changes in the predictions of the returns and investment risk sharing. Cost-of-living increases to the lump sum benefit are not included in the benefit increases and are excluded from our setting.

The benefit increase $\delta(k)$ in the starting phase is calculated by equating the discounted benefits to the asset value and re-arranging to get

$$1 + \delta(k) = \frac{A(k_{-})}{\sum_{g=0}^{k-1} N(g) B^{(g)}(k-1) \prod_{\ell=k+1}^{g+T} \frac{1}{1+i(\ell,k)}}, \quad \text{for } k \in \{1, 2, \ldots, T-1\}. \tag{2}$$

The benefit increase is calculated just before the contributions from the newly entering generation have been received.

Once $\delta(k)$ is calculated, the next generation joins and the asset value is

$$A\left(k\right) = A\left(k_{-}\right) + N(k) C(k), \quad \text{for } k \in \{1, 2, \ldots, T-1\}.$$

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The benefit increase formula (2) shows how investment returns are shared in the scheme. It is through this formula that the scheme is a collective one. It takes the total value of assets in the scheme and divides by the total discounted value of benefits. Each member who has been in the scheme for at least 1 year gets the same increase on their accrued benefit.

The asset value is a function of the past alone, including a past which no current member has lived in as a scheme member, once the scheme has been running for more than T years. The discounted value represents the predicted future, including a future which the older members will not experience as scheme members, as well as the past through the benefit increases. Thus, most members are exposed to past returns and future predicted returns, and their consequences, which lie outside the time that they are in the scheme. It is solely the first and last generations who are exposed to only one of the future and past outside of their time in the scheme, respectively.

3.3.2. Middle phase of the lump sum benefit scheme

In the middle phase of the scheme, generations start retiring but generations continue to join. In a scheme in which the same number of members join at each time, i.e. $N(0) = N(1) = N(2) = \cdots$, a membership steady-state has been reached, where the number of members retiring equals the number of members joining.

At time $k \in \{T, T+1, \ldots, M-1\}$, generation k joins and generation $(k-T)$ retires, with a single benefit payment $B^{(k-T)}(k)$ per member.

The asset value at time $k \in \{T, T+1, \ldots, M-1\}$ is

$$A\left(k_-\right) = A\left(k-1\right)\ (1 + R(k)), \quad \text{for } k \in \{T, T+1, \ldots, M-1\}.$$

The benefit increase in the middle phase is calculated via a slight adjustment to equation (2) to exclude the generations who have retired, i.e.

$$1 + \delta(k) = \frac{A(k_-)}{N(k-T)B^{(k-T)}(k-1) + \sum_{g=k-T+1}^{k-1} N(g)B^{(g)}(k-1) \prod_{\ell=k+1}^{g+T} \frac{1}{1+i(\ell,k)}}, \tag{3}$$

for $k = T, T+1, \ldots, M-1$. The benefit increase is calculated just before both the contributions from the newly entering generation have been received and before a generation retires. Just after these contributions have been received and the benefit payments made, the asset value changes to

$$A(k) = A(k_-) + N(k)C(k) - N(k-T) \cdot B^{(k-T)}(k), \quad \text{for } k \in \{T, T+1, \ldots, M-1\}.$$

3.3.3. End phase of the lump sum benefit scheme

In the end phase of the scheme, generations start retiring and no new generation joins. The last generation joins at time $M-1$.

As before, the asset value is

$$A\left(k_-\right) = A\left(k-1\right)\ (1 + R(k)), \quad \text{for } k \in \{M, M+1, \ldots, M+T-1\}.$$

The benefit increase in the end phase is calculated as

$$1 + \delta(k) = \frac{A(k_-)}{N(k-T)B^{(k-T)}(k-1) + \sum_{g=k-T+1}^{M-1} N(g)B^{(g)}(k-1) \prod_{\ell=k+1}^{g+T} \frac{1}{1+i(\ell,k)}}, \tag{4}$$

for $k = M, M+1, \ldots, M+T-2$.

Just after the benefit payment paid to each member of the generation retiring at time k, the asset value falls to

$$A(k) = A(k_-) - N(k-T) \cdot B^{(k-T)}(k), \quad \text{for } k \in \{M, M+1, \ldots, M+T-2\}.$$

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Just before the last generation, generation $(M-1)$, retires, the asset value is $A\left((M+T-1)_{-}\right)=A(M+T-2)\cdot(1+R(k))$. Each member of the last generation retires with benefit payment $B^{(M-1)}(M+T-1)=A\left((M+T-1)_{-}\right)/N(M-1)$. Once this last tranche of benefits are paid out, the asset value falls to zero and the scheme ceases to exist.

3.3.4. Individual defined contribution (IDC) lump sum scheme

The benefit payments under the lump sum CDC scheme can be compared to those of an IDC scheme. An important reason for doing so is to see what are the financial advantages of being in a CDC scheme instead of an IDC scheme.

In the IDC scheme, each member of generation g invests C at time g to earn the same investment return as the CDC scheme. Thus, their IDC investment accumulates at their retirement time $g+T$ to

$$B^{\mathrm{IDC},(g)}(g+T)=C\prod_{\ell=g+1}^{g+T}(1+R(\ell))\,.$$

As in the CDC scheme, a prediction $B^{\mathrm{IDC},(g)}(k)$ can be made at each time $k\in\{g,g+1,\ldots,g+T\}$ of the value of the final benefit $B^{\mathrm{IDC},(g)}(g+T)$, for each generation g. When generation g first joins the IDC scheme, the initial predicted retirement benefit is the accumulation of the contribution paid by them at the predicted returns, i.e.

$$B^{\mathrm{IDC},(g)}(g)=C\prod_{\ell=g+1}^{g+T}\big(1+i(\ell,g)\big)\,.$$

At time $k\in\{g+1,g+2,\ldots,g+T-1\}$, their predicted retirement benefit accumulates partially with actual returns and partially with predicted returns, i.e.

$$B^{\mathrm{IDC},(g)}(k)=C\prod_{\ell=g+1}^{k}(1+R(\ell))\prod_{\ell=k+1}^{g+T}\big(1+i(\ell,k)\big)\,.$$

A consideration of the annual change in the predicted benefit leads to an analogy of the CDC benefit increase, which is called the *IDC factor*. It is defined for each $g\in\{0,1,\ldots,M-1\}$ and each $k\in\{g+1,g+2,\ldots,g+T-1\}$ as

$$\mathrm{IDC}^{(g)}(k):=\frac{1+R(k)}{1+i(k,k-1)}\prod_{\ell=k+1}^{g+T}\frac{1+i(\ell,k)}{1+i(\ell,k-1)}-1. \tag{5}$$

The annual change in the predicted IDC benefit at retirement is driven by the product of two factors, one concerned with the past and the other with the future. The first factor, $(1+R(k))/(1+i(k,k-1))$, gives the impact due to the actual return being different to its last predicted value. The second factor, $\prod_{\ell=k+1}^{g+T}(1+i(\ell,k))/(1+i(\ell,k-1))$, shows the effect of change in predictions of future returns.

3.4. Attribution analysis for the lump sum CDC scheme

The benefit increases (2)–(4) are each expressed as the product of three factors, motivated by an analysis which is outlined below. In a lump sum CDC scheme, the benefit increase $\delta(k)$ can be decomposed into three components, namely

$$1+\delta(k)=\left(1+\mathrm{IDC}^{(g)}(k)\right)\left(1+\beta^{(g)}(k)\right)\left(1+\gamma(k)\right). \tag{6}$$

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The first factor is the IDC factor, which gives the change in the predicted benefit in the IDC scheme at time k. The second factor, $\beta^{(g)}(k)$, is the *investment risk-sharing factor*. It communicates the change in the benefit increase once the IDC factor has been stripped out and ignoring the effect of any unfair funding. The third factor, $\gamma(k)$, is the *unfair predictions factors*. It reflects the impact of unfair funding, i.e. if each initial target benefit is not the fair accumulation of its corresponding contribution. In the fair CDC scheme, $\gamma(k) = 0$ for all k.

3.4.1. Motivation for the attribution

The attribution is motivated by an examination of the benefit increases when the return predictions at each maturity time do not change over time, i.e. $i(\ell, k) := i(\ell)$ for some constant $i(\ell)$, for $\ell > k$ and $k = 0, 1, 2, \ldots$. For example, the unchanging predictions could be that the return over the time period [0,1) is $i(1, 0) = i(1) = 5\%$, the return over the time period [1,2) is $i(2, 0) = i(2, 1) = i(2) = 7\%$, and so on.

In this economic scenario of unchanging predictions, there is no investment risk sharing in the fair CDC scheme and it collapses to an IDC scheme. This is regardless of whether returns are accurately predicted or not. The benefit increases in the fair CDC scheme in this scenario are $\delta(k) = (1 + R(k))/(1 + i(k)) - 1$ for each k. The latter result can be seen from the expression (5) for the IDC factor, since there the second term in the product on the right-hand side is unity when return predictions do not change over time. Thus, the benefit increase in the fair CDC scheme is exactly equal to $\mathrm{IDC}^{(g)}(k)$ in this particular economic scenario, and there is no investment risk-sharing.

However, there is investment risk sharing in the unfair CDC scheme when return predictions at each maturity time do not change over time. In the unfair lump sum CDC scheme, the contribution C and target benefit of every generation g satisfies

$$B^{(g)}(g) = C \prod_{\ell=1}^{T} (1 + i(\ell)).$$

To calculate the benefit increase, as in equation (2), the discounted value of each generation's accrued benefit is calculated and summed up. In both the fair and unfair CDC schemes, the first benefit increase is always $\delta(1) = (1 + R(1))/(1 + i(1))$.

The benefit increase at time 2 of

$$\delta(2) = \frac{N(0)(1 + R(1))(1 + R(2)) + N(1)(1 + R(2))}{N(0)(1 + R(1))(1 + i(2)) + N(1)(1 + i(2))\frac{1+i(1)}{1+i(T+1)}},$$

in which the discounted value at time 2, of the target benefit of generation 1 – which is due to the paid at time $T + 1$ – is

$$C\,(1 + i(2))\,\frac{1 + i(1)}{1 + i(T + 1)}.$$

Even if $R(1) = i(1)$ and $R(2) = i(2)$, the benefit increase at time 2, $\delta(2)$, is non-zero. The use of a constant contribution and constant initial target benefit leads to an asymmetry in discount factor for generation 1's accrued benefit, which in turns affects the benefit increase.

Consideration of these results motivate the factors in the benefit attribution. Note that the attribution is not unique. A different fair CDC scheme could be used as the comparator CDC scheme, one in which the initial target benefit is fixed at $\bar{b}$ and the contributions are the fair discounted value of the initial target benefit. The factors are not independent either, in the sense that changing the predicted returns will change all of the factors, albeit some are more sensitive to such changes than others. The non-independence of the factors can easily be seen by examining the explicit algebraic form of the decomposition of, for example, the second benefit increase $\delta(2)$.

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3.4.2. Calculation of the attribution in a lump sum CDC scheme

The calculation of each factor in (6) is as follows.

- The IDC factor is calculated from (5), independently of the amount of the contribution in either the CDC or IDC scheme.
- The fair CDC scheme is used to calculate the investment risk-sharing factor $\beta^{(g)}(k)$ for each generation g. Denote the benefit increase applied at time k for the fair scheme by $\tilde{\delta}(k)$. Then the investment risk sharing factor in the fair scheme at time k is

$$\tilde{\beta}^{(g)}(k) = \frac{1+\tilde{\delta}(k)}{1+\mathrm{IDC}^{(g)}(k)} - 1.$$

  This is also the investment risk sharing factor for the unfair scheme, i.e. $\beta^{(g)}(k) := \tilde{\beta}^{(g)}(k)$.
- The unfair predictions factor $\gamma(k)$ of the unfair scheme reflects the additional benefit increase when the lump sum contributions do not each fairly fund their corresponding initial target lump sum benefit. The unfair predictions factor

$$\gamma(k) = \frac{1+\delta(k)}{1+\tilde{\delta}(k)} - 1.$$

  The factor $\gamma(k)$ is the difference in the benefit increase in the fair and unfair CDC schemes.

3.4.3. Use of the attribution in a lump sum CDC scheme

The attribution (6) of the benefit increase in a lump sum CDC scheme allows the quantitative analysis of various scheme design choices, such as

- The choice of the considered lump sum CDC scheme over the lump sum IDC scheme. Another way of looking at this is the overall effect of investment risk sharing in the CDC scheme over not investment risk sharing in an IDC scheme, through the product $\left(1+\beta^{(g)}(k)\right)\left(1+\gamma(k)\right)$. This enables a quantitative comparison of the considered CDC scheme with an IDC scheme under different economic scenarios and with different membership profiles;
- Using the prevailing predictions of returns to calculate the initial target benefit in conjunction with a fixed contribution, as is done in the fair lump sum CDC scheme. The effect of this design choice on the benefits is measured through the investment risk-sharing factor $\beta^{(g)}(k)$;
- Using the same initial target benefit and the same contribution for all members, as is done in the unfair lump sum CDC scheme. The additional benefit increase in the unfair CDC scheme, over and above that in the fair CDC scheme, is expressed through the unfair predictions factor $\gamma(k)$. How the factor $\gamma(k)$ changes under different economic scenarios and different membership profiles can be determined.

Through an examination of the factors of the decomposition, the importance of the different design choices can be evaluated. For example, the numerical illustration below indicates that the fair CDC scheme is close to an IDC scheme in terms of the benefit paid to each retiring member. The investment risk-sharing factor $\beta^{(g)}(k)$ is usually only a small proportion of the overall benefit increase. This suggests that there may not be much financial benefit from implementing a fair CDC scheme compared to an IDC scheme.

In contrast, the unfair CDC scheme gives very different benefits to either the fair CDC scheme or the IDC scheme. The benefit smoothing effect, one of the often-touted advantages of CDC schemes, is clearly seen in the unfair CDC scheme. It arises from choosing the same initial target benefit and the same contribution for all members, and thus effectively using the same set of predicted returns to calculate the initial target benefit as the accumulation of the contribution.

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The attribution of the benefit increase to the various design choices in the CDC scheme, its interpretation and the understanding it brings to the considered CDC schemes, are the key contributions of the paper.

3.5. Numerical example in the lump sum scheme

Now turn to a numerical example of the attribution of the benefit increase in the unfair lump-sum CDC scheme. The benefit increase in the fair lump sum CDC scheme is a subset of the latter. Suppose there are $M = 100$ generations in the scheme, who join at time $0, 1, \ldots, 99$. Each generation makes a single contribution of 100 units immediately upon entry in exchange for a benefit paid exactly $T = 20$ years after they join. The initial target benefit is the constant amount

$$B^{(g)}(g) := 100 \prod_{\ell=1}^{10} (1 + i(\ell, 0))$$

for all generations g. The programming of the evolution of the amount of the target benefit, as detailed in section 3.3, was done in the statistical software programme R. Further assumptions needed for each numerical illustration are detailed in the relevant section below.

3.5.1. Investigation of changes in the predicted returns

Let exactly $N(g) = 1$ member join at each integer time $g = 0, 1, \ldots, 99$. Assume that the initial prediction of investment returns is a constant return at all future times, i.e. $i(\ell, 0) = 0.1$ for $\ell = 1, 2, \ldots$. Future predicted investment returns are assumed to be one of the following:

- A parallel shift upwards each year, so that $i(\ell, k+1) = i(\ell, k) + 0.001$, or
- A parallel shift downwards each year, so that $i(\ell, k+1) = i(\ell, k) - 0.001$,

for $\ell = k+1, k+2, \ldots$ and $k = 0, 1, 2, \ldots$. The predictions are not the same constant value at all times. If they were the same constant value at all times, then there is nothing of interest to show, since (i) the three CDC schemes coincide and (ii) there is no investment risk-sharing. In what follows, the expression 'predicted returns are increasing' refers to the parallel shift upwards in the predicted returns each year, as detailed above. Similarly for the expression 'predicted returns are decreasing'. The expression 'actual return are as last predicted' means that $R(k) = i(k, k-1)$ for $k = 1, 2, \ldots$.

3.5.2. Retirement benefit attribution in the lump sum scheme

To decompose the lump sums paid at retirement, begin with the amount attributed to the initial target benefit, which is the value of $B^{(g)}(g)$ for each member of generation g. The additional amount paid out at retirement due to the IDC factor is

$$B^{(g)}(g) \prod_{k=g+1}^{g+T} \left(1 + \mathrm{IDC}^{(g)}(k)\right) - B^{(g)}(g).$$

The additional amount attributed to the investment risk-sharing factor is

$$B^{(g)}(g) \prod_{k=g+1}^{g+T} \left(1 + \mathrm{IDC}^{(g)}(k)\right) \left(1 + \beta^{(g)}(k)\right) - B^{(g)}(g) \prod_{k=g+1}^{g+T} \left(1 + \mathrm{IDC}^{(g)}(k)\right)$$

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and the amount left over, which is non-zero only in the unfair lump-sum CDC scheme, is the additional amount due to the unfair predictions factor;

$$B^{(g)}(g+T) - B^{(g)}(g) \prod_{k=g+1}^{g+T} \left(1 + \mathrm{IDC}^{(g)}(k)\right)\left(1 + \beta^{(g)}(k)\right).$$

3.5.3. Illustration of the benefit attribution in the lump sum CDC schemes

The illustration indicates that it is the fixing of the initial target benefit and the contributions which result in the smoothing effect of CDC schemes. When the initial target benefit is the accumulation of the contributions, accumulated using the prevailing predictions of returns, then the resultant fair CDC scheme appears to be very close to being an IDC scheme, under the considered economic scenarios. (The stochastic model used to illustrate the annuity CDC scheme shows that this is not true, but is an artifice – unintended – of the deterministic economic model chosen here.) It is the unfair CDC scheme which exhibits one of the key features of investment risk sharing: the smoothing of benefit payments over time.

The unfair CDC scheme pays benefits which can be very different to those offered by the fair CDC scheme. This is due to the initial target benefit being fixed for all generations. The unfair predictions factor has a significant effect on the retirement benefit paid, in addition to the initial target benefit and the IDC factor. It is this factor which distinguishes the fair from the unfair scheme, in the benefit attribution. It is the primarily the unfair predictions factor which does most of the work of allocating returns among generations, rather than the investment risk-sharing factor.

In detail, the illustration of the retirement lump sum benefit attribution shows that:

- It is the unfair predictions factor which is most important for cross-generational smoothing. It reduces the sensitivity of the target benefits in the unfair CDC scheme to changes in the prediction of returns, compared to the fair CDC scheme.

  Consider first the scenario in which predicted returns increase over time and the return earned on assets is the last predicted value (left-hand plots in Figure 1). The asset value is the same in both CDC schemes until the first benefit payment out, since the same contributions are made in each scheme. In the fair CDC scheme, the initial target benefit will increase over time as new generations face higher predicted returns in this scenario (the blue triangles in Figure 1(a)). This acts to increases the value of the discounted benefits compared to the unfair CDC scheme which has a constant initial target benefit (Figure 1(c)). Thus, the benefit increase is lower in the fair CDC scheme and hence the unfair predictions factor will be positive in this situation.

  Despite the positivity of the unfair predictions factor, the amount of the retirement benefit in the unfair CDC scheme will, with the exception of the first generations to join, be less than that in the fair CDC scheme. The additional benefit increase in the unfair CDC scheme, represented by the unfair predictions factor, cannot compensate for the higher initial target benefit in the fair CDC scheme. As the calculation of the benefit increase relies on the total discounted value of the accrued target benefits, it is weighted down by this averaging.
- The retirement benefits paid out in the unfair lump sum CDC scheme are higher for the first generations to join, compared to the fair lump sum CDC scheme. The lower discounted value of the benefits in the unfair CDC scheme induces a higher benefit increase at the beginning, compared to the fair CDC scheme. The benefit increase in the fair scheme, even when coupled with the higher initial target benefits in this scenario of increasing predicted returns, are initially not large enough to compensate for a higher benefit increase in the unfair CDC scheme. It is only for later generations in the fair CDC scheme that the higher initial target benefit compensates for the lower benefit increase, to give an overall larger retirement benefit.

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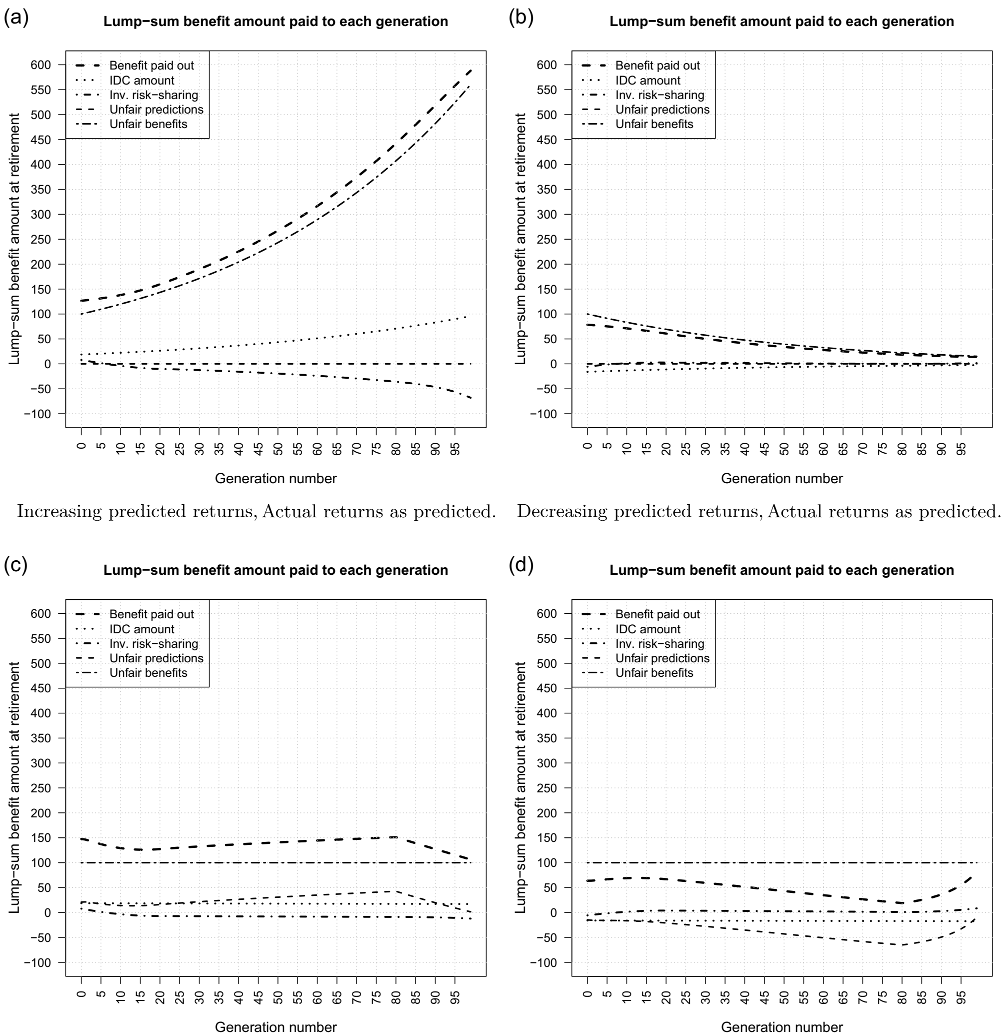

(a) Increasing predicted returns, Actual returns as predicted. (b) Decreasing predicted returns, Actual returns as predicted.

(c) Increasing predicted returns, Actual returns as predicted. (d) Decreasing predicted returns, Actual returns as predicted.

Figure 1. Lump sum benefits paid out at retirement and their decomposition, for the fair lump sum CDC scheme (top plots) and unfair lump sum CDC scheme (bottom plots) with constant contributions of 14.86 units.

Once new members stop joining, the unfair predictions factor declines to zero. As no members join, no further distortions in the value of the discounted benefits are introduced in the unfair CDC scheme compared to the fair CDC scheme. The effects of earlier distortions are somewhat reduced by earlier benefit increases, and these continue to eliminate the difference in the benefit increase between the two CDC schemes.

- The larger the increase in the predicted returns compared to the time 0 predicted returns (which are used to calculate the initial target benefit for the unfair CDC scheme and generation 0 in the fair CDC scheme), the bigger will be the unfair predictions factor.

The opposite happens when predicted returns decrease over time and the return earned on assets is the last predicted value (right-hand plots in Figure 1). In this case, the unfair predictions factor acts to give a lower benefit increase in the unfair CDC scheme. In this case, the fair CDC scheme pays a higher retirement benefit to the first generations to join.

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However, once the initial target benefit in the fair CDC scheme becomes too low, the higher benefit increases in the fair CDC scheme cannot compensate for it. Then the unfair CDC scheme pays a higher retirement benefit.

- The investment risk-sharing factor is generally of a smaller magnitude than the unfair predictions factor. Recall that the investment risk-sharing factor is the difference in the benefit increase in a fair CDC scheme and the notional benefit increase in an IDC scheme. The initial target benefit is the same in the fair CDC scheme as in the IDC scheme. However, the notional benefit increase in an IDC scheme is individual to the member in question. It compares the value of the member's own accumulated contribution to the member's discounted notional benefit value in the IDC scheme. If the ratio of these is close to the fair CDC scheme benefit increase, then the investment risk-sharing factor will be close to zero.

  The investment risk-sharing factor is therefore larger when the member is not close to the average. For example, if the member is one of a few younger members among many older members, or *vice versa*. It is most pronounced for the youngest and oldest generations (e.g. the green squares in Figure 1), as these members are in a either a growing or shrinking membership.

The analysis, in the considered, deterministic economic model, indicates that the fair lump sum CDC scheme is very close to an IDC scheme. However, the unfair lump sum CDC scheme is not. The source of the difference between the schemes is in the fixing of the initial target benefit in the unfair lump sum CDC scheme, assuming that the same contribution is paid in both schemes by all members.

However, as will be seen in the next section, the analysis of the annuity CDC schemes within a stochastic economic model, modifies these conclusions. It is not true that the fair annuity CDC scheme is close to an IDC scheme. It is true that the median benefit paid out from the fair annuity CDC scheme is close in value to the median benefit paid out from an IDC scheme. However, there is quite a lot of variability around the benefit payments.

4. Annuity CDC Schemes

Now turn to a CDC scheme in which members make regular contributions every year until retirement and in return receive a fixed term annuity payment in retirement. As before, the first generation (generation 0) to join the scheme at time 0 comprises $N(0) > 0$ members. In general, generation g consists of $N(g) > 0$ members who join at time g, for $g = 0, 1, 2, \ldots, M-1$, for some integer $M \geq 1$. Every member of generation g pays a regular contribution every year until their retirement exactly T years after they join. While the contributions paid could vary over time, it is assumed here that they are constant. No one dies or leaves the scheme early; the only exit from the scheme is at the end of the retirement period which is S years' long.

4.1. Benefits and contributions in the annuity CDC scheme

4.1.1. Benefits in the annuity CDC scheme

In exchange for their contributions, each member of generation g receives in retirement an annuity benefit of amount $B^{(g),\mathrm{cum}}(k)$ at times $k = g+T, g+T+1, \ldots, g+T+S-1$, for each $g \in \{0, 1, 2, \ldots, M-1\}$. For schemes in which each contribution fairly funds part of the overall annuity benefit, the target benefit accrued due to the contribution paid at time k is calculated by equating the contribution with the discounted value of paying an S-year term annuity of unknown annual amount $B^{(g)}(k)$ from that member's retirement date, and then solving for $B^{(g)}(k)$. For schemes in which each contribution does not fairly fund the benefit accrued by that contribution, the amount $B^{(g)}(k)$ is a constant.

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No new members join on and after some known future time $M > 0$, the same as in the lump-sum CDC scheme. Thus from time M onwards, the end phase, the scheme is declining in terms of its membership since members leave and none join. The last generation joins at time $M - 1$, retires at time $M + T - 1$ at which time they start receiving their annuity benefit. They exit the scheme at time $M + T + S - 2$.

As in the lump sum CDC scheme, the accrued benefits receive an increase every year, with the first increase starting exactly one year after a benefit is accrued. For example, with $\delta(k)$ denoting the benefit increase applied at time k to benefits accrued on or before time $k - 1$, $B^{(g)}(k) = (1 + \delta(k))B^{(g)}(k - 1)$. As the benefits are in the form of an annuity, the increases continue throughout retirement.

The cumulative benefit accrued by generation g at time k, just after any contribution due at time k has been paid, is

$$B^{(g),\mathrm{cum}}(k) = \begin{cases} B^{(g)}(g) & k = g, \\ \sum_{\ell=g}^{k-1} B^{(g)}(\ell) \prod_{m=\ell+1}^{k} (1 + \delta(m)) + B^{(g)}(k) & k = g+1, g+2, \ldots, g+T-1, \\ \sum_{\ell=g}^{g+T-1} B^{(g)}(\ell) \prod_{m=\ell+1}^{k} (1 + \delta(m)) & k = g+T, g+T+1, \ldots, g+T+S-1. \end{cases}$$

4.1.2. Contributions in the annuity CDC scheme

In the annuity CDC scheme, each member pays a series of regular contributions in exchange for receiving a regular benefit payment at retirement. If each contribution fairly funds the new accrued benefit, then the scheme is fair. Otherwise it is said to be unfair even if lifetime fairness holds, i.e. the total discounted value of the contributions equals the total discounted benefits. This is for reason of allowing us to distinguish easily between the two types of schemes.

Let a member of generation g pay the contribution $C^{(g)}(k)$ at time $k \in \{g, g+1, \ldots, g+T-1\}$. Such a member retires at time $g + T$ and receives an S-year annuity benefit from time $g + T$, paid annually in advance until the last payment at time $g + T + S - 1$. Thus, they receive their first retirement benefit immediately upon retirement and their last contribution is 1 year before retirement.

It is assumed here that contributions are the same constant value at all times and for all generations, i.e. $C^{(g)}(k) = C$ for some fixed $C > 0$. While there are other reasonable choices, such as compound or age-related contributions, it seems most likely in practice that contributions are constant, albeit a constant proportion of salary. When the contributions are constant and the scheme is a fair one, the overall target benefit varies between generation. This is because the amount of target benefit funded by a contribution made at time k will depend on the predictions of returns at time k, which vary over time.

4.1.3. Three annuity CDC schemes

To reduce notation, define the annuity discounting factor

$$\nu^{(g)}(k) := \begin{cases} \sum_{m=g+T}^{g+T+S-1} \prod_{\ell=k+1}^{m} \frac{1}{1+i(\ell,k)} & \text{for } k \in \{g, g+1, \ldots, g+T-1\}, \\ 1 + \sum_{m=k+1}^{g+T+S-1} \prod_{\ell=k+1}^{m} \frac{1}{1+i(\ell,k)} & \text{for } k \in \{g+T, g+T+1, \ldots, g+T+S-2\}, \end{cases} \tag{7}$$

with the two right-hand expressions required to value the annuity benefit either before or after it is in payment. The annuity discounting factor $\nu^{(g)}(k)$ uses the returns predicted to hold at time k for the discounting.

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For the contribution $C^{(g)}(k)$ at time k to fund fairly an annuity of unknown annual amount $B^{(g)}(k)$, it must hold that

$$C^{(g)}(k) = \nu^{(g)}(k)\, B^{(g)}(k), \quad \text{for } k = g, g+1, \ldots, g+T-1.$$

As in the lump sum CDC scheme, the benefit increases account for actual returns not turning out as expected, changes in the predictions of the returns and investment risk sharing. They are not cost-of-living pension increases, which is why projected benefit increases are not included in the valuation of the annuity in the previous expression.

Three different choices for the accrual of benefits result in three different annuity CDC schemes. As in the lump sum CDC schemes, fairness refers to financial fairness. A contribution is fair if it equals the discounted value of the benefit it accrues, when the prevailing discount rates are used.

The three annuity CDC schemes are:

- Unfair annuity CDC scheme, in which the target benefit accrued by each contribution is fixed to be the same constant value regardless of when the contribution is made. Fix the cumulative target benefit $\bar{b} > 0$ and set $B^{(g)}(k) := \bar{b}/T$ for $k = g, g+1, \ldots, g+T-1$ and $g = 0, 1, \ldots, M-1$. Under a constant interest rate (i.e. $i(\ell, k) := i$ for $\ell \geq k$ and $k = 0, 1, 2, \ldots$), compound, i.e. age-related, contributions are financially fair contributions, i.e. $C^{(g)}(k+1) = C^{(g)}(k) \times (1+i)$ for all $k \in \{g+1, g+2, \ldots, g+T-1\}$. However, the assumption here is that interest rates are not constant and contributions are constant, so the benefit accrued by each contribution is not fairly funded. In the unfair annuity CDC scheme, younger members accrue the same target benefit as older members, for the same contribution.
- Partially fair annuity CDC scheme, in which the accrued target benefits are calculated as the accumulation of each contribution, but the accumulation uses the set of predicted returns holding at time 0. In this scheme, set $B^{(g)}(k) := C^{(g)}(k) / \sum_{m=T}^{T+S-1} \prod_{\ell=k-g+1}^{m} \frac{1}{1+i(\ell,0)}$. The constant contributions result in compound benefit accrual, i.e. $B^{(g)}(k+1) = (1 + i(k-g+1, 0))B^{(g)}(k)$. In the partially fair annuity CDC scheme, younger members accrue more target benefit for their contributions compared to older members, since they have a longer time to earn returns on their contributions.
- Fair annuity CDC scheme, in which the accrued target benefits are calculated as the accumulation of each contribution, with the accumulation using the set of predicted returns holding at the time each contribution is made. Set $B^{(g)}(k) := C^{(g)}(k)/\nu^{(g)}(k)$. In this scheme as in the previous one, younger members accrue more target benefit for their contributions compared to older members. However, the amount of target benefit accrued by each contribution is not known until the contribution is made at time k since it depends on the set of predicted returns holding at time k.

The constant contribution paid by all members before retirement is calculated by reference to the unfair annuity CDC scheme. It is calculated so that the discounted lifetime contributions equals the discounted lifetime benefits of a member in generation 0, i.e.

$$C\left(1 + \sum_{m=1}^{T-1} \prod_{\ell=1}^{m} \frac{1}{1+i(\ell,0)}\right) = \frac{\bar{b}}{T} \nu^{(0)}(0). \tag{8}$$

The unfair annuity CDC scheme is closest to a UK DB scheme, in terms of the presentation of the benefits and contributions. It is quite possible (as evidenced by the proposed Royal Mail CDC scheme and given both the type of DB schemes in existence in the UK, and that DB pension actuaries are the architects of the new CDC schemes) that UK CDC schemes could look similar. An important question is to know if there are substantial implications for choosing one of the above model CDC scheme structures, from an investment risk-sharing perspective.

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4.2. Evolution of the annuity CDC scheme

The evolution of the annuity CDC scheme is similar to the lump sum CDC scheme, and is relegated to Appendix A. Similarly, consideration of an annuity IDC scheme, shown in Appendix A.4, results in the annuity IDC factor

$$\mathrm{IDC}^{(g)}(k) := \begin{cases} \dfrac{B^{\mathrm{IDC},(g)}(k) - \frac{C^{(g)}(k)}{v^{(g)}(k)}}{B^{\mathrm{IDC},(g)}(k-1)} - 1, & \text{for } k = g+1, g+2, \ldots, g+T \\ \dfrac{B^{\mathrm{IDC},(g)}(k)}{B^{\mathrm{IDC},(g)}(k-1)} - 1, & \text{for } k = g+T+1, g+T+2, \ldots, g+T+S-1, \end{cases} \tag{9}$$

for each generation $g = 0, 1, \ldots, M-1$.

4.3. Extending the benefit decomposition to the annuity CDC scheme

The decomposition of the benefit increase in the lump sum CDC scheme can be extended to the annuity CDC scheme too. As before, this allows the impact of both investment risk sharing and the additional risk sharing due to contributions not funding the accrued benefits.

The benefit increase $\delta(k)$ in the annuity CDC scheme is expressed as the product of four factors, namely

$$1 + \delta(k) = \left(1 + \mathrm{IDC}^{(g)}(k)\right)\left(1 + \beta^{(g)}(k)\right)\left(1 + \gamma^{(g)}(k)\right)\left(1 + \kappa^{(g)}(k)\right). \tag{10}$$

- The IDC factor represents the benefit increase which the member would get if they had, instead, invested their contributions in an IDC scheme. Its expression is given in (9).
- The investment risk-sharing factor $\beta^{(g)}(k)$ is the core investment risk-sharing factor and is calculated for the fair annuity CDC scheme. The contributions are rolled up at the predicted returns prevailing at the time each contribution is made, using the number of years from when the contributions are made until retirement. The benefit increase in the fair annuity CDC scheme is set equal to $\left(1 + \mathrm{IDC}^{(g)}(k)\right)\left(1 + \beta^{(g)}(k)\right) - 1$, from which the value of $\beta^{(g)}(k)$ can be extracted.
- The unfair predictions factor $\gamma^{(g)}(k)$ gives the effect of using the predicted returns at time 0 to calculate the target benefits accrued by each contribution, rather than the predicted returns at the time each contribution is made. The benefit increase in the partially fair annuity CDC scheme is set equal to $\left(1 + \mathrm{IDC}^{(g)}(k)\right)\left(1 + \beta^{(g)}(k)\right)\left(1 + \gamma^{(g)}(k)\right) - 1$, from which the value of $\gamma^{(g)}(k)$ can be extracted, once the values of $\mathrm{IDC}^{(g)}(k)$ and $\beta^{(g)}(k)$ are determined. In the fair annuity CDC scheme, $\gamma^{(g)}(k) = 0$ for all g and k.
- The unfair benefit factor $\kappa^{(g)}(k)$ shows the residual impact of the contributions not funding the accrued benefits, once the effect of using the time 0 predicted returns to acccumulate has been calculated via the unfair predictions factor $\gamma^{(g)}(k)$. It is extracted from the benefit increase in the unfair annuity CDC scheme, using the expression (10) and the other now-calculated factors. In the fair and partially fair annuity CDC schemes, $\kappa^{(g)}(k) = 0$ for all g and k.

4.4. Numerical illustration of the annuity CDC benefit decomposition

The benefit decomposition in the annuity CDC scheme is illustrated using the Wilkie model (Wilkie, 1986, 1995) to generate the economic scenarios. The model is used to generate an inflation index, equity and long-term bond returns. Price inflation is stripped out of the returns and yields generated from the model. This is to compensate for the simplifying assumption in the presented CDC schemes being that cashflows are not inflation-linked, which is unrealistic over long-term periods.

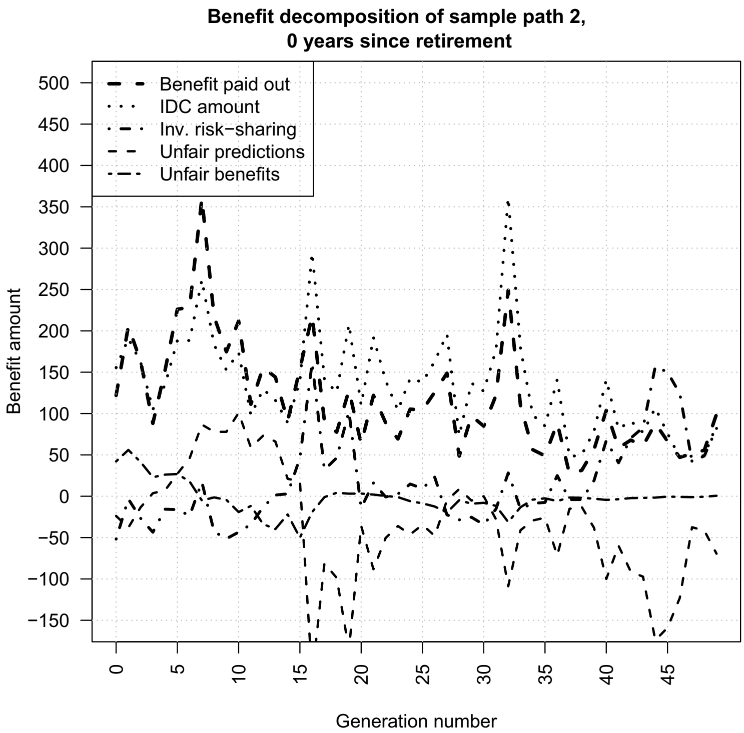

Figure 2. A sample path of the retirement benefit paid out in the unfair annuity CDC scheme, under the Wilkie model, for the payment made at the time of retirement. In term of notation, the chart shows a sample path of $B^{(g),\mathrm{cum}}(g+30)$ plotted against the generation g. Contributions are a constant 9.86 units paid by each member for $T=30$ years and the benefit payment is for $S=20$ years. The initial target benefit is a constant 100 units per annum. The inflation-adjusted investment returns earned on assets are generated from the Wilkie model.

The Wilkie model is a mean-reverting, discrete time model with prices and indices reported annually. The parameter values used are the ones from fitting the model to UK market data from 1923 to 2009 and are taken from Wilkie *et al.* (2010). The model and parameter values used are fully detailed in Appendix B.

4.4.1. Scheme assets 100% invested in shares

Scheme assets are assumed to be 100% invested in shares. The actual investment return on the scheme assets is the real return on the total return index of shares generated by the Wilkie model. The model generating the returns is assumed to be unknown to the managers of the CDC scheme. Instead, the predicted returns are taken as the current long-term bond returns – which are observed in the market and generated by the Wilkie model – plus an additional constant known as the equity risk premium, with the sum of these two values then adjusted to remove price inflation. The latter approach, albeit without stripping out price inflation, is a standard one in the valuation of UK DB pension schemes.

4.4.2. Annual contribution paid for $T=30$ years

A constant annual contribution of $C=17.06$ units is paid by each member into the CDC scheme for $T=30$ years, based on a predicted return of 5.37% per annum. The contribution is calculated

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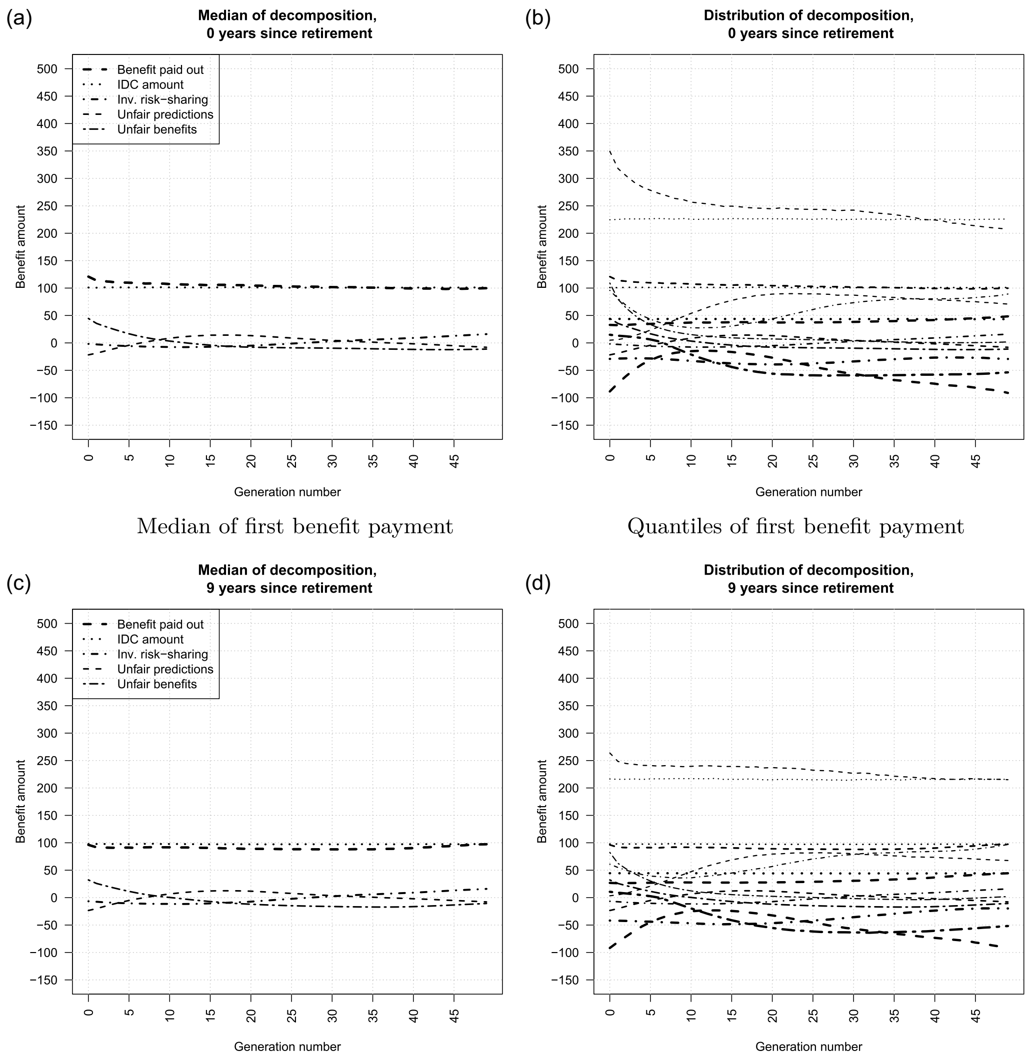

Median of first benefit payment — Quantiles of first benefit payment

Median of tenth benefit payment — Quantiles of tenth benefit payment

Figure 3. The median (left-hand plots) and median, 10% and 90% quantiles (right-hand plots) of the annuity benefits paid out at retirement and their decomposition, for the unfair CDC scheme, using the Wilkie model. Contributions are a constant 17.06 units paid by each member for $T = 30$ years and the benefit payment is for $S = 20$ years. The initial target benefit is a constant 100 units per annum.

via equation (8), using the starting values of the Wilkie model which are detailed in Appendix B. The starting values in the Wilkie model are the long-term expected values.

In retirement, each member receives a fixed-term annuity benefit for $S = 20$ years, paid annually in advance.

4.4.3. Observations from the figures

The benefit payments calculated under 100,000 simulations from the Wilkie model were analysed, with a sample path of these payments upon retirement shown in Figure 2. It is observed that the

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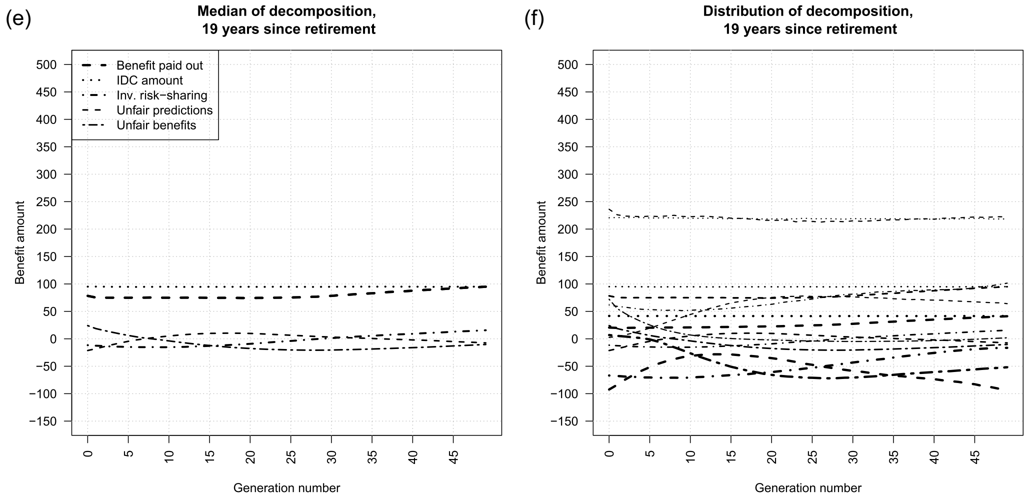

Median of last benefit payment

Quantiles of last benefit payment

Figure 3. Continued.

path is very difficult to understand. More useful to examine are the statistics of the payments, shown for the first, tenth and last payments in retirement in Figures 3–4.

The figures indicate the following.

- A constant benefit accrual, used only in the unfair CDC scheme, results in the first generations getting higher benefit payments compared to the IDC scheme, and the later generations getting lower payments. This is due to the unfair predictions factor, with the benefit amount attributed to this factor denoted by the blue line in the figures.

  For all generations, their initial contributions are too high relative to the concomitant accrued benefits in the unfair annuity CDC scheme. They are too high because benefit accrual is uniform across all ages, meaning that each generation pays a financially unfair contribution when they are younger. However, when the first generations join, the existing asset value is relatively small. The benefit increase compensates for the difference between the relatively large contributions (the numerator in the benefit increase formula) and the relatively small accrued benefits (the denominator) when the first generations are younger. The higher the contribution, the larger the benefit increase for the first generations (left-hand plots in Figure 4) as the numerator in the benefit increase formula is higher.

  The effect diminishes with later benefit payments as more members join, the existing asset value grows larger and hence the impact of the too-high contributions paid by younger members diminishes (blue lines in the left-hand plots of Figure 3).

  Particularly for the last generations, they pay their initially too-high contributions when the unfair CDC scheme's assets are already very sizeable in value. Hence, the last generations do not get the large, positive benefit increases given to the first generations. Moreover, a too-high benefit payment has already been either accrued by or paid out to the first generations. It is the last generations who pay for this anomaly, receiving much lower initial benefit increases compared to the first generations' initial benefit increases.
- The distribution of the benefit payouts among generations is sensitive to the contribution amounts. If the interest rate implied by the contributions is lower than the returns achieved by the assets, then the first generations are better off in the unfair or partially fair CDC scheme and the last generations are better off in either the IDC scheme or fair CDC scheme (left-hand

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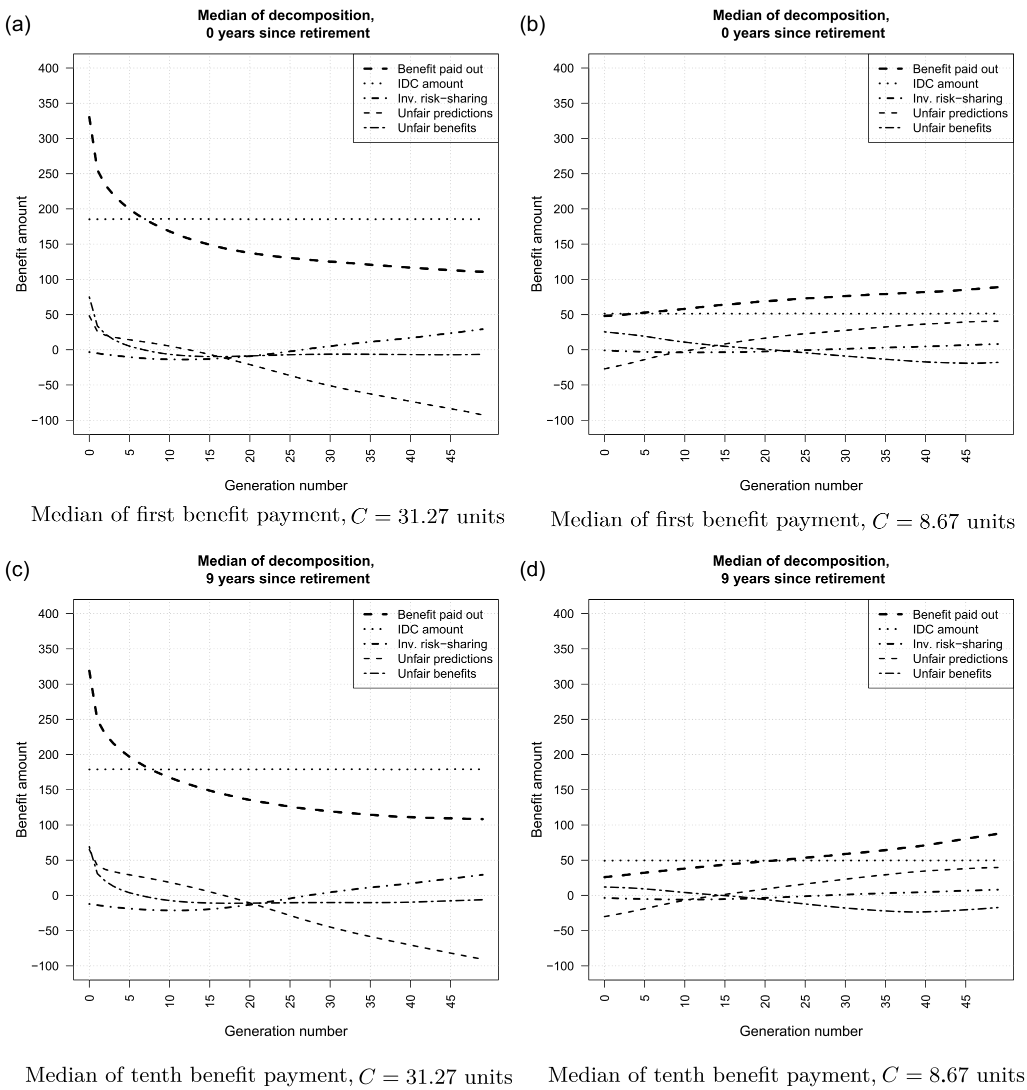

Figure 4. The median annuity payments paid out at retirement and their decomposition, when the contribution paid by each member before retirement is $C = 31.27$ units (left-hand plots) and $C = 8.67$ units (right-hand plots), for the unfair CDC scheme, using the Wilkie model. The initial target benefit is a constant 100 units per annum.

plots in Figure 4). The reverse is true when the interest rate implied by the contributions is higher than the achieved returns (right-hand plots in Figure 4).

With the contributions calculated using a return close to the mean return in the Wilkie model, the median benefit payments in the unfair CDC scheme (left-hand plots in Figure 3, black line) are close to those of the IDC scheme (orange lines in Figure 3). The median payments decline slowly over time due to paying out slightly too much, too soon. This is simply because the predicted returns are slightly higher than the realised returns, for both the CDC and IDC schemes. The decline is larger for the unfair CDC scheme (black lines), compared to the IDC scheme (orange lines), as more benefit is paid out at earlier times resulting in less money available to pay out the later benefit payments.

A contribution amount different to the fairly model-neutral value of $C = 17.06$ units leads to the median payments being quite different in the unfair CDC scheme compared to the IDC

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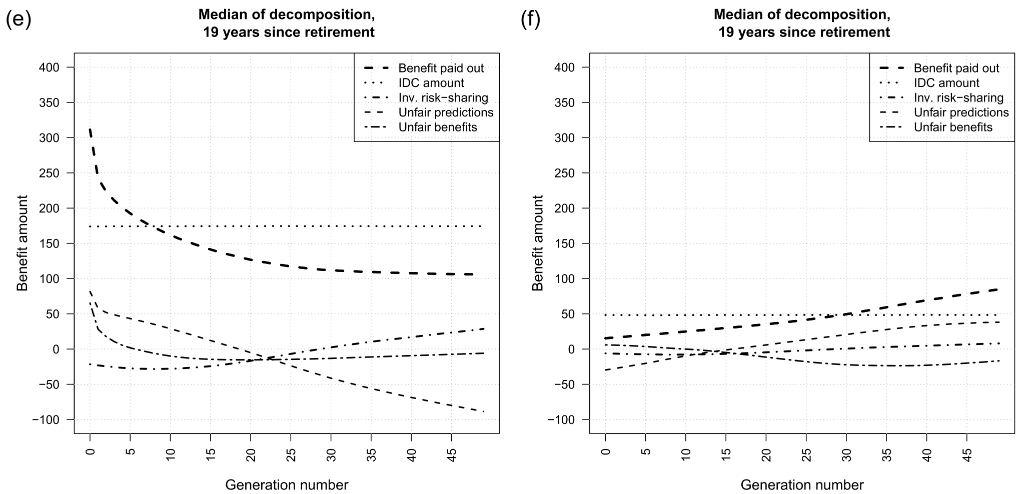

Median of last benefit payment, $C = 31.27$ units

Median of last benefit payment, $C = 8.67$ units

Figure 4. Continued.

scheme (Figure 4). The figures show that it is both the unfair predictions factor (red lines) and unfair benefit factors (blue lines) which are the culprits, with the unfair predictions factor (red lines) being the largest contributor.

When the predicted returns are low relative to the returns which are achieved, the benefit increases awarded are high for the earlier generations (left-hand plots in Figure 4). Their contributions earn a much higher return than predicted. The benefit increases are, once again, too large, as there are no or little existing assets to temper the effect of the mis-match between actual and predicted returns. Again, the later generations do not receive the same level of increase as they pay their contributions when the asset value is relatively large. Instead, they end up paying for the too-large benefits of the earlier generations. The reverse is true when predicted returns are too high (right-hand plots in Figure 4).

A constant benefit accrual amplifies this effect when the predicted returns are too low relative to the returns actually achieved (left-hand side of Figure 3, blue lines) and reduces the effect when the predicted returns are too high (right-hand side of Figure 3, blue lines).

Additionally, a changing number of members in each generation can also exacerbate or alleviate the effects. For example, suppose there are a large number of members in the first generations who accrue a too-high benefit payment. If the number of members drop in later generations, then the cost of the too-high benefit payments accrued by the earlier generations is shared among a smaller group of people in the later generations. Thus, the members of the later generations end up with an even lower pension, compared to those in a scheme with a constant number of members in each generations.

- The fair annuity CDC scheme is not close to the IDC scheme. Looking at the median payments (Figures 3–4, sum of the orange IDC line and green investment risk-sharing line) it seems that they are close as the median benefit attributed to investment risk sharing is relatively small.

  However, the quantiles shown in the right-hand plots of Figure 3(e) (green dashed lines) indicate caution, as there is some significant variability in the distribution of the benefits attributed to investment risk sharing. The sample path in Figure 2 bears this out; the benefits paid from the fair annuity CDC scheme can differ notably from the IDC scheme.

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5. Conclusion

The nascent CDC scheme industry in the UK is at a crossroads. Will pensions actuaries build a strong, vibrant CDC scheme industry, instead of only advising on the cleanest way for a DB pension scheme to die? Will the Royal Mail CDC scheme be a bohemian outlier or does it herald a splendorous era of collectivisation?

If CDC schemes are to gain traction in the UK, the implications of different design features should be understood. This is highly important because in CDC schemes, the scheme manager makes decisions on behalf of the members, but the members bear the full financial consequences of those decisions. Members do not decide on their own investment strategy, as they do in DC schemes. Nor do they have an employer picking up the cost of underfunding, as they do in DB schemes. While it can be convincingly argued that most DC fund members do not understand the financial consequences of their chosen investment strategy, and DB schemes can fail or cut their future benefits, the power and knowledge imbalance between members and CDC scheme manager is large. It is crucial that scheme design decisions are justified and in the interests of all members.

This paper focuses on the CDC design decisions which have implications for the extent of investment risk-sharing between generations. One of the main contributions of the paper is an attribution of the benefits obtained in a CDC scheme to each design decision and an analysis of the attribution.

One optional feature of a UK CDC scheme is to fix the benefit accrual. It is well-known that a constant benefit accrual is financially unfair on an accrued basis, in the sense that members pay too much when they are young for their accrued benefit and too little when they are old. While such an approach may be acceptable in a DB scheme in which employers pay the balance of costs, the justification becomes weaker in a CDC scheme. The members bear the consequences of cross-subsidies between generations, in the form of benefits which are lower than anticipated.

This paper shows that, for a new scheme, a constant benefit accrual results in benefit payments which are too high for earlier generations, with the cost being borne by later generations. This cross-subsidy between generations is known in advance and thus seems genuinely unfair. Such an effect can be further amplified if there are more members in earlier generations compared to later generations.

Fundamentally, the issue is that a CDC operates on a year-by-year basis, due to the calculation of the benefit increase formula. This is at odds with the use of lifetime contributions and lifetime benefit accrual, implied by a constant benefit accrual.

Which generation benefits the most from investment risk sharing, can also depend on the contributions paid, assuming that these are constant. Too high a contribution relative to the actual returns achieved on assets, means that earlier generations do well at the expense of later generations. Too low a contribution, and the reverse holds. Therefore, it is important to review contributions at regular intervals against expected future returns, and modify them as appropriate, to avoid excessive implicit wealth transfers between generations.

Adding in longevity risk would alter the results. Qiao and Sherris (2013) discuss the implications of longevity risk sharing in a pooled annuity fund – which is a decumulation-only CDC plan with no investment risk sharing – in which successive generations join the fund. Longevity risk can be split into idiosyncratic longevity risk, in which the distribution of deaths is known but not observed empirically, and systematic longevity risk, in which the wrong distribution of deaths is used to calculate the annuity values and longevity credits. Having too few people in the scheme to pool adequately idiosyncratic longevity risk at some time would increase the volatility of the income paid out at that time. Systematic longevity risk in one generation is borne by both that generation and later generations. Indeed, the existence of systematic longevity risk in one generation will increase the risk faced by later generations, as they face a wider range of possible income values compared to earlier generations as the financial impact of paying out too much or too little income to a generation is compounded by returns over time. Olivieri & Pitacco (2020) discuss ways of dealing with systemic longevity risk in the context of life annuities.

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The limitations of the paper are many and the topic is highly important to the development of CDC schemes in the UK. Further research to analyse the implications of investment risk sharing in a more realistic setting, for example to include longevity risk sharing, is warranted in order to give a fuller picture.

Acknowledgements. This research was funded by the Institute and Faculty of Actuaries' Actuarial Research Centre grant "Minimizing longevity and investment risks while optimising future pension plans". The authors are grateful for this funding.

The authors thank two anonymous referees for helpful comments on the submitted manuscript, which have led to improvements in the paper.

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A. Annuity CDC scheme evolution

A.1. Starting phase of the annuity CDC scheme

In the starting phase of the annuity benefit scheme, new generations join each year and no generation has yet retired. The scheme is growing in terms of number of members, and its assets are anticipated to grow as it receives contributions and has not yet made any payments out.

The asset value of the scheme just before anyone joins is $A(0_-)=0$. Once generation 0 joins, $A(0)=N(0)C^{(0)}(0)$. Letting $R(k)$ denote the random investment return earned on assets over $(t-1,t]$, the asset value just before the new contribution from generation 1 is $A(1_-)=A(0)(1+R(1))$.

The asset value just before any payments received or paid out at time k is

$$A\left((k+1)_-\right)=A(k)(1+R(k)),\quad \text{for } k\in\{1,2,\ldots,T-1\}.$$

Denote the cumulative benefit accrued by generation g at time k by $B^{(g),\text{cum}}(k)$, with $B^{(g),\text{cum}}(g)=B^{(g)}(g)$. The cumulative benefit accrued up to retirement can be calculated recursively as

$$B^{(g),\text{cum}}(k+1)=B^{(g),\text{cum}}(k)\left(1+\delta(k+1)\right)+B^{(g)}(k+1),\quad \text{for } k=g,g+1,\ldots,g+T-2. \tag{A.1}$$

The cumulative benefit accrued once generation g retires and is no longer accruing new benefits is calculated recursively as

$$B^{(g),\text{cum}}(k+1)=B^{(g),\text{cum}}(k)\left(1+\delta(k+1)\right),\quad \text{for } k=g+T-1,g+T,\ldots,g+S-1. \tag{A.2}$$

To determine the first benefit increase $\delta(1)$, equate the asset value with the discounted value of the accrued benefits, i.e.

$$\delta(1)=\frac{A(1_-)}{N(0)\,B^{(0),\text{cum}}(0)\nu^{(0)}(1)}-1.$$

Once $\delta(1)$ has been calculated, the benefit accrued by generation 0 is increased to

$$B^{(0),\text{cum}}(1)=B^{(g),\text{cum}}(0)\,(1+\delta(1))+B^{(0)}(1),$$

allowing for the new accrued benefit at time 1.

In general, the benefit increase at time $k\in\{1,2,\ldots,T-1\}$ is

$$\delta(k)=\frac{A(k_-)}{\sum_{g=0}^{k-1}N(g)\,B^{(g),\text{cum}}(k-1)\nu^{(g)}(k)}-1.$$

Once the benefit increase is known, the accrued benefits are increased, allowing for the new accrued benefit at time k, using equation (A.1). The asset value after the new contributions are received at time k is

$$A(k)=A(k_-)+\sum_{g=0}^{k}N(g)C^{(g)}(k),\quad \text{for } k\in\{1,2,\ldots,T-1\}.$$

A.2. Middle phase of the annuity benefit scheme

At time T, the scheme enters its middle phase when some generations start retirement but other generations continue to accrue benefits. The scheme may either be growing or shrinking in terms of numbers of members, depending on how many members are in each generation.

The asset value is $A\left((k+1)_-\right)=A(k)(1+R(k))$ at time $k\in\{T,T+1,\ldots,M-1\}$. As benefit increases continue to be granted while a pension is in payment, the benefit increase awarded at time k is

$$\delta(k) = \frac{A(k_-)}{\sum_{g=\max\{0,k-(T+S-1)\}}^{k-1} N(g)\nu^{(g)}(k)B^{(g),\text{cum}}(k-1)} - 1, \quad \text{for } k \in \{T, T+1, \ldots, M-1\}.$$

The benefits accrued at time k are calculated using either equation (A.1) or (A.2), according to whether generation g is retired or not. Then new contributions are received, including from the newly joining generation k, and benefits paid out, so that the asset value after these payments is

$$A(k) = A(k_-) + \sum_{g=k-T+1}^{k} N(g)C^{(g)}(k) - \sum_{g=\max\{0,k-(T+S-1)\}}^{k-T} N(g)B^{(g),\text{cum}}(k),$$

for $k \in \{T, T+1, \ldots, M-1\}$.

A.3. End phase of the annuity benefit scheme

From time $M > T$, the scheme enters its end phase as no new generation joins the scheme. The scheme is overall declining in terms of number of members. While some existing generations are not-yet-retired, and thus continue to make contributions until their retirement,

The asset value is $A\left((k+1)_-\right) = A(k)(1+R(k))$ at integer time k. The benefit increase awarded is

$$\delta(k) = \frac{A(T_-)}{\sum_{g=\max\{0,k-(T+S-1)\}}^{M-1} N(g)\nu^{(g)}(k)B^{(g),\text{cum}}(k-1)} - 1,$$

$$\text{for } k \in \{M, M+1, \ldots, M+T+S-2\}.$$

Once again, the benefits accrued at time k are calculated using either equation (A.1) or (A.2). Then new contributions are received from the not-yet-retired generations and benefits are paid out. The asset value after these payments is for $k \in \{M, M+1, \ldots, M+T-2\}$,

$$A(k) = A(k_-) + \sum_{g=k-T+1}^{M-1} N(g)C^{(g)}(k) - \sum_{g=\max\{0,k-(T+S-1)\}}^{k-T} N(g)B^{(g),\text{cum}}(k)$$

and for $k \in \{M+T-1, M+T, \ldots, M+T+S-2\}$

$$A(k) = A(k_-) - \sum_{g=\max\{0,k-(T+S-1)\}}^{M-1} N(g)B^{(g),\text{cum}}(k).$$

By simplifying the expression for the benefit increase at time $M+T+S-2$, the last tranche of benefits paid out at time $M+T+S-2$ to generation $(M-1)$ is

$$B^{(M-1),\text{cum}}(M+T+S-2) = \frac{A((M+T+S-2)_-)}{N(M-1)}.$$

In other words, the last generation receives an equal share of the final asset value as their last benefit payment. Once the final benefit is paid out, the asset value falls to zero, i.e. $A(M-2+S) = 0$, and the scheme ceases to exist.

A.4. Annuity IDC scheme

The benefit payments under the annuity CDC scheme are compared to those of an IDC scheme which aims to provide an income in retirement. In the IDC scheme, each member of generation g invests $C^{(g)}(k)$ at time $k = g, g+1, \ldots, g+T-1$ to earn the same investment return as the CDC scheme.

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Let $B^{\mathrm{IDC},(g)}(k)$ represent the predicted cumulative benefit accrued at time k by each member of generation g in the IDC scheme. Their IDC investments accumulate at their retirement time $g+T$ to give a first benefit payment at time $g+T$ of

$$B^{\mathrm{IDC},(g)}(g+T) = \frac{\sum_{m=g}^{g+T-1} C^{(g)}(m) \prod_{\ell=m+1}^{g+T} (1+R(\ell))}{\nu^{(g)}(g+T)}.$$

The benefit amount $B^{\mathrm{IDC},(g)}(g+T)$ is also the prediction of the benefit to be paid out to each member of generation g beyond time $g+T$. The benefit paid out at time $k = g+T+1, g+T+2, \ldots, g+T+S-2$ is

$$B^{\mathrm{IDC},(g)}(k) = \frac{\sum_{m=g}^{g+T-1} C^{(g)}(m) \prod_{\ell=m+1}^{k} (1+R(\ell)) - \sum_{m=g+T}^{k-1} B^{\mathrm{IDC},(g)}(m) \prod_{\ell=m+1}^{k} (1+R(\ell))}{\nu^{(g)}(k)}.$$

and the benefit paid out at time $g+T+S-1$ is whatever remains in the IDC pot, i.e.

$$B^{\mathrm{IDC},(g)}(g+T+S-1) = \sum_{m=g}^{g+T-1} C^{(g)}(m) \prod_{\ell=m+1}^{g+T+S-1} (1+R(\ell)) - \sum_{m=g+T}^{g+T+S-2} B^{\mathrm{IDC},(g)}(m) \prod_{\ell=m+1}^{g+T+S-1} (1+R(\ell)).$$

As in the CDC scheme, a prediction $B^{\mathrm{IDC},(g)}(k)$ can also be made at each earlier time $k \in \{g, g+1, \ldots, g+T\}$ of the benefit stream $\{B^{\mathrm{IDC},(g)}(k); k = g+T, g+T+1, \ldots, g+T+S-1\}$, for each generation g. When generation g first joins the IDC scheme, the initial predicted retirement benefit is

$$B^{\mathrm{IDC},(g)}(g) = \frac{C^{(g)}(g)}{\nu^{(g)}(g)}.$$

At time $k \in \{g+1, g+2, \ldots, g+T-1\}$, their predicted retirement benefit accumulates partially with actual returns and partially with predicted returns, i.e.

$$B^{\mathrm{IDC},(g)}(k) = \frac{\sum_{m=g}^{k-1} C^{(g)}(m) \prod_{\ell=m+1}^{k} (1+R(\ell))}{\nu^{(g)}(k)} + \frac{C^{(g)}(k)}{\nu^{(g)}(k)},$$

in which the second term on the right-hand side is the predicted benefit accrued by the contribution made at time k.

Define the IDC benefit increase, which is the annuity IDC factor, as

$$\mathrm{IDC}^{(g)}(k) := \begin{cases} \dfrac{B^{\mathrm{IDC},(g)}(k) - \frac{C^{(g)}(k)}{\nu^{(g)}(k)}}{B^{\mathrm{IDC},(g)}(k-1)} - 1, & \text{for } k = g+1, g+2, \ldots, g+T \\ \dfrac{B^{\mathrm{IDC},(g)}(k)}{B^{\mathrm{IDC},(g)}(k-1)} - 1, & \text{for } k = g+T+1, g+T+2, \ldots, g+T+S-1, \end{cases}$$

according to whether contributions are being made, and hence whether a new accrued benefit must be stripped out of the predicted benefit.

The first expression for the IDC factor can be written, in an analogous way to the IDC factor in the lump-sum scheme, as

$$\mathrm{IDC}^{(g)}(k) = \frac{1+R(k)}{1+i(k,k-1)} \frac{\sum_{m=g+T}^{g+T+S-1} \prod_{\ell=k+1}^{m} \left(1+i(\ell,k-1)\right)^{-1}}{\sum_{m=g+T}^{g+T+S-1} \prod_{\ell=k+1}^{m} \left(1+i(\ell,k)\right)^{-1}} - 1,$$

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for $k = g + 1, g + 2, \ldots, g + T$. Thus the annual change in the predicted-before-retirement IDC benefit is the product of two factors, one concerned with the past and the other with the future. The first factor, $(1 + R(k))/(1 + i(k, k - 1))$, gives the impact due to the actual return being different to its prediction. The second factor shows the effect of change in predictions of future returns on the valuation of the retirement benefit stream. A more complicated expression is obtained for the IDC factor once the benefit is in payment.

B. Stochastic financial market model: the Wilkie model

The Wilkie model is in discrete time with prices and indices reported annually. The parameter values given below are the ones from fitting the model to UK market data from 1923 to 2009 and are found in Wilkie *et al.* (2010). The model and parameter values used are given here for completeness, with the model notation and exposition following closely that in Hardy (2003).

Price inflation is stripped out of the returns derived from the model, since the simplifying assumption in the presented CDC schemes is that cashflows are neither wage nor price inflation-linked but the main factor driving the Wilkie model is price inflation.

The stochastic process $Z_X := \{Z_X(k) : k = 1, 2, \ldots\}$ is a sequence of independent random variables which each have a standard normal distribution, for each $X \in \{c, d, q, w, y\}$. Each of these stochastic processes is independent of the other.

A total of 100,000 simulated values were generated from the Wilkie model detailed below. The time unit is one year.

B.1. Force of inflation

For integer k, the force of inflation over the time period $[k - 1, k)$ is

$$\delta_q(k) = \mu_q + a_q\big(\delta_q(k-1) - \mu_q\big) + \sigma_q Z_q(k), \qquad \delta_q(0) = \mu_q \quad \text{a.s.}$$

in which (Wilkie *et al.*, 2010; Paragraph 2.12)

$$\mu_q = 0.043, \qquad a_q = 0.58, \qquad \sigma_q = 0.04.$$

B.2. Share prices and dividends

For integer k, the share dividend yield over the time period $[k - 1, k)$ is

$$y(k) = \exp\big(w_y\delta_q(k) + \mu_y + yn(k)\big)$$

where

$$yn(k) = a_y \cdot yn(k-1) + \sigma_y Z_y(k), \qquad yn(0) = 0 \quad \text{a.s.}$$

The parameter values used are (Wilkie *et al.*, 2010; Paragraph 4.8)

$$w_y = 1.55, \qquad \mu_y = \ln(0.0375), \qquad a_y = 0.63, \qquad \sigma_y = 0.155.$$

The force of dividend growth over the time period $[k - 1, k)$ is

$$\delta_d(k) = w_d \cdot DM(t) + (1 - w_d)\delta_q(k) + d_y\sigma_y Z_y(k-1) + \mu_d + b_d\sigma_d Z_d(k-1) + \sigma_d Z_d(k)$$

where

$$DM(k) = d_d\delta_q(k) + (1 - d_d)\, DM(k-1), \qquad DM(0) = \mu_q \quad \text{a.s.}$$

The parameter values used are (Wilkie *et al.*, 2010; Paragraph 5.7)

$$w_d = 0.43, \quad d_y = -0.22, \quad \mu_d = 0.011, \quad b_d = 0.43, \quad \sigma_d = 0.07, \quad d_d = 0.16.$$

The dividend index is calculated recursively as

$$D(k) = D(k-1)\exp\left(\delta_d(k)\right), \qquad D(0) = 1 \quad \text{a.s.}$$

As the dividend yield $y(k)$ at time k is the dividend index value $D(k)$ divided by the share price index value $P(k)$, it follows that

$$P(k) = \frac{D(k)}{y(k)}, \qquad P(0) = D(0)/y(0) \quad \text{a.s.}$$

A total return index on shares over the time period $[k-1, k)$ is derived by assuming that dividends received are re-invested in shares, e.g.

$$py(k) = py(k-1)\cdot\frac{P(k)+D(k)}{P(k-1)}, \qquad py(0) = 1 \quad \text{a.s.}$$

The return on assets used in the numerical illustrations in section 4.4 is

$$R(k) = \frac{py(k)}{py(k-1)}e^{-\delta_q(k)} - 1, \quad \text{for } k = 1, 2, \ldots.$$

The mean of $\prod_{k=1}^{M+T+S}\left(1 + R(k)\right) - 1$ is 5.2% per annum.

B.3. Long-term bond yield

For simplicity, only the long-term bond yield $c(k)$ is used in the simulations. It is used as the prediction of the constant future long-term bond yield at time k. It is determined by

$$c(k) = cm(k) + cn(k),$$

in which

$$cm(t) = d_c\delta_q(k) + (1-d_c)\cdot cm(k-1), \qquad cm(0) = \mu_q \quad \text{a.s.}$$

and

$$cn(k) = \mu_c\exp\left(a_c\, cn(k-1) + y_c\sigma_y Z_y(k) + \sigma_c Z_c(k)\right), \qquad cn(0) = \mu_c \quad \text{a.s.}$$

The parameter values used are (Wilkie *et al.*, 2010; Paragraph 6.11)

$$d_c = 0.045, \quad \mu_c = 0.0223, \quad a_c = 0.92, \quad y_c = 0.37, \quad \sigma_c = 0.255.$$

B.4. Predicted return on assets and equity risk premium

The predicted return on assets is taken as the long-term bond yield $c(k)$ plus the equity risk premium, then this net sum is adjusted to strip out price inflation. Choosing the equity risk premium to be 3% per annum, the predicted return per annum at time $k = 0, 1, 2, \ldots$ is

$$i(\ell, k) = \exp\left(c(k) + 0.03 - \delta_q(k)\right) - 1, \quad \text{for } \ell = k+1, k+2, \ldots.$$

Cite this article: Barajas-Paz A and Donnelly C (2023). An attribution analysis of investment risk sharing in collective defined contribution schemes, *Annals of Actuarial Science*, **17**, 385–414. https://doi.org/10.1017/S1748499523000015