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Existence of a bounded function of the maximal spectral type

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Let us denote by $L^2_\mu(\Omega)$ the Hilbert space of all functions on the measure space Ω, square-integrable with respect to the measure μ.

Let us consider in $L^2_\mu(\Omega)$ the spectral measure $E(M)$, i.e. a family of projection operators depending in the σ-additive way on a Borel measurable subset of the real line. It is well known that in this case the set function

$$\sigma_f(M) = (E(M)f, f)$$

is a generalized measure.

In the study of certain questions connected to the spectral theory of dynamical systems and stationary random processes it is interesting to know whether there is a function in $L^2_\mu(\Omega)$ which has the maximal spectral type. That means that, for every $g \in L^2_\mu(\Omega)$, $\sigma_g(M) \ll \sigma_f(M)$ where the sign $\ll$ means the absolute continuity. The following theorem answers this question in the case of the separable space.

THEOREM. *If the space $L^2_\mu(\Omega)$ is separable then for every function $f \in L^2_\mu(\Omega)$ and for every $\varepsilon > 0$ there exists a bounded function $g \in L^2_\mu(\Omega)$ such that $\|f-g\| < \varepsilon$ and $\sigma_f \ll \sigma_g$. In particular, if f has the maximal spectral type then g has the same property.*

Proof. Let us remember several facts from the spectral theory of linear operators. It is well known (see [**1**, **2**]) that in our case

$$L^2_\mu(\Omega) = \sum_n \oplus H_n$$

so that every H_n is isometric to a space $L^2_\sigma(M_n)$ where σ is a measure belonging to the maximal spectral type and

$$(-\infty, +\infty) = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots.$$

Here the index n goes through a finite or countable set of values. In other words, every $f(P) \in L^2_\mu(\Omega)$ determines a sequence $\{\tilde{f}_n(\lambda)\}$ where $f_n \in L^2_\sigma(M_n)$. We shall denote this one-to-one correspondence by

$$f \leftrightarrow \{\tilde{f}_n\}. \tag{1}$$

Let us denote the set $\bigcup_n [\{\lambda ; |\tilde{f}_n(\lambda)| \neq 0\} \cap M_n]$, by A_f. Then the relation $\sigma_f \gg \sigma_g$ holds if and only if $\sigma(A_g \backslash A_f) = 0$.

Now let f be a given function and $\varepsilon > 0$. Let us denote

$$f^{(a,b)}(P) = \begin{cases} f(P), & \text{if } a \le |f(P)| < b \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Let us choose K large enough so that the following inequality holds:

$$\|f^{(K,\infty)}\| < \varepsilon/2. \tag{3}$$

According to (1) we can write

$$f^{(0,K^m)} = f^m \leftrightarrow \{\tilde{f}_n^m\}.$$

Since correspondence (1) is isometric then if $K^m \to \infty$ we have

$$\int_{M_n} |\tilde{f}_n(\lambda) - \tilde{f}_n^m(\lambda)|^2 \sigma(d\lambda) \to \infty.$$

Since convergence in the mean implies the almost everywhere convergence for a subsequence we can construct by a diagonal process a sequence $\{m_p\}$, $m_0 = 1$ and $m_p \to \infty$ and sets $N_n \subseteq M_n$ such that

$$\tilde{f}_n^{m_p}(\lambda) \to f_n(\lambda) \text{ for every } \lambda \in N_n, \sigma(M_n \backslash N_n) = 0. \tag{4}$$

Let us now construct the following function

$$f(z, P) = f^{(0,K)} + \sum_{p=1}^{\infty} z^{m_p} f^{(K_{m_{p-1}}, K_{m_p})}(P). \tag{5}$$

The orthogonality of the functions $f^{(K_{m_{p-1}}, K_{m_p})}$ and inequality (3) imply that series (5) converges in the mean for $|z| < 1$ and moreover,

$$\|f(z, P) - f^{(0,K)}(P)\| < \varepsilon/2.$$

Comparing the last inequality with (3) we obtain

$$\|f(z, P) - f(P)\| < \varepsilon \quad \text{for all } z, |z| < 1. \tag{6}$$

Moreover, for $|z| < 1/K$ we shall have

$$|f(z, P)| < \max[K, 1]. \tag{7}$$

We are now going to investigate the spectral type of functions $f(z, P)$. Let

$$f(z, P) \leftrightarrow \{\tilde{f}_n(\lambda, z)\} \quad \text{and} \quad A_{n,z} = \{\lambda : \tilde{f}_n(\lambda, z) \neq 0\}.$$

It follows from formula (5) that

$$\tilde{f}_n(\lambda, z) = \tilde{f}_n^0(\lambda) + \sum_{p=1}^{m} z^{m_p}[\tilde{f}_n^{m_p}P(\lambda) - \tilde{f}_n^{m_{p-1}}(\lambda)] \tag{8}$$

and the series converges in the mean. We deduce from (4) that for $\lambda \in N_n$ and for $z = 1$ the series converges in the usual sense. Hence by the Abel convergence theorem we conclude that $\tilde{f}_n(\lambda, z)$ is an analytic function in the open disk $|z| < 1$. Hence, for $\lambda \in N_n$ one of two possibilities holds: either $\tilde{f}_n(\lambda, z) \equiv 0$ or this function has at most a countable number of zeroes in the disk $|z| < 1$. Let us denote by B_n the set of all $\lambda \in N_n$ for which the second possibility holds. Since the identity $\tilde{f}_n(\lambda, z) \equiv 0$ implies that all the coefficients of series (8) vanish, then $f_n^{m_p}(\lambda) = 0$ for all m_p and $\lambda \notin B_n$ so that

$$\tilde{f}_n(\lambda, 1) = \tilde{f}_n(\lambda) = 0 \quad \text{for every } \lambda \in N_n \backslash B_n.$$

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This means that $B_n \supset A_{n,1}$. Thus, for all $\lambda \in A_{n,1}$ the second possibility holds.

Let us consider now the Cartesian product $A_{n,1} \times S$ of the set $A_{n,1}$ with the measure σ and the disk $S = \{z, |z| < 1\}$ with Lebesgue measure m. In that product the set $\{(\lambda, z): \tilde{f}_n(z, \lambda) = 0\}$ has $(\sigma \times m)$-measure 0 because every λ-section of that set has m-measure zero. In fact, such a section consists of at most countably many points. Therefore, for almost every z we have $\tilde{f}_n(\lambda, z) \neq 0$ for almost every $\lambda \in A_{n,1}$. That implies that for almost all z

$$\sigma(A_{n,1} \backslash A_{n,z}) = 0. \tag{9}$$

Choosing a $z_0, |z_0| < 1/K$ such that (9) holds for all n and using the fact that $A_{n,z} \subseteq N_n \subseteq M_n$ and $\sigma(M_n \backslash N_n) = 0$ we obtain that

$$\sigma\Big(\bigcup_n A_{n,1} \Big\backslash \bigcup_n A_{n,z_0}\Big) = 0.$$

That in turn implies that $\sigma_f = \sigma_{f(1,P)} \ll \sigma_{f(z_0,P)}$. If follows from (7) that $f(z_0, P)$ is a bounded function. This finishes the proof of the theorem. □

One can see from the proof of the theorem that in general the measures σ_f and $\sigma_{f(z_0,P)}$ are non-equivalent. It would be interesting to answer the following questions.

(1) Does there exist a bounded function of any given spectral type?

(2) Does there exist a bounded function in any subspace invariant with respect to $E(M)$?

(3) Does there exist a sequence f_q such that:

(a) all f_q are bounded;

(b) $(E(M)f_q, f_{q'}) = 0$ for $q \neq q'$ for all M;

(c) if $(E(M)f_q, g) = 0$ for all q and M, then $g \equiv 0$.

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