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RESEARCH ARTICLE

Equidimensionality of universal pseudodeformation rings in characteristic p for absolute Galois groups of p-adic fields

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Abstract
Let K be a finite extension of the p-adic field $\mathbb{Q}_p$ of degree d, let $\mathbb{F}$ be a finite field of characteristic p and let $\overline{D}$ be an n-dimensional pseudocharacter in the sense of Chenevier of the absolute Galois group of K over the field $\mathbb{F}$. For the universal mod p pseudodeformation ring $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ of $\overline{D}$, we prove the following: The ring $\overline{R}^{\mathrm{ps}}_{\overline{D}}$ is equidimensional of dimension dn^2+1. Its reduced quotient $\overline{R}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ contains a dense open subset of regular points x whose associated pseudocharacter D_x is absolutely irreducible and *nonspecial* in a certain technical sense that we shall define. Moreover, we will characterize in most cases when K does not contain a p-th root of unity the singular locus of Spec $\overline{R}^{\mathrm{univ}}_{\overline{D}}$. Similar results were proved by Chenevier for the generic fiber of the universal pseudodeformation ring $R^{\mathrm{univ}}_{\overline{D}}$ of $\overline{D}$.

Contents

1. Introduction

Let p be a prime number, and let K be a finite extension of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ with absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$. In [Che11], Chenevier establishes the following results on the rigid variety $\mathcal{X}_n$ of continuous pseudocharacters of G_K of dimension n with coefficients in $\mathbb{Q}_p^{\mathrm{alg}}$.

Theorem (Chenevier).

(a) *The open locus of regular points of $\mathcal{X}_n$ contains $\mathcal{X}_n^{\mathrm{irr}}$.*
(b) *The open subvariety $\mathcal{X}_n^{\mathrm{irr}} \subset \mathcal{X}_n$ of irreducible pseudocharacters is Zariski dense in $\mathcal{X}_n$.*
(c) *The variety $\mathcal{X}_n$ is equidimensional of dimension $dn^2 + 1$.*

Moreover, [Che11] gives a precise description of the singular locus of the varieties $\mathcal{X}_n$ in terms of representation-theoretic data. Note that from any continuous pseudocharacter x of dimension n of G_K with coefficients in $\mathbb{Q}_p^{\mathrm{alg}}$, one can reconstruct a semisimple n-dimensional continuous representation $\rho_x : G_K \to \mathrm{GL}_n(\mathbb{Q}_p^{\mathrm{alg}})$ that is unique up to conjugation; one calls x irreducible if ρ_x is irreducible.

The above results can be reinterpreted as results on the generic fibers of universal rings for pseudodeformations of a fixed residual pseudocharacter of G_K as introduced by Chenevier in [Che14]. Note that we use the term *pseudocharacter* for what Chenevier in [Che14] calls *determinant law* and what is called *pseudorepresentation* in [WE13]; the term *Taylor-pseudocharacter* we use for what in [Tay91] was called a *pseudocharacter*. Pseudocharacters of dimension n of a group Γ are certain polynomial laws in the sense of [Rob63] that model the formal properties of the characteristic polynomial of n-dimensional representations of Γ. The simpler notion of Taylor-pseudocharacter refers to maps that model the formal properties of the trace of n-dimensional representations of Γ. The two notions agree for coefficient fields of characteristic zero or of characteristic $p > 2n$; see [Che14, Proposition 1.29]. Taylor-pseudocharacters have some defects in characteristic $p \leq n$. Pseudocharacters behave well independently of the characteristic (and n). Also, a pseudocharacter D of G_K of dimension n with coefficients in an algebraically closed field κ is the pseudocharcter attached to a semisimple representation $\rho_D : G_K \to \mathrm{GL}_n(\kappa)$ that is unique up to conjugation.

Let now $\mathbb{F}$ be a finite field of characteristic p with ring of Witt vectors $W(\mathbb{F})$, and let $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ be the category of Noetherian $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$. Let $\overline{D}$ be a continuous pseudocharacter of G_K of dimension n with values in $\mathbb{F}$. If $\mathbb{F}$ is sufficiently large, $\overline{D}$ can be thought of as the pseudocharacter attached to a representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$, that is, morally as the characteristic polynomial law attached to $\bar{\rho}$. It is shown in [Che14] that any residual $\overline{D}$ admits a universal pseudodeformation ring $R_{\overline{D}}^{\mathrm{univ}}$ that represents the functor $\widehat{\mathcal{A}r}_{W(\mathbb{F})} \to \mathit{Sets}$, that to any object R of $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ assigns the set of pseudocharacter D_R of G_K with values in R and with residual pseudocharacter $\overline{D}$. The above theorem now asserts that the absolutely irreducible locus of $\operatorname{Spec} R_{\overline{D}}^{\mathrm{univ}}[1/p]$ is dense open in $\operatorname{Spec} R_{\overline{D}}^{\mathrm{univ}}[1/p]$ and contained in the regular locus of Spec $R^{\mathrm{univ}}_{\overline{D}}[1/p]$, and that $R^{\mathrm{univ}}_{\overline{D}}[1/p]$ is equidimensional of dimension dn^2+1; here x in $X^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} R^{\mathrm{univ}}_{\overline{D}}$ with corresponding residue field $\kappa(x)$ and pseudocharacter D_x is called irreducible if the semisimple representation $\rho_x := \rho_{D_x\otimes_{\kappa(x)}\kappa(x)^{\mathrm{alg}}}\colon G_K \to \mathrm{GL}_n(\kappa(x)^{\mathrm{alg}})$ is irreducible.

The present work concerns the special fiber of Spec $R^{\mathrm{univ}}_{\overline{D}}$, that is, the mod p reduction $\overline{R}^{\mathrm{univ}}_{\overline{D}} = R^{\mathrm{univ}}_{\overline{D}}/(p)$ of the ring $R^{\mathrm{univ}}_{\overline{D}}$ and the corresponding special fiber scheme $\overline{X}^{\mathrm{univ}}_{\overline{D}} := \operatorname{Spec} \overline{R}^{\mathrm{univ}}_{\overline{D}}$. Our main results are natural analogs of the assertions in the above theorem of Chenevier. Before giving them, we point to some differences to the results of Chenevier and introduce some notions to deal with them.

Let $\zeta_p \in K^{\mathrm{alg}}$ denote a primitive p-th root of unity. A first simple observation is that, already for $n=1$, the space $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ has empty regular locus, whenever ζ_p lies in K. To address this problem, we study the natural determinant map $\det_{\overline{D}}\colon \overline{X}^{\mathrm{univ}}_{\overline{D}} \to \overline{X}^{\mathrm{univ}}_{\det\overline{D}}, D\mapsto \det D$, where $\det D$ is the constant coefficient of the pseudocharacter D; that is, if D is attached to an n-dimensional representation ρ, then $\det D$ is attached to the one-dimensional representation $\det\rho$. Eventually, we show that $\det_{\overline{D}}$ is formally smooth when restricted to a dense open subset of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. This subset is (slightly) smaller than the open of locus $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$ of irreducible points of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. There is a closed subset of the irreducible locus (of relatively small dimension) spanned by points that we call *special* such that when we restrict $\det_{\overline{D}}$ to the dense open subscheme of nonspecial irreducible points of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ it is formally smooth. Because $\overline{X}^{\mathrm{univ}}_{\det\overline{D}}$ and its induced reduced subscheme $\overline{X}^{\mathrm{univ}}_{\det\overline{D},\mathrm{red}}$ are explicit and well understood, base change to reduced structures gives us access to $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ for which we deduce Chenevier's dimension formula.

There are several equivalent ways to describe the locus of special points of $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$; see Subsection 5.1 and some basic results on Clifford theory explained in Section 2; each has its benefits. Let ad_{ρ_x} denote the adjoint representation of ρ_x, and let $\mathrm{ad}^0_{\rho_x}$ be its subrepresentation on trace zero matrices. Let x be a dimension 1 point of $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$. The deformation theory as introduced by Mazur in [Maz89] yields that the map $\det_{\overline{D}}$ is formally smooth at x if $H^2(G_K,\mathrm{ad}^0_{\rho_x})$ vanishes. We call such an x *special*, if $H^2(G_K,\mathrm{ad}^0_{\rho_x})\neq 0$. An important observation is that special points are induced from representations of smaller dimension of the group $G_{K'}$ for K' a suitable extension of K. This link and induction give a strong dimension bound for the special locus $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ in $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$, that is, the Zariski closure of the special points of dimension 1 therein. More precisely, one has: If $\zeta_p\notin K$, then x is special if and only if ρ_x is induced from a representation of $G_{K(\zeta_p)}$; if $\zeta_p\in K$, then x is special if and only if there exists a degree p Galois extension K' of K such that ρ_x is induced from a representation of $G_{K'}$. To state our main results, we also abbreviate $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}} := (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}\setminus(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ and $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} := \overline{X}^{\mathrm{univ}}_{\overline{D}}\setminus(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$.

Theorem 1 (Theorem 5.5.1, **equidimensionality**). *The following assertions hold:*

(a) $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}}\subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is open and Zariski dense.*

(b) *If $\zeta_p\notin K$, then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}}$ is regular.*

(c) *If $\zeta_p\in K$, then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\text{n-spcl}}_{\mathrm{red}}$ is regular, and $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{reg}}$ is empty.*

(d) $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ *is equidimensional of dimension $[K:\mathbb{Q}_p]n^2+1$.*

Theorem 2 (Theorem 5.5.5, **singular locus**). *If $\zeta_p\notin K$, then the following hold:*

(a) *The closure of $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ lies in $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$.*

(b) *If $n>2$ or $[K:\mathbb{Q}_p]>1$, then $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}\subset(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$.*

(c) *If $n=2$, $K=\mathbb{Q}_p$, and $x\in(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ is a direct sum $D_1\oplus D_2$ of one-dimensional characters D_i, then $x\in(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{sing}}$ if and only if $D_2=D_1(m)$ for $m\in\{\pm1\}$.*

Theorem 3 (Theorem 5.5.7, **Serre regularity**). *The ring $\overline{R}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ satisfies Serre's condition (R_2) [1], unless $n = 2$, $K = \mathbb{Q}_2$ and $\overline{D}$ is trivial.*

We in fact determine the exact dimension of $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ and $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{spcl}}$ in Lemma 5.5.2 and Corollary 5.5.3. From this, depending on n and $[K : \mathbb{Q}_p]$, one can in general establish Serre's condition (R_m) for some $m = m_{K,n} > 2$.

It is a foundational and natural question to study the equidimensionality of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ and to better understand some geometric properties of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$, extending [Che11] to the special fiber. However, our true motivation was the expectation, that the equidimensionality proved here should help proving expected ring theoretic properties of the universal lifting ring $R^{\square}_{\bar{\rho}}$ attached to any continuous homomorphism $\bar{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F})$ by Kisin in [Kis09] as an important technical device to understand deformation rings as introduced in [Maz89] by Mazur. There should be a bootstrap argument to deduce from the dimension for $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ found here, that $R^{\square}_{\bar{\rho}}$ is flat over $W(\mathbb{F})$ of expected relative dimension $(d+1)n^2$. This in turn would give the local complete intersection property of $R^{\square}_{\bar{\rho}}$, the normality of the special fiber ring $R^{\square}_{\bar{\rho}}/(p)$, and it should allow one to deduce a bijection between the irreducible components of $R^{\square}_{\bar{\rho}}$ and of $R^{\square}_{\det\bar{\rho}}$, as expected from computations by us in the case $n = 2$ in [BJ15], and then in further cases in [CDP15], [Bab19] and [Iye20]. This in turn might prove, in light of recent results of [EG19], the Zariski density of crystalline points, and thereby to complete work of many others, notably [Nak14, §4] by Nakamura, extending previous important work of Chenevier, in the case where $\overline{\rho}$ is absolutely irreducible, and of [Iye20] when $\overline{\rho}$ is trivial, under some technical hypotheses on K, p and n. This program has now been completed in work of the first author with A. Iyengar and V. Paškūnas in [BIP21] and [BIP22].

Let us give some further ideas of the proofs and indicate some of the auxiliary results and techniques developed in this article. Our overall strategy is similar to [Che11]. But we face new phenomena that have to be dealt with.

Above, we already mentioned special points x of $(\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{irr}}$. They can exist when the cyclotomic character has finite order, that is, on the special but not the generic fiber. At such x, the representations ρ_x is induced from a representation of $G_{K'}$ for a proper cyclic extension K' of K of degree dividing n. So it is important for us to define an induction for pseudocharacters. This we work out in Subsection 4.6; our present approach incorporates significant improvements due to the referee. Using induction of pseudorepresentation, we show that the locus of special representations can be covered by finitely many $\overline{X}^{\mathrm{univ}}_{\overline{D}'}$, where the $\overline{D}'$ are continuous pseudocharacters of $G_{K'}$ for the K' just mentioned, and in particular they are of dimension $n/[K' : K] < n$. Now, in an inductive procedure, the space $\overline{X}^{\mathrm{univ}}_{\overline{D}'}$ is known to have dimension $(d[K' : K])(n/[K' : K])^2 + 1 = dn^2/[K' : K] + 1$, and this is much smaller than the lower dimension bound $dn^2 + 1$ that we establish for (all components of) $\overline{X}^{\mathrm{univ}}_{\overline{D}}$. In particular, the special locus is nowhere dense in $\overline{X}^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$. Another operation on pseudocharacters that we introduce in Subsection 4.5 is twisting by one-dimensional representations. We use it to prove the closedness of the special locus in the case $\zeta_p \notin K$.

A further important ingredient in our inductive argument to establish Theorem 1 is the proof that every neighborhood of some x in the reducible locus $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ contains a point of $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$. Here, we follow the argument used by Chenevier [Che11, Theorem 2.1], using, however, étale topology in place of rigid geometry. The key point in our setting is that étale locally $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}} \hookrightarrow \overline{X}^{\mathrm{univ}}_{\overline{D}}$ is a closed immersion. Hence, if a neighborhood U of some $x \in (\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ does not intersect $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$, the local behavior at x in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ is similar to that of $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \times \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ for pseudocharacters such that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$, after completion at x. This will ultimately yield a contradiction by comparing dimensions (of tangent spaces); see Theorem 5.2.1 and its proof. Following a suggestion of the referee, in Subsection 5.3 we give a second independent proof of Theorem 5.2.1.

[1] see Definition A.1.3

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On the technical side, we shall often work with dimension 1 points $x \in \overline{X}_{\overline{D}}^{\mathrm{univ}}$. The set of these is Zariski dense in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$, so they allow us to see all irreducible components. At the same time, their residue fields $\kappa(x)$ are Laurent series fields over a finite field, and so finite-dimensional $\kappa(x)$-algebras carry a unique topology compatible with that of $\kappa(x)$. We make use of this in considering deformation functors at such points. This is especially useful if D_x is irreducible or at least multiplicity free. This technique of studying deformation rings at dimension 1 points was introduced by Kisin over p-adic fields. We need to reprove some basic results, for instance in Subsection 3.3 and Subsection 4.8, building on [Che14], [WE18] for pseudodeformations and on [Nek06] for Tate local duality over general coefficient rings such as $\kappa(x)$ or its ring of integers.

Outline

We now give an outline of this work. Section 2 presents parts of Clifford theory to be used in Subsection 5.1 when defining and characterizing special points. Section 3 reviews the theory of deformations of Galois representation in the sense of Mazur with a strong emphasis on result related to deformation rings at dimension 1 points where the residue field is a local equicharacteristic field. Section 4 is a detailed review of pseudocharacters following largely [Che14] with some noteworthy additions that are crucial for the main results of this work: We consider the locus of reducibility in the context of pseudocharacters, we introduce twisting and induction of pseudocharacters, and we give a special treatment to some elementary facts on equicharacteristic dimension 1 points on pseudodeformation rings.

The final Section 5 contains the proof of the main results of this work, Theorems 1 to 3 on the special fiber of universal pseudodeformation rings. We follow Chenevier's proof for the generic fiber [Che11] and explain how to overcome all complications that arise in the special fiber. Much of these complications are packed into our definition of special points in Subsection 5.1; see Definition 5.1.2. nonspecial (irreducible) points will take the role of irreducible points in Chenevier's work; they describe that part of the irreducible locus of the special fiber of the pseudodeformation space over which the determinant map is relatively formally smooth.

Subsection 5.1 also contains some technical result on the comparison of universal pseudodeformation and universal deformation rings over local fields where the residual pseudocharacter is a sum of two irreducible ones; see Lemma 5.1.6. In Subsection 5.2, we describe the induction procedure that proves the main result: Given a suitable induction hypothesis, we shows that the reducible locus is nowhere dense. In Subsection 5.4, we show that the nonspecial points are open and Zariski dense in the irreducible locus under some inductive hypotheses. By combining the previous subsections, it is then in Subsection 5.5 straightforward to prove Theorems 1 to 3.

Let us also note that in an appendix, we provide some results on commutative rings, on algebras over a field and on absolutely irreducible mod p representations of the absolute Galois group of a p-adic field. These results are mostly standard and they serve as a convenient reference. In addition, in Subsection A.4 we prove a variant of an important result of Vaccarino that we use in the construction of induction for pseudocharacters in Subsection 4.6.

Some notation and conventions

- Throughout, we fix a prime number p and a finite field $\mathbb{F}$ of characteristic p.
- For any field E, we denote by E^{alg} an algebraic closure of E and by $G_E = \mathrm{Gal}(E/E^{\mathrm{alg}})$ its absolute Galois group.
- We write $\mathbb{Q}_p$ for the p-adic completion of $\mathbb{Q}$ and fix an algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ of $\mathbb{Q}_p$. All algebraic extension fields of $\mathbb{Q}_p$ will be considered as subfields of $\mathbb{Q}_p^{\mathrm{alg}}$.
- We fix a finite extension field K of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$ inside $\mathbb{Q}_p^{\mathrm{alg}}$.
- Throughout, κ will denote a finite field of characteristic p or a local field of residue characteristic p. It will take the role of a coefficient field for deformations and pseudodeformations. If such a coefficient field is meant to be finite, we usually write $\mathbb{F}$.

- For a point x on a scheme X, we write $\mathcal{O}_{X,x}$ for the local ring at x and $\kappa(x)$ for its residue field; the latter is the second way in which the letter κ occurs; note that $\kappa(x)$ can be any field.
- For a complete Noetherian local ring R with finite residue field $\mathbb{F}$, we call $x \in \operatorname{Spec} R$ with corresponding prime ideal $\mathfrak{p}_x \subset R$ a *point of dimension* 1 if $R/\mathfrak{p}_x$ has Krull dimension 1. The residue field $\kappa(x)$ will then either be a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}((x))$.
- By a ring, we mean a unital commutative ring. Algebras over a ring A do not need to be commutative. To make clear that an A-algebra is commutative, we will always speak of it as a commutative A-algebra.
- The categories $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$ of certain (pro-)Artinian local Λ-algebras, that have the same residue field as Λ, are introduced at the beginning of Subsection 3.1.
- The category $\mathcal{A}dm_\Lambda$ of admissible Λ-algebras is introduced at the beginning of Subsection 4.4.

2. Clifford theory

Clifford theory provides a crucial input in determining conditions that characterize the special points that we will introduce in Definition 5.1.2, building on Lemma 5.1.1. In this section, we give the representation theoretic background. We also include some results for coefficient fields that are not algebraically closed. The results in Subsection 2.2, and most importantly Corollary 2.2.2, are probably well known. Those in Subsection 2.3, and in particular Lemma 2.3.1, seem of more exotic nature to us. We give proofs whenever we could not locate the results in the literature.

Throughout this section, G denotes a (possibly infinite) group and H a subgroup of finite index. If G is a topological group, we assume H to be open in G. We define $N := \bigcap_{g\in G/H} H^g$. It is of finite index and normal in G and the largest subgroup of H with this property. If H is normal, then $N = H$; if G is a topological group, then N is open in G. All representations will act on a free module of finite rank over some ring or field.

2.1. Generalities

Definition 2.1.1. For a representation $\rho\colon N \to \mathrm{GL}_m(A)$ over a ring A and $g \in G$, we define the conjugate of ρ by g as the representation

$$\rho^g : N \longrightarrow \mathrm{GL}_m(A), \qquad n \longmapsto \rho(gng^{-1}).$$

Remark 2.1.2. Conjugation in the sense of Definition 2.1.1 defines an action of G on the set $\{[\rho^g] : g \in G\}$ of isomorphism classes $[\rho^g]$ of representations ρ^g of N. Since N acts trivially, the action factors via G/N and so, up to isomorphism, there are only finitely many conjugates of ρ.

For the remainder of this subsection, let E denote a field of characteristic $p \geq 0$. Unless said otherwise, any representation will be of finite dimension over E.

The following lemma will be used repeatedly.

Lemma 2.1.3 (Mackey's tensor product theorem for induced representations; [CR62, Corollary 44.4]). *Let ρ and ρ' be representations of G and of H, respectively. Then*

$$\rho \otimes \mathrm{Ind}_H^G \rho' \cong \mathrm{Ind}_H^G((\mathrm{Res}_H^G \rho) \otimes \rho').$$

We will also need:

Lemma 2.1.4. *Let ρ be a semisimple representation of H, and let $\nu = \mathrm{Res}_N^H \rho$. Assume for parts (f) to (h) that H is normal in G, and so in turn $N = H$ and $\nu = \rho$. Then the following hold:*

(a) *For any separable field extension $F \supset E$ the representation $\rho \otimes_E F$ is semisimple.*
(b) *One has $\mathrm{Res}_N^G \mathrm{Ind}_H^G \rho \cong \bigoplus_{g\in G/H} (\mathrm{Res}_N^H \rho)^g$.*
(c) *If τ is an irreducible representation of G, then $\mathrm{Res}_N^G \tau$ is semisimple, and all irreducible summands of $\mathrm{Res}_N^G \tau$ are conjugate to one another in the sense of Definition 2.1.1.*

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(d) *If* $[G:N]$ *is not a multiple of* p*, then* $\mathrm{Ind}_H^G\rho$ *is semisimple.*
(e) *If* ν *is irreducible and if* G/N *acts freely on* $\{[\nu^g] : g \in G\}$*, then* $\mathrm{Ind}_H^G\rho$ *is irreducible.*
(f) *The representation* $\mathrm{Ind}_H^G\rho$ *is absolutely irreducible if and only if* ρ *is absolutely irreducible and* G/H *acts freely on* $\{[\rho^g] : g \in G\}$*.*
(g) *Let* ρ' *be a second representation of* H*. Then* $\mathrm{Ind}_H^G\rho \cong \mathrm{Ind}_H^G\rho'$ *if and only if*

$$\bigoplus_{g\in G/H} \rho^g \cong \bigoplus_{g\in G/H} (\rho')^g. \tag{1}$$

(h) *If* ρ *is irreducible in (g), then the isomorphism in (1) is equivalent to* $\rho' \cong \rho^g$ *for some* $g \in G$*.*

Proof. We denote by V the E-vector space underlying ρ. Part (a) is [CR62, Corollary 69.8] with the ring A from there being the image of $E[H]$ in $\mathrm{End}_E(V)$. Part (b) holds by [Ser77, Proposition 22]. Part (c) follows from [CR62, Theorem 49.2]. For (d), note that ρ is a subrepresentation of $\mathrm{Ind}_N^H\mathrm{Res}_N^H\rho$, and hence $\mathrm{Ind}_H^G\rho$ is a subrepresentation of $\mathrm{Ind}_N^G\mathrm{Res}_N^H\rho$. By (c) applied to the irreducible summands of ρ, we see that $\mathrm{Res}_N^H\rho$ is semisimple. Now, the result can be found in [Web16, Chapter 5, Exercise 8].

To prove Part (e), let $V' \subset \mathrm{Ind}_H^G\rho$ be an irreducible G-subrepresentation. Then by (b) the representation $\mathrm{Res}_N^G V'$ contains ν^g for some $g \in G$. As V' is a G-representation, we deduce $\nu^g \subset \mathrm{Res}_N^G V'$ for all $g \in G$. By hypothesis, the $\nu^g, g \in G/N$, are pairwise nonisomorphic, and hence $\bigoplus_{g\in G/N} \nu^g \subset \mathrm{Res}_H^G V'$. By (b) the left-hand side is isomorphic to $\mathrm{Res}_N^G\mathrm{Ind}_H^G\rho$ so that for dimension reasons we must have $V' = \mathrm{Ind}_H^G\rho$.

We next prove Part (f). Because the solution space of a linear system of equations has the same dimension over its field of definition and over any extension, one has

$$\mathrm{Hom}_{E[N]}(\rho, \rho^g) \otimes_E E^{\mathrm{alg}} \cong \mathrm{Hom}_{E^{\mathrm{alg}}[N]}(\rho \otimes_E E^{\mathrm{alg}}, \rho^g \otimes_E E^{\mathrm{alg}}).$$

This allows one by base change $E \longrightarrow E^{\mathrm{alg}}$ to reduce one direction of (f) to (e). For the converse, assume that $\mathrm{Ind}_H^G\rho$ is absolutely irreducible. Because Ind_H^G is an exact functor, ρ must be absolutely irreducible and hence also ρ^g for all $g \in G$. Because $\mathrm{Ind}_H^G\rho$ is absolutely irreducible, Frobenius reciprocity yields

$$E \cong \mathrm{End}_{E[G]}(\mathrm{Ind}_H^G\rho) \cong \mathrm{Hom}_{E[G]}(\rho, \mathrm{Res}_H^G\mathrm{Ind}_H^G\rho) \overset{(c)}{=} \mathrm{Hom}_{E[G]}(\rho, \oplus_{g\in G/H}\rho^g).$$

Hence, ρ is isomorphic to ρ^g if and only if $g \in H$, and this completes the proof of (f).

We now prove Part (g). Note that by (b) the only if direction is clear. For the other direction, note first that by [CR81, Lemma 10.12] we have $\mathrm{Ind}_H^G\rho \cong \mathrm{Ind}_H^G\rho^g$ for all $g \in G$. Since induction and direct sum commute, we also have

$$\mathrm{Ind}_H^G\Big(\bigoplus_{g\in G/H} \rho^g\Big) = \bigoplus_{g\in G/H} (\mathrm{Ind}_H^G\rho^g) = \big(\mathrm{Ind}_H^G\rho\big)^{\oplus[G:H]}.$$

The same formula applies to ρ', and so our hypothesis gives $(\mathrm{Ind}_H^G\rho)^{\oplus[G:H]} \cong (\mathrm{Ind}_H^G\rho')^{\oplus[G:H]}$. The Krull–Schmidt theorem (see [CR62, Theorem 14.5]) now yields $\mathrm{Ind}_H^G\rho \cong \mathrm{Ind}_H^G\rho'$. Part (h) follows from the uniqueness of composition factors and the irreducibility of the ρ^g. □

2.2. *Some results when p does not divide* $[G:H]$

Suppose now that $\chi\colon G \longrightarrow E^\times$ is a character of finite order m so that E contains a primitive m-th root of unity ζ and $m \cdot 1 \in E^\times$. We also set $H := \ker\chi$ so that H is normal in G and note that $p \nmid m = [G:H]$. The following is a standard result of Clifford theory, for example, [CR62, Theorem 49.2, Corollary 50.6].

Theorem 2.2.1. *Let* $\rho\colon G \longrightarrow \mathrm{GL}_n(E)$ *be an absolutely irreducible representation such that* $\rho \cong \rho \otimes \chi$*. Then the following hold:*

(a) *The order m of χ divides the degree n of ρ.*
(b) *There exists a Kummer extension $E' = E(\sqrt[m]{\lambda})$ of E for some $\lambda \in E^\times$ and an absolutely irreducible representation $\rho' \colon H \to \mathrm{GL}_{n/m}(E')$ such that*

$$\rho \otimes_E E' \cong \mathrm{Ind}_H^G \rho'.$$

(c) *The representations $(\rho')^g$, $g \in G/H$, are pairwise nonisomorphic and absolutely irreducible, and one has $\mathrm{Res}_H^G \rho \otimes_E E' \cong \bigoplus_{g \in G/H} (\rho')^g$.*
(d) *If E is local field, G is a topological group and ρ is continuous, then so is ρ'.*
(e) *If in addition to (d), G is compact, then ρ can be defined over the ring of integers $\mathcal{O}_E$ of E and ρ' can be defined over $\mathcal{O}_{E'}$.*

Proof. Lacking a precise reference, we give a proof. Let A be an invertible $n \times n$-matrix over E such that

$$A\rho(g)A^{-1} = \chi(g)\rho(g) \text{ for all } g \in G. \tag{2}$$

From equation (2), one deduces $A^m \rho(g) A^{-m} = \chi^m(g)\rho(g) = \rho(g)$ for all $g \in G$. Since ρ is absolutely irreducible, [CR62, (29.13)] implies that $A^m = \lambda \cdot \mathbb{1}_n$ for some $\lambda \in E$. Define $E' := E(\sqrt[m]{\lambda})$. Let $A' = \sqrt[m]{\lambda}^{-1} A$ in $\mathrm{GL}_n(E')$ so that equation (2) also holds for A' and also $(A')^m = \mathbb{1}_n$. Since $m \cdot 1$ is invertible in E, it follows, using the Jordan form, that A' is semisimple. Moreover, A' is diagonalizable over E' since E contains a primitive m-th root of unity.

After a change of basis over E', we may write A as a block diagonal matrix with diagonal blocks $A_1, \ldots, A_m$ such that for $i = 1, \ldots, m$ each A_i is a scalar matrix $\zeta^i \mathbb{1}_{n_i}$ with $n_i \geq 0$ and $\sum_{i=1,\ldots,m} n_i = n$. For all $g \in G$ and $i, j = 1, \ldots, m$, we decompose $\rho(g)$ correspondingly into blocks $\rho_{i,j}(g)$ so that equation (2) turns into

$$\zeta^{i-j} \rho_{i,j}(g) = \chi(g)\rho_{i,j}(g). \tag{3}$$

Choose $g \in G$ such that $\chi(g) = \zeta$. Then $\rho_{i,j}(g)$ is zero unless $i - j \equiv 1 (\mathrm{mod}\ m)$. Since $\rho(g)$ is invertible, all $\rho_{i+1,i}(g)$ and $\rho_{m,1}(g)$ must be invertible and hence square matrices and of nonzero size. We deduce that all n_i are equal, hence nonzero, and hence equal to n/m. In particular, m divides n, proving (a).

Next, for $h \in H$ and for all $i, j = 1, \ldots, m$, equation (3) becomes $\zeta^{i-j}\rho_{i,j}(h) = \rho_{i,j}(h)$ so that $\rho(h) = \bigoplus_{i=1}^m \rho_{i,i}(h)$ is a block diagonal matrix and each $\rho_{i,i} \colon H \to \mathrm{GL}_{n/m}(k)$, $h \mapsto \rho_{i,i}(h)$, is a representation of dimension n/m. In particular, the restriction satisfies

$$\mathrm{Res}_H^G \rho \otimes_E E' = \bigoplus_{i=1}^m \rho_{i,i}.$$

We choose $\rho' = \rho_{1,1}$ and consider $\mathrm{Ind}_H^G \rho'$. By [CR62, (10.8) Frobenius Reciprocity Theorem], we have

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho', \rho \otimes_E E') = \mathrm{Hom}_H(\rho', \mathrm{Res}_H^G \rho \otimes_E E') \neq 0.$$

Let $f \colon \mathrm{Ind}_H^G \rho' \to \rho \otimes_E E'$ be a nonzero G-homomorphism. Since ρ is irreducible, it must be surjective, and because $\dim \rho = n = m \cdot n/m = \dim \mathrm{Ind}_H^G \rho'$, its kernel must be zero so that f is an isomorphism. Next, note that Ind_H^G is an exact functor; see [CR81, §10, Exercise 20]. Hence, ρ' is absolutely irreducible because ρ is so. This completes the proof of (b).

Part (c) follows from Lemma 2.1.4(b) and (f). Part (d) easily follows from the continuity of $\mathrm{Res}_H^G \rho \otimes_E E' \cong \bigoplus_{g \in G/H} (\rho')^g$, using that all linear topologies on a finite-dimensional vector space over E' that are compatible with the topology on E' are equivalent.

Concerning (e), we only prove the first assertion; the proof of the second then follows from (d). For this, let V be the E-vector space underlying ρ, and let T be an $\mathcal{O}_E$-lattice in V. The stabilizer of T is an open subgroup of $\mathrm{GL}_n(E)$ and hence, by the continuity of ρ, the latice T is fixed by an open subgroup G'

of G. Therefore, G/G' is finite. Thus, $T' := \bigcap_{g\in G/G'} gT$ is an $\mathcal{O}_E$-lattice in V, and this lattice is clearly G-stable. Choosing an $\mathcal{O}_E$-basis of T', that is then also an E-basis of V, assertion (e) for ρ is clear. □

Corollary 2.2.2. *Suppose that $\rho\colon G \to \mathrm{GL}_n(E)$ is a representation that is absolutely semisimple; this holds for instance if E is perfect. Then $\rho \cong \rho\otimes\chi$ holds if and only if there is a separable extension E' of E of degree less than $m^n\cdot(n^2)!$ and a representation $\rho'\colon H \to \mathrm{GL}_{n/m}(E')$ such that $\rho\otimes_E E' \cong \mathrm{Ind}_H^G\rho'$. Furthermore, any such ρ' is absolutely semisimple, and one has $\mathrm{Res}_H^G\rho\otimes_E E' = \bigoplus_{g\in G/H}(\rho')^g$.*

Proof. If $\rho\otimes_E E' \cong \mathrm{Ind}_H^G\rho'$, then Lemma 2.1.3 implies

$$(\rho\otimes_E E')\otimes\chi \cong (\mathrm{Ind}_H^G\rho')\otimes\chi \cong \mathrm{Ind}_H^G(\rho'\otimes\mathrm{Res}_H^G\chi) \cong \mathrm{Ind}_H^G\rho' \cong \rho\otimes_E E',$$

and this implies $\rho\otimes\chi\cong\rho$ by [CR62, 29.7].

Conversely, suppose that $\rho\cong\rho\otimes\chi$. After replacing E by a separable extension of degree at most $(n^2)!$ (see Lemma A.2.7 and Remark A.2.8) we may assume that ρ is an absolutely completely reducible G-representation over E', that is, $\rho = \oplus_{j\in J}\rho'_j$ for absolutely irreducible representations ρ'_j for $j\in J$. We regroup this decomposition according to orbits under iterated twisting by χ. This gives rise to a decomposition

$$\rho \cong \bigoplus_{i\in I}\Big(\bigoplus_{j=0}^{m_i-1}\rho_i\otimes\chi^j\Big)^{\oplus r_i}, \tag{4}$$

for integers $r_i>0$, absolutely irreducible representations $\rho_i\colon G\to\mathrm{GL}_{n_i}(E')$, and divisors m_i of m, for $i\in I$ so that $\rho_i\otimes\chi^{m_i}\cong\rho_i$, and no ρ_i is isomorphic to $\rho_{i'}\otimes\chi^j$ for some $j\in\{0,\dots,m_{i'}-1\}$ and $i'\in I$. We have $G\supset H_i := \ker\chi^{m_i}\supset H$, $[H_i:H]=m_i$, and $\mathrm{Res}_{H_i}^G\chi$ is a character of order m_i.

By Theorem 2.2.1, we find Kummer extensions E'_i of E' of degree dividing m_i and representations $\rho''_i\colon H_i\to\mathrm{GL}_{n_i/m_i}(E'_i)$ such that $\mathrm{Ind}_{H_i}^G\rho''_i\cong\rho_i\otimes_{E'}E'_i$. Let $\mathbb{1}_H$ be the trivial representation of H on E'. Then

$$\begin{aligned}\Big(\bigoplus_{j=0}^{m_i-1}\rho_i\otimes\chi^j\Big)\otimes_{E'}E'_i &\cong \mathrm{Ind}_{H_i}^G\rho''_i\otimes\Big(\bigoplus_{j=0}^{m_i-1}\chi^j\Big)\cong\mathrm{Ind}_{H_i}^G\Big(\rho''_i\otimes\bigoplus_{j=0}^{m_i-1}\mathrm{Res}_{H_i}^G\chi^j\Big)\\ &\cong\mathrm{Ind}_{H_i}^G\big(\rho''_i\otimes\mathrm{Ind}_H^{H_i}\mathbb{1}_H\big)\cong\mathrm{Ind}_{H_i}^G\mathrm{Ind}_H^{H_i}\big(\mathrm{Res}_H^{H_i}\rho''_i\otimes\mathbb{1}_H\big)\\ &\cong\mathrm{Ind}_H^G\big(\mathrm{Res}_H^{H_i}\rho''_i\big),\end{aligned}$$

where the second and fourth isomorphism follows from Lemma 2.1.3. Let E'' be the composite of the E'_i, and set $\rho' := \bigoplus_{i\in I}(\mathrm{Res}_H^{H_i}\rho''_i\otimes_{E'_i}E'')^{\oplus r_i}$ so that clearly $[E'':E']<m^n$. The first assertion of the corollary is now evident from the above and from (4). The remaining assertions follow from Lemma 2.1.4(b) and (d). □

2.3. *Some results when p divides* $[G:H]$

Suppose for the remainder of this subsection that $p=\mathrm{Char}\,E>0$. Let $V=E^n$, and let $\rho\colon G\to\mathrm{Aut}_E(V)$ be a representation such that the canonical map $E\to\mathrm{End}_G(V)$ is an isomorphism.

Lemma 2.3.1. *Suppose that ρ is absolutely irreducible. Let $H\subset G$ be a normal subgroup of index p, and set $V_H := \mathrm{Res}_H^G\rho\otimes_E E^{\mathrm{alg}}$. Then the following hold:*

(a) *If V_H is reducible, then $V\otimes_E E^{\mathrm{alg}}\cong\mathrm{Ind}_H^G W$ for any irreducible submodule $W\subset V_H$.*
(b) *If V_H is irreducible, then we have:*

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(1) *Any* $E[G]$*-module* W *with* $\mathrm{Res}^G_H W \cong \mathrm{Res}^G_H V$ *is isomorphic to* V*.*
(2) $\mathrm{Ind}^G_H V_H$ *is indecomposable, its socle is isomorphic to* $V \otimes_E E^{\mathrm{alg}}$*,*
(3) $V \otimes_E E^{\mathrm{alg}}$ *is not induced from any* H*-module,*
(4) *All irreducible subquotients of* $\mathrm{Ind}^G_H V_H$ *are isomorphic to* $V \otimes_E E^{\mathrm{alg}}$*.*

Proof. By Lemma 2.1.4(c), we have $V_H = \oplus_{g\in G/H^*} W^g$ for some irreducible H-module W over E^{alg} and some subgroup $H^* \subset G$ with $H \subset H^*$. Since $G/H \cong \mathbb{Z}/p\mathbb{Z}$, in Part (a) of the present lemma, we must have $H^* = H$, and then the assertion follows from Lemma 2.1.4(e).

We now prove (b). Let W be as in (1). By choosing the same underlying E vector space, we assume that $\mathrm{Res}^G_H W = \mathrm{Res}^G_H V$. Let $g \in G$ be a generator of G/H, and let $A, B \in \mathrm{Aut}_E(\mathrm{Res}^G_H V)$ be the automorphisms given by the action of g on W and V, respectively. Because $\mathrm{Res}^G_H V$ is absolutely irreducible, there exists a nonzero scalar $\lambda \in E$ such that $B = \lambda A$. As $g^p \in H$, we find $A^p = B^p = \lambda^p A^p$. Because $\mathrm{Char}\, E = p > 0$, we must have $\lambda = 1$, and so (1) is proved.

For (2), write $\mathrm{Ind}^G_H V_H = \bigoplus_{i\in I} W_i$ with indecomposable G-modules W_i. Let W_i' be an irreducible quotient of W_i as a G-submodule. From Lemma 2.1.4(b) and (c) and the irreducibility of V_H, we deduce $\mathrm{Res}^G_H W_i' \cong V_H$ for all i, and by (1), we find $W_i' \cong V \otimes_E E^{\mathrm{alg}}$. The following inequality implies $\#I = 1$ and the uniqueness of W_1' and thus gives (2):

$$\begin{aligned}\#I \le \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_G\Big(\bigoplus_i W_i, V \otimes_E E^{\mathrm{alg}}\Big) &= \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_G(\mathrm{Ind}^G_H V_H, V \otimes_E E^{\mathrm{alg}}) \\ = \dim_{E^{\mathrm{alg}}} \mathrm{Hom}_H(V_H, V_H) &= 1.\end{aligned}$$

To see (3), observe that if $V \otimes_E E^{\mathrm{alg}}$ was induced, then by Lemma 2.1.4(b) then V_H had to be reducible. For (4), note that we have $\mathrm{Ind}^G_H V_H \cong (V \otimes_E E^{\mathrm{alg}}) \otimes_E \mathrm{Ind}^G_H E$. Now, clearly the semisimplification of $\mathrm{Ind}^G_H E$ is the trivial module E^p, and this shows (4). □

For the remainder of this subsection, we shall also assume that E is a topological field, that ρ is continuous and that G is topologically finitely generated, and we let $\Phi(G) = G^p\overline{[G,G]}$ so that $G/\Phi(G)$ is the maximal p-elementary abelian Hausdorff quotient of G, and we set $m := \dim_{\mathbb{F}_p} G/\Phi(G)$. We note that the hypothesis on G holds for $G = G_K$ with K a p-adic field by [Jan83, Satz 3.6].

In the sequel, we shall write $\overline{\mathrm{End}}(V)$ for the cokernel of the natural inclusion $E \hookrightarrow \mathrm{End}_E(V)$. We shall relate the nonvanishing of the module of G-invariants $\overline{\mathrm{End}}_G(V)$ of this cokernel to V being induced from a subgroup of G of p-power index. We **assume** that p divides n, since otherwise the trace splits the inclusion $E \to \mathrm{End}_E(V)$ G-equivariantly so that $\overline{\mathrm{End}}_G(V) = 0$ by our hypothesis $E = \mathrm{End}_G(V)$.

Let $\mathrm{End}'_G(V)$ be the subset of $A \in \mathrm{End}_E(V)$ such that there exists a map $\lambda_A \colon G \to E, g \mapsto \lambda_A(g)$ with

$$\forall g \in G\colon \rho(g)A\rho(g)^{-1} = A + \lambda_A(g)1_n. \tag{5}$$

Again, because $E = \mathrm{End}_G(V)$, one has the short exact sequence

$$0 \longrightarrow E = \mathrm{End}_G(V) \longrightarrow \mathrm{End}'_G(V) \longrightarrow \overline{\mathrm{End}}_G(V) \longrightarrow 0. \tag{6}$$

We write $\overline{A} \in \overline{\mathrm{End}}_G(V)$ for the class of $A \in \mathrm{End}'_G(V)$ under this map. Recall that $f \in E[T]$ is p-linear if $f = \sum_i a_i T^{p^i}$ and that the set $E[T]^{\text{p-lin}}$ of p-linear polynomials in $E[T]$ is a ring under addition and composition. The following lemma provides some basic properties of $\mathrm{End}'_G(V)$.

Lemma 2.3.2. *Let* A *be in* $\mathrm{End}'_G(V)$*. For* $\lambda \in E^{\mathrm{alg}}$*, let* V_λ *and* V'_λ *denote the eigenspace and generalized eigenspace of* A *for* λ*. Suppose from Part (i) on that* ρ *is absolutely irreducible.*

(a) *Each* λ_A *is a continuous homomorphism* $G \to (E, +)$*.*
(b) *The groups* $H_A := \mathrm{Ker}\,\lambda_A$ *and* $H_\rho := \bigcap\{H_A \mid A \in \mathrm{End}'_G(V)\}$ *contain* $\Phi(G)$*.*
(c) $\Lambda_A := \lambda_A(G) \subset (E, +)$ *is a finite-dimensional* $\mathbb{F}_p$ *vector space.*
(d) *The multiset of eigenvalues of* A *with multiplicities is a torsor under* $\Lambda_A := \lambda_A(G)$*.*

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(e) *The map* $\lambda_\bullet\colon A \mapsto \lambda_A$ *factors via an injective homomorphism*

$$\overline{\mathrm{End}}_G(V) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(G, (E, +)), \overline{A} \mapsto \lambda_{\overline{A}}.$$

(f) $\mathrm{End}'_G(V)$ *is a module for* $E[T]^{\text{p-lin}}$ *under* $(f, A) \mapsto f(A)$ *and one has* $\lambda_{f(A)} = f \circ \lambda_A$.
(g) *If* $\overline{\mathrm{End}}_G(V) \neq 0$, *then there exist* $A \in \mathrm{End}'_G(V)$ *such that* $\Lambda_A \cong (\mathbb{F}_p, +)$ *and* $[G : H_A] = p$.
(h) *The restriction* $\rho|_{H_A}$ *commutes with A; it preserves* V_λ *and* V'_λ *for all* $\lambda \in E^{\mathrm{alg}}$.
(i) *A is semisimple over* E^{alg}.
(j) *One has* $\rho \otimes_E E^{\mathrm{alg}} \cong \mathrm{Ind}_{H_A}^{G_K} V_\lambda$ *for any eigenvalue* $\lambda \in E^{\mathrm{alg}}$ *of A.*
(k) *Over* E^{alg} *the set* $\mathrm{End}'_{G_K}(V)$ *is simultaneously diagonalizable.*
(l) *Suppose* $\dim_{\mathbb{F}_p} E \geq m$, *then there exists* $A \in \mathrm{End}'_G(V)$ *with* $H_\rho = H_A$.

Proof. (a) The continuity of λ_A follows from that of ρ. It is a homomorphism because of

$$A + \lambda_A(gh)1_n = ghAh^{-1}g^{-1} = g(A + \lambda_A(h)1_n)g^{-1} = gAg^{-1} + \lambda_A(h)1_n = A + (\lambda_A(g) + \lambda_A(h))1_n.$$

To see (b) note that the image of λ_A is p-elementary abelian because $p \cdot 1 = 0$ in E. By (a) $H_A = \mathrm{Ker}\,\lambda_A \supseteq \Phi(G)$, and hence $H_\rho \supseteq \Phi(G)$. Part (c) is clear from (a) and (b) since by assumption, G is topologically finitely generated and hence so is $G/\Phi(G)$. For Part (d), denote by $\chi_A(T) \in E[T]$ the characteristic polynomial of A. Then equation (5) implies $\chi_A(T) = \chi_A(T + \lambda_A(g))$ for all $g \in G_K$, and this proves (d).

For (e), one easily verifies that $\lambda_\bullet$ is the boundary map of cohomology $H^0(G, \overline{\mathrm{End}}_E(V)) \longrightarrow H^1(G, E)$ induced from the sequence (6); Part (a) shows that the target module is $\mathrm{Hom}_{\mathrm{cont}}(G, (E, +))$; the cocyle condition is easily verified. Moreover, λ_A is trivial if and only if $A \in \mathrm{End}_G(V)$. Hence, $\overline{A} \to \lambda_{\overline{A}}$ is defined and injective. The homomorphism property is straightforward.

(f) Raising equation (5) to the power p and using $\mathrm{Char}(E) = p$ we find

$$\forall g \in G_K : \rho(g)A^p\rho(g)^{-1} = A^p + \lambda_A(g)^p 1_n.$$

Since $\mathrm{End}'_G(V)$ is clearly an E-vector space and $\lambda_\bullet$ is E-linear, Part (f) follows. To see (g), let A be in $\mathrm{End}'_G(V) \backslash E$ so that $\Lambda_A \subset (E, +)$ is nontrivial and finite. Let $\Lambda \subset \Lambda_A$ be a sub $\mathbb{F}_p$-vector space of codimension 1, and let f be the p-linear polynomial $\prod_{\lambda \in \Lambda}(T - \lambda)$. Then $\Lambda_{f(A)}$ has order p by (f), and H_A has index p by its definition in (b).

In (h), the asserted commutativity is clear from equation (5); the assertion on the V_λ and V'_λ is then immediate. For (i), choose an eigenvalue $\lambda \in E^{\mathrm{alg}}$ for which $\dim V_\lambda$ is minimal. By (c), we have $\dim V_\lambda \cdot \#\Lambda_A \leq n$ with equality if and only if A is semisimple. Because A and $\rho|_{H_A} \otimes_E E^{\mathrm{alg}}$ commute, the action of H_A preserves V_λ. Let $V_{E^{\mathrm{alg}}} := V \otimes_E E^{\mathrm{alg}}$. Frobenius reciprocity gives a nonzero homomorphism in

$$\mathrm{Hom}_G(\mathrm{Ind}_{H_A}^G V_\lambda, V_{E^{\mathrm{alg}}}) \cong \mathrm{Hom}_{H_A}(V_\lambda, V_{E^{\mathrm{alg}}}|_{H_A}).$$

Since $V_{E^{\mathrm{alg}}}$ is an irreducible G_K-representation, it follows that

$$n \leq \dim \mathrm{Ind}_{H_A}^G V_\lambda = [G : H_A] \cdot \dim V_\lambda = \#\Lambda_A \cdot \dim V_\lambda \leq n.$$

Hence, we must have equality and so A is semisimple. For (j), all but the last assertion follow from the proof for (i).

(k) Using equation (5), we compute for $A, B \in \mathrm{End}'_{G_K}(V)$ and all $g \in G_K$ that

$$g(AB - BA)g^{-1} = (A + \lambda_A(g)1_n)(B + \lambda_B(g)1_n) - (B + \lambda_B(g)1_n)(A + \lambda_A(g)1_n) = (AB - BA).$$

Since $E = \mathrm{End}_G(V)$, we conclude that $AB - BA$ is a scalar matrix. We also know that A (and B) is semisimple. To conclude, we may work over E^{alg} so that we may assume that A has diagonal form.

But then it is elementary to see that $AB - BA$ has entries 0 along the diagonal and hence this scalar matrix must be zero. It follows that any $A, B \in \mathrm{End}'_G(V)$ commute, and we conclude using (j).

(l) We need to show that for all $A, B \in \mathrm{End}'_G(V)$ there exist $\mu, \nu \in E\backslash\{0\}$ such that $H_{\mu A+\nu B} = H_A \cap H_B$. Let W be the $\mathbb{F}_p$-vector space $G/(H_A \cap H_B)$, and regard λ_A and λ_B as $\mathbb{F}_p$-linear maps $W \longrightarrow E$. Note that $d := \dim_{\mathbb{F}_p} W \leq m$. Let $\underline{B} := (b_1, \ldots, b_d)$ be an $\mathbb{F}_p$ basis of W. Suppose also without loss of generality that $\dim_{\mathbb{F}_p} G_K/H_A, \dim_{\mathbb{F}_p} G_K/H_B < d$ since otherwise we are done.

For $\nu \in E^{\mathrm{alg}}$ set $C_\nu := \lambda_A + \nu\lambda_B$. Since the common kernel of λ_A and λ_B is $0 \subset W$, there exists $\nu \in E^{\mathrm{alg}}$ such that C_ν is injective, that is, such that the vectors $(C_\nu b_i)_{i=1,\ldots,d}$ are $\mathbb{F}_p$-linearly independent in E. This means that the Moore determinant of these vectors is nonzero. In other words, the determinant of the $d \times d$-square matrix with (i, j)-entry given by $(\lambda_A(b_i) + \nu\lambda_B(b_i))^{p^{j-1}}$ is nonzero. As a function of ν, this is a polynomial of degree at most $(p^d - 1)/(p - 1)$. It is not identically zero because of its value at ν, and hence it can have at most $p^d - 1 < p^m - 1$ zeros. It follows that for some $\nu' \in E$ it is nonzero because $\#E \geq p^m$, and this completes the proof of (l). □

We will later also need the following particular result:

Corollary 2.3.3. *Suppose ρ is a nontrivial extension of an absolutely irreducible representation ρ_2 by an absolutely irreducible representation ρ_1. Suppose further that ρ_1 and ρ_2 are not isomorphic and that the ρ_i are not induced from any normal index p subgroup of G. Then $\overline{\mathrm{End}}_G(V) = 0$.*

Proof. Assume on the contrary that we can find $A \in \mathrm{End}'_G(V)\backslash E$. We may assume that Λ_A has order p by Lemma 2.3.2(g), and we also may assume $E = E^{\mathrm{alg}}$ since this leaves $\dim_E \overline{\mathrm{End}}_G(V)$ unchanged. Then H_A has index p in G and we have p distinct subspaces V_λ of $V_{E^{\mathrm{alg}}}$ that are stabilized by H_A. By hypotheses and Lemma 2.3.1, the restrictions $\rho_i|_{H_A}$ are absolutely irreducible. This already implies $p = 2$. We also find that the extension of ρ_2 by ρ_1 becomes trivial when restricted to H_A, and that these restrictions must agree with the two distinct V_λ. It follows by Frobenius reciprocity that we have a nonzero map $\mathrm{Ind}_{H_A}^G(\rho_2|_{H_A}) \longrightarrow \rho$ for $i = 1, 2$. By Lemma 2.3.1(b)(4) all simple subquotients of $\mathrm{Ind}_{H_A}^G(\rho_2|_{H_A})$ are isomorphic to ρ_2. But this is absurd, since ρ is nonsplit and ρ_2 is not a submodule of ρ. We reach a contradiction even for $p = 2$. □

3. Deformations of Galois representation

This section recalls and augments the classical deformation theory [Maz89] of Mazur. Throughout we fix a profinite group G which often is G_K for K a p-adic field.

In Subsection 3.1, we fix the basic categories relevant for all deformations functors that we shall study. We also recall some results on formal smoothness. Subsection 3.2 recalls Mazur's deformation theory and some extensions for residual representations $G \longrightarrow \mathrm{GL}_n(\kappa)$ where κ is a finite or a local field. Subsection 3.3 studies dimension 1 points on universal deformation rings. Except for some results on equicharacteristic dimension 1 points, all results are well documented in the literature. Subsection 3.4 gives a criterion for the determinant functor to be smooth. For this, we recall Tate local duality for coefficient modules over local fields.

3.1. Basic categories and functors and formal smoothness

The field κ and the ring Λ: From now on, κ is either (a) a finite field of characteristic p that carries the discrete topology, or (b) a local field with its natural topology and with residue characteristic p. In the latter case, κ is a finite extension of $\mathbb{Q}_p$ or a finite extension of the formal Laurent series field $\mathbb{F}_p((t))$. Depending on case (a) or (b), we define a topological ring Λ. In case (b), we set $\Lambda = \kappa$. In case (a), Λ is a Noetherian complete local ring with residue field κ and equipped with the topology defined by its maximal ideal $\mathfrak{m}_\Lambda$.

The categories $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$: By $\mathcal{A}r_\Lambda$, we denote the category of Artinian local Λ-algebras A with residue field isomorphic to κ and with local Λ-algebra homomorphisms as morphisms. For any local ring A, we denote by $\mathfrak{m}_A$ its maximal ideal. We regard any object A of $\mathcal{A}r_\Lambda$ as a topological ring. In case (a), we give A the discrete topology. In case (b), the ring A is a κ-algebra of finite κ-dimension and we give A the unique topology that arises from any structure on A as a normed κ-vector space. This topology on A is relevant whenever we talk about continuous maps to A or to any $\mathrm{GL}_n(A)$. We further define $\widehat{\mathcal{A}r}_\Lambda$ as the category of complete Noetherian local Λ-algebras with residue field κ and with local homomorphisms as morphisms. Any object of $\widehat{\mathcal{A}r}_\Lambda$ is a limit of objects of $\mathcal{A}r_\Lambda$. We equip an object A of $\widehat{\mathcal{A}r}_\Lambda$ with the weakest topology such that all maps $A \to A/\mathfrak{m}_A^m$, $m \geq 1$, are continuous. In case (a), this simply means that A carries the $\mathfrak{m}_A$-adic topology.

In $\mathcal{A}r_\Lambda$, the coproduct of two objects A, A' is their tensor product $A \otimes_\Lambda A'$. For $A, A' \in \widehat{\mathcal{A}r}_\Lambda$, the coproduct is the completed tensor product $A \widehat{\otimes}_\Lambda A' := \varprojlim_n A/\mathfrak{m}_A^n \otimes_\Lambda A'/\mathfrak{m}_{A'}^n$. Note that by [Gro64, Lemma 0_{IV}.(19.7.1.2)] the ring $A \widehat{\otimes}_\Lambda A'$ lies again in $\widehat{\mathcal{A}r}_\Lambda$. From the discussion around the Cohen structure theorem in [Sta18, §0323], one also deduces:

Proposition 3.1.1. *Let $A \in \widehat{\mathcal{A}r}_\Lambda$ and $h := \dim_\kappa \mathfrak{m}_A/(\mathfrak{m}_\Lambda, \mathfrak{m}_A^2)$. Then there exists a surjective continuous homomorphism in $\widehat{\mathcal{A}r}_\Lambda$ from the power series ring $\Lambda[[x_1, \ldots, x_h]]$ onto A. Moreover, h is minimal with this property.*

Further properties of $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$ can be found in [Sta18, §06GB] and [Sta18, §06GV].

Functors on $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$: We follow [Sch68]; see also [Sta18, Chapter 06G7]. By $\kappa[\varepsilon] := \kappa[X]/(X^2) \in \mathcal{A}r_\Lambda$, we denote the *ring of dual numbers over κ*. Recall from [Sch68] that a *small extension in $\mathcal{A}r_\Lambda$* is a surjection $f\colon B \to A$ in $\mathcal{A}r_\Lambda$ whose kernel $\ker f$ is isomorphic to κ as a B-module, and in particular $\ker f$ is annihilated by $\mathfrak{m}_B$ and thus $(\ker f)^2 = 0$.

In the following, we consider covariant functors F from $\mathcal{A}r_\Lambda$ or $\widehat{\mathcal{A}r}_\Lambda$ to *Sets* such that $F(\kappa)$ is a singleton.

Definition 3.1.2. A covariant functor $F\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ is called *continuous* if the canonical map $F(A) \to \varprojlim_n F(A/\mathfrak{m}_A^i)$ is bijective for all $A \in \widehat{\mathcal{A}r}_\Lambda$.

It is straightforward to see that there is a bijection between continuous functors $\widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ and functors $\mathcal{A}r_\Lambda \to \mathit{Sets}$ given by restriction. From now on, all functors on $\widehat{\mathcal{A}r}_\Lambda$ will be continuous and we use the same symbol to denote them and their restriction to $\mathcal{A}r_\Lambda$. For any $B \in \widehat{\mathcal{A}r}_\Lambda$, we denote by $h_B\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ the functor which is given by $h_B(A) := \mathrm{Hom}_{\widehat{\mathcal{A}r}_\Lambda}(B, A)$. A functor $F\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ is representable if it is isomorphic to h_B for some $B \in \widehat{\mathcal{A}r}_\Lambda$.

Definition 3.1.3 [Sch68, Definitions 2.2 – 2.7]. Let $F, F'\colon \widehat{\mathcal{A}r}_\Lambda \to \mathit{Sets}$ be functors.

(a) The *tangent space of F* is $t_F := F(\kappa[\varepsilon])$.
(b) A natural transformation $F' \to F$ is called *smooth* if for all small extensions $B \to A$ in $\mathcal{A}r_\Lambda$, the map $F'(B) \to F'(A) \times_{F(A)} F(B)$ is surjective; cf. [Sch68, Definition 2.2].
(c) A pair (A, ξ) consisting of an object A in $\widehat{\mathcal{A}r}_\Lambda$ and a smooth natural transformation $\xi\colon h_A \to F$ is called a *hull of F* if the induced map $t_{h_A} \to t_F$ on tangent spaces is bijective; note that by Yoneda ξ corresponds to some element of $F(A)$.

Hulls are unique up to isomorphism but in general not up to unique isomorphism. If F is representable by some $A \in \widehat{\mathcal{A}r}_\Lambda$, it clearly has a hull. Moreover, one has $t_F \cong \mathrm{Hom}_\kappa(\mathfrak{m}_A/(\mathfrak{m}_A^2, \mathfrak{m}_\Lambda), \kappa)$.

Definition 3.1.4 (Formal smoothness). We recall two notions of formal smoothness

(a) A homomorphism $R_1 \to R_2$ of topological rings with R_1 and R_2 linearly topologized is called *formally smooth* if for every commutative solid diagram

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$$\begin{tikzcd} R_2 \arrow[r] & A \\ R_1 \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

of homomorphisms of topological rings with B a discrete ring and $B \to A$ surjective with square zero kernel, a dotted arrow exists which makes the diagram commute, cf. [Sta18, Definition 07EB]

(b) A morphism $\varphi\colon Y \to X$ of locally Noetherian schemes is called *formally smooth* at $y \in Y$, if the induced morphism $\widehat{\mathcal{O}}_{X,\varphi(x)} \to \widehat{\mathcal{O}}_{Y,y}$ of topological rings is formally smooth.

Formal smoothness is related to smoothness of natural transformations between representable functors:

Proposition 3.1.5 [Sch68, Proposition 2.5(i)]. *Let $R_1 \to R_2$ be a morphism in $\widehat{\mathcal{A}r}_\Lambda$, and set $h =$* $\dim R_2 - \dim R_1$. *Then the following assertions are equivalent:*

(i) *$R_1 \to R_2$ is formally smooth.*
(ii) *The induced map of functors $h_{R_2} \to h_{R_1}$ is smooth.*
(iii) *There is an isomorphism $R_1[[x_1, \dots, x_h]] \to R_2$ of R_1-algebras.*

If any of (i)–(iii) holds, then h is called the relative dimension of R_2 over R_1.

Note that (ii)⇒(iii) is from [Sch68] and that (iii)⇒(i)⇒(ii) are straightforward. A consequence of (ii)⇒(i) of Proposition 3.1.5 is that a morphism in $\widehat{\mathcal{A}r}_\Lambda$ is formally smooth if the lifting property in Definition 3.1.4(a) holds for all small extensions in $\mathcal{A}r_\Lambda$.

3.2. Mazur's deformation theory and extensions

Our presentation of deformation functors follows Mazur [Maz89] and Kisin [Kis09]. Consider a continuous representation

$$\overline{\rho}\colon G \to \mathrm{GL}_n(\kappa). \tag{7}$$

We write $\mathrm{ad}_{\bar\rho}$ for $\mathrm{Mat}_{n\times n}(\kappa)$ together with the action of G induced by $\bar\rho$ composed with the conjugation action of $\mathrm{GL}_n(\kappa)$ on $\mathrm{Mat}_{n\times n}(\kappa)$. By $\mathrm{ad}^0_{\bar\rho}$, we denote the subrepresentation on trace zero matrices and by $\overline{\mathrm{ad}}_{\bar\rho}$ the quotient modulo the center κ. In the following, for a representation ρ into $\mathrm{GL}_n(A_1)$ and a ring homomorphism $A_1 \to A_2$ we write $\rho \otimes_{A_1} A_2$ for the composition of ρ with $\mathrm{GL}_n(A_1) \to \mathrm{GL}_n(A_2)$, cf. [Kis03, p. 433].

Definition 3.2.1. [Gou01, Definitions 2.1 and 2.2]. Let A be in $\mathcal{A}r_\Lambda$ with residue map $A \to \kappa$.

(a) A *lifting of $\overline{\rho}$ to A* is a continuous homomorphism $\rho\colon G \to \mathrm{GL}_n(A)$ with $\rho \otimes_A \kappa = \overline{\rho}$.
(b) The symbol $\Gamma_n(A)$ denotes the kernel of the canonical homomorphism $\mathrm{GL}_n(A) \to \mathrm{GL}_n(\kappa)$.
(c) A *deformation of $\overline{\rho}$ to A* is a $\Gamma_n(A)$-conjugacy class of a liftings of $\bar\rho$ to A.
(d) The *deformation functor $\mathcal{D}_{\bar\rho}$, or $\mathcal{D}_{\Lambda,\bar\rho}$ if we wish to indicate Λ, of $\overline{\rho}$* is defined as

$$\mathcal{D}_{\overline{\rho}}\colon \mathcal{A}r_\Lambda \longrightarrow Sets, \quad A \longmapsto \{\rho\colon G \longrightarrow \mathrm{GL}_n(A) : \rho \text{ is a deformation of } \overline{\rho}\},$$

In the following, for a profinite group H and a continuous H-module M we denote by $H^i(H, M)$ the i-th continuous group cohomology of H with coefficients in M. If M is discrete, details can be found in [NSW00, Chapter 1]. For other coefficients we refer to the introduction of Subsection 3.4.

The next definition is important in relation to the finiteness of $\mathcal{D}_{\bar\rho}(\mathbb{F}[\varepsilon])$.

Definition 3.2.2. The following finiteness conditions go back to [Maz89, §1.1].

(a) A profinite group G has property Φ_p if $H^1(H, \mathbb{F}_p)$ is finite for all open subgroups H of G.
(b) The representation $\overline{\rho}$ satisfies condition $\Phi_{\overline{\rho}}$ if $\dim_\kappa H^1(G, \mathrm{ad}_{\overline{\rho}})$ is finite.

Proposition 3.2.3. *The following assertions hold:*

(a) *The profinite group G_K satisfies Mazur's condition Φ_p.*
(b) *If a profinite group G satisfies Φ_p, then $\Phi_{\overline{\rho}}$ holds for any residual representation $\overline{\rho}$ of G.*

Proof. Part (a) is immediate from class field theory. If κ is finite, then Part (b) is well-known; to deduce it one applies the inflation restriction sequence to $G \supset H := \operatorname{Ker} \operatorname{ad}_{\overline{\rho}}$. If κ is a local field, the assertion is proved later in Corollary 3.3.6. We invite the reader to check that there is no circular reasoning involved. □

The versal hull of $\mathcal{D}_{\bar{\rho}}$: Let $\bar{\rho}$ be as in (7). The following result for finite κ is due to Mazur, with an extension due to Ramakrishna. For κ a p-adic field, a proof is given in [Kis03, Lemma 9.3] by Kisin. The proof for local fields of positive characteristic is analogous.

Theorem 3.2.4 [Maz89, 1.1–1.6], [Gou01, Theorem 3.3, p. 53, Theorem 4.2]. *Assuming condition $\Phi_{\overline{\rho}}$, the following hold:*

(a) *One has $t_{\mathcal{D}_{\bar{\rho}}} \cong H^1(G, \operatorname{ad}_{\bar{\rho}})$, and $h := \dim_\kappa H^1(G, \operatorname{ad}_{\bar{\rho}})$ is finite.*
(b) *The functor $\mathcal{D}_{\overline{\rho}}$ has a hull; we write $\rho^{\mathrm{ver}}_{\overline{\rho}} \colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{\Lambda,\overline{\rho}})$ for a representative of its versal deformation and $R^{\mathrm{ver}}_{\Lambda,\bar{\rho}} \in \widehat{\mathcal{A}r}_\Lambda$ for a versal deformation ring of $\bar{\rho}$.*
(c) *If $\kappa = \mathrm{End}_\kappa \overline{\rho}$, then $\mathcal{D}_{\overline{\rho}}$ is representable; we write $\rho^{\mathrm{univ}}_{\overline{\rho}} \colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{univ}}_{\Lambda,\overline{\rho}})$ for a representative of its universal deformation and $R^{\mathrm{univ}}_{\Lambda,\bar{\rho}} \in \widehat{\mathcal{A}r}_\Lambda$ for the universal deformation ring of $\bar{\rho}$.*
(d) *There is a surjection $\varphi \colon \Lambda[[x_1, \dots, x_h]] \to R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$ in $\widehat{\mathcal{A}r}_\Lambda$ such that $\operatorname{Ker}\varphi$ is generated by at most $\dim_\kappa H^2(G, \operatorname{ad}_{\bar{\rho}})$ elements.*
(e) *If $H^2(G, \operatorname{ad}_{\bar{\rho}}) = 0$, then the natural map $\Lambda \to R^{\mathrm{ver}}_{\Lambda,\bar{\rho}}$ is formally smooth of relative dimension h, or equivalently, the natural transformation $\mathcal{D}_{\bar{\rho}} \to h_\Lambda$, $[\rho \colon G \to \mathrm{GL}_n(A)] \mapsto A$ is smooth.*

Remark 3.2.5. The existence of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ (and of $\rho^{\mathrm{ver}}_{\overline{\rho}}$) as a profinite topological ring does not require condition $\Phi_{\overline{\rho}}$. The latter is needed for the rings to be Noetherian, that is, to lie in $\widehat{\mathcal{A}r}_\Lambda$.

We shall later need to understand the change of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}$ under maps $\Lambda \to \Lambda'$:

Lemma 3.2.6 (Cf. [Wil95, p. 457]). *Let $\Lambda \to \Lambda'$ be a finite injective homomorphism of complete Noetherian local rings with finite residue fields κ and κ', respectively. Let $\overline{\rho}' := \overline{\rho} \otimes_\kappa \kappa'$. Let R_Λ be a hull for $\mathcal{D}_{\Lambda,\overline{\rho}}$ and $R_{\Lambda'}$ for $\mathcal{D}_{\Lambda',\rho'}$. Then $R_{\Lambda'} \cong R_\Lambda \otimes_\Lambda \Lambda'$.*

3.3. Deformation rings at dimension 1 points

Suppose κ is finite. Then for R in $\widehat{\mathcal{A}r}_\Lambda$ and we call $x \in \operatorname{Spec} R$ with corresponding prime ideal $\mathfrak{p}_x \subset R$ a *point of dimension* 1 if $R/\mathfrak{p}_x$ has Krull dimension 1. For such x the field $\kappa(x)$ is either a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$. It has been first exploited by Kisin, for example, [Kis03], that points x of dimension 1 with $\kappa(x) \supset \mathbb{Q}_p$ on universal deformation rings can be much easier to understand than the closed point $\operatorname{Spec}\mathbb{F}$. We recall this method and work it out further for dimension 1 points x with $\kappa(x) \supset \mathbb{F}_p((t))$. The latter points will be an essential tool in our work.

Let κ be a finite field of characteristic p with the discrete topology. Let Λ be the ring of integers of a finite totally ramified extension L of $W(\kappa)[1/p]$. Let $\bar{\rho} \colon G \to \mathrm{GL}_n(\kappa)$ be a continuous representation that satisfies $\Phi_{\overline{\rho}}$ with versal deformation

$$\rho^{\mathrm{ver}}_{\overline{\rho}} \colon G \longrightarrow \mathrm{GL}_n(R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}).$$

Consider a continuous homomorphism $f \colon R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \to E$ of Λ-algebras for a local field E. Denote by $\mathfrak{p}$ the prime ideal $\operatorname{Ker} f$. Then via f the field E is a finite extension of the fraction field E_f of $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}}/\mathfrak{p}$, and $\rho^{\mathrm{ver}}_{\overline{\rho}}$ induces a representation $\rho_E \colon G \to \mathrm{GL}_n(E)$. We may and will assume that $E = E_f$ and that $\rho_E(G) \subset \mathrm{GL}_n(\mathcal{O})$ for $\mathcal{O}$ the ring of integers of E – the latter by using strict equivalence.

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Suppose first that E is of characteristic 0, in which case we follow [Kis03, §9]. Then f factors via a map $f[1/p]\colon R^{\mathrm{ver}}_{\Lambda,\bar\rho}[1/p] \to E$ which is an L-algebra homomorphism, and E is a finite extension field of L. We denote by $\widehat{R}$ the completion of $R^{\mathrm{ver}}_{\Lambda,\bar\rho}[1/p]$ at the kernel of $f[1/p]$. Then E is the residue field of $\widehat{R}$. From the finiteness of $L \to E$, one easily deduces that in fact $\widehat{R}$ is naturally a E-algebra. Moreover, we have a continuous homomorphism $\widehat{\rho}\colon G \to \mathrm{GL}_n(\widehat{R})$ induced from $\rho^{\mathrm{ver}}_{\bar\rho}$. Clearly, $\widehat{\rho}$ is a deformation of ρ_E. Using Remark 3.2.5, this provides one with a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{E,\rho_E} \longrightarrow \widehat{R}.$$

Suppose now that E is of characteristic p. Then E is isomorphic to a Laurent series field $\kappa'((x))$ for a finite extension κ' of the finite field κ and with ring of integers $\mathcal{O} \cong \kappa'[[x]]$. Denote by $\rho_{\mathcal{O}}$ the map ρ_E with the range restricted to $\mathrm{GL}_n(\mathcal{O})$. It is a deformation of $\bar\rho' := \bar\rho \otimes_\kappa \kappa'$. Let $\Lambda' = \Lambda \otimes_{W(\kappa)} W(\kappa')$, and consider the map

$$f_E\colon R^{\mathrm{ver}}_{\Lambda',\bar\rho'} \otimes_{\Lambda'} E \overset{\text{Lemma 3.2.6}}{\cong} R^{\mathrm{ver}}_{\Lambda,\bar\rho} \otimes_\Lambda E \xrightarrow{f\otimes_{\mathbb{F}}\mathrm{id}_E} E, r\otimes e \mapsto f(r)\cdot e.$$

In the present case, we define $\widehat{R}$ as the completion of $R^{\mathrm{ver}}_{\Lambda',\bar\rho'} \otimes_{\Lambda'} E$ at $\ker f_E$. By definition, $\widehat{R}$ is a E-algebra with residue field E. Moreover, $\rho^{\mathrm{ver}}_{\bar\rho} \otimes_{R^{\mathrm{ver}}_{\Lambda,\bar\rho}} \widehat{R}$ defines a continuous representation

$$\hat\rho\colon G \longrightarrow \mathrm{GL}_n(\widehat{R})$$

which is a deformation of ρ_E. Again, this yields a homomorphism

$$\varphi\colon R^{\mathrm{ver}}_{E,\rho_E} \longrightarrow \widehat{R}.$$

Theorem 3.3.1. *The map φ is formally smooth. If $R^{\mathrm{ver}}_{\Lambda,\bar\rho}$ is universal, it is formally étale, and hence an isomorphism, by Proposition 3.1.5.*

Remark 3.3.2. Before we give the proof, let us explain a difference depending on the characteristic of E, that will be resolved in Lemma 3.3.5. Suppose that $R^{\mathrm{ver}}_{\Lambda,\bar\rho}$ is a universal ring. Let $X^{\mathrm{univ}}_{\Lambda,\bar\rho} = \operatorname{Spec} R^{\mathrm{ver}}_{\Lambda,\bar\rho}$, and let $x \in X^{\mathrm{univ}}_{\Lambda,\bar\rho}$ be a point of dimension one so that $E = \kappa(x)$ is a local field.

If E has characteristic zero, that is, if x lies on the generic fiber of $X^{\mathrm{univ}}_{\Lambda,\bar\rho}$, then $\widehat{R}$ is the completion of the local ring $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\bar\rho},x}$. Now, by Theorem 3.3.1 this completion is isomorphic to the universal ring $R^{\mathrm{ver}}_{E,\rho_E}$, and so from the latter one can transfer many ring-theoretic properties to $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\bar\rho},x}$.

If on the other hand E has characteristic p, that is, x lies on the special fiber $\overline{X}^{\mathrm{univ}}_{\Lambda,\bar\rho}$, then $\widehat{R}$ is not isomorphic to the completion $\widehat{\mathcal{O}}_{\overline{X}^{\mathrm{univ}}_{\Lambda,\bar\rho},x}$ of the local ring $\mathcal{O}_{\overline{X}^{\mathrm{univ}}_{\Lambda,\bar\rho},x}$. It follows, however, from Lemma 3.3.5 below that we have an isomorphism $\widehat{R} \cong \widehat{\mathcal{O}}_{\overline{X}^{\mathrm{univ}}_{\Lambda,\bar\rho},x}[[T]]$. Via this route, Theorem 3.3.1 allows one again to deduce ring-theoretic properties of $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\bar\rho},x}$ from $R^{\mathrm{ver}}_{E,\rho_E}$.

Remark 3.3.3. In Corollary 4.8.8, we provide an analog of Theorem 3.3.1 for pseudodeformations.

Proof of Theorem 3.3.1. If $\operatorname{Char} E = 0$, then this is [Kis03, Proposition 9.5]. We give the proof if $p = \operatorname{Char} E > 0$. It closely follows that of [Kis03, Proposition 9.5]. We consider a commutative diagram

$$\begin{tikzcd} R^{\mathrm{ver}}_{E,\rho_E} \arrow[r] \arrow[d] & A \arrow[d] \\ \widehat{R} \arrow[r] \arrow[ur, dashed, "g"] & A/I \end{tikzcd} \tag{8}$$

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with $A \in \mathcal{A}r_E$ and $I \subset A$ is a square zero ideal, with the solid arrows given, and we seek to construct a dashed arrow g so that the two triangular subdiagrams commute. If $R^{\mathrm{ver}}_{E,\rho_E}$ is universal, we also have to show that the dashed arrow is unique. Note that A and I are finite-dimensional E-vector spaces. Also, the bottom arrow induces a pair of homomorphism $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A/I$ and $E \to A/I$, where the second one is simply the E-algebra structure map.

By possibly conjugating $\hat{\rho}$ by some matrix in $\Gamma_n(\widehat{R})$, we can assume that $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} \widehat{R} = \hat{\rho}$. Following the proof in [Kis03, Proposition 9.5], one shows that there exists an $\mathcal{O}$-subalgebra A° of A such that

(a) A° is free over $\mathcal{O}$ of rank equal to $\dim_E A$ and $A^\circ \otimes_{\mathcal{O}} E = A$,
(b) the image of A° under $A \to E$ is $\mathcal{O}$, and so $A^\circ \in \widehat{\mathcal{A}r}_{\kappa'}$
(c) the image of $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} A$ lies in $\mathrm{GL}_n(A^\circ)$,
(d) the homomorphism $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A/I$ factors via A°/I° where $I^\circ = I \cap A^\circ$.

Write ρ_{A° for $\rho^{\mathrm{ver}}_{\rho_E} \otimes_{R^{\mathrm{ver}}_{E,\rho_E}} A$ considered with its image in $\mathrm{GL}_n(A^\circ)$. Then ρ_{A° reduces to $\rho^{\mathrm{ver}}_{\bar{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}} A^\circ/I^\circ$ modulo I°, and thus by the versality of $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ there is a homomorphism $g^\circ \colon R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A^\circ$ such that $\rho^{\mathrm{ver}}_{\bar{\rho}'} \otimes_{R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}} A^\circ$ is strictly equivalent to ρ_{A°. Let $g \colon \widehat{R} \to A$ be the homomorphism obtained from $g^\circ \otimes \mathrm{id}$ under completion. It is now not difficult to see that both triangles in diagram (8) commute with this choice of g.

It remains to show the uniqueness of g if $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ is universal. The argument in [Kis03, Proposition 9.5] shows that there is in fact a directed system A°_n, $n \in \mathbb{N}_{\geq 1}$, satisfying (a) – (d) such that $\bigcup_n A^\circ_n = A$. Now if one has g_1, g_2 completing the diagram (8) to two commutative diagrams, there have to be homomorphisms $g^\circ_1, g^\circ_2 \colon R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'} \to A^\circ_n$ for n sufficiently large that give rise to g_1 and g_2, respectively. The corresponding deformations $G \to \mathrm{GL}_n(A^\circ_n)$ of $\bar{\rho}'$ do agree over A and hence they will agree for n sufficiently large, that is, they represent the same strict equivalence class. Because $R^{\mathrm{ver}}_{\Lambda',\bar{\rho}'}$ is universal, they define the same ring maps $g^\circ_1 = g^\circ_2$ and hence $g_1 = g_2$. □

Applying Theorem 3.3.1 in the simplest nontrivial case will deduce the following corollary that will be used in the proof of Lemma 3.3.5.

Corollary 3.3.4. *Let κ be finite, let κ' be a finite extension of κ and let $L = \kappa'((s))$ be the Laurent series field over κ' with uniformizer s. Let $\mathfrak{q}$ be the kernel of the multiplication map $L \otimes_\kappa L \to L$. Then $X \mapsto s \otimes 1 - 1 \otimes s$ induces a continuous L-algebra isomorphism*

$$\psi \colon L[[X]] \xrightarrow{\simeq} L \widehat{\otimes} L := \varprojlim_n (L \otimes_\kappa L)/\mathfrak{q}^n.$$

Proof. We first show that ψ is an isomorphism in the case $\kappa' = \kappa$. Let $G = \widehat{\mathbb{Z}}$ be the free profinite group on one topological generator γ, and let $\bar{\rho} \colon G \to \mathrm{GL}_1(\kappa)$ be the trivial representation given by $\gamma \mapsto 1$. Because $H^1(G,\kappa) \cong \kappa$ and $H^2(G,\kappa) = 0$, we have $R^{\mathrm{univ}}_{\Lambda,\bar{\rho}} = \Lambda[[T]]$ for the resulting universal ring with universal deformation $\rho^u \colon G \to \mathrm{GL}_1(\Lambda[[T]])$ given by $\gamma \mapsto 1 + T$.

Let $f \colon R^{\mathrm{univ}}_{\Lambda,\bar{\rho}} = \Lambda[[T]] \to L$ be the specialization that is given by reduction mod p composed with the injection $\iota \colon \kappa[[T]] \hookrightarrow L = \kappa((s)), T \mapsto s$. The corresponding representation at L is $\rho_L \colon \Gamma \to \mathrm{GL}_1(L), \gamma \mapsto 1 + s$, and for its universal ring we find $R^{\mathrm{univ}}_{L,\rho_L} = L[[X]]$ with universal representation

$$\rho^u_L \colon \Gamma \to \mathrm{GL}_1(L[[X]]), \quad \gamma \mapsto 1 + s + X.$$

Let $\widehat{R}$ be the completion of $R^{\mathrm{univ}}_{\Lambda,\bar{\rho}} \otimes_\Lambda L = \kappa[[T]] \otimes_\kappa L$ at the kernel $\mathfrak{q}$ of the homomorphism $f_L \colon \kappa[[T]] \otimes_\kappa L \to L$ that maps $g(T) \otimes h \in \kappa[[T]] \otimes_\kappa L$ to $g(s)h \in L$. Under f_L, the nonzero elements of $\kappa[[T]] \otimes 1$ map to $L^\times$, and therefore $\kappa[[T]] \otimes_\kappa L \to \widehat{R}$ extends to $\kappa((T)) \otimes_\kappa L \to \widehat{R}$, and completion gives an isomorphism $\kappa((T)) \widehat{\otimes}_\kappa L \xrightarrow{\simeq} \widehat{R}$. We now invoke Theorem 3.3.1. It asserts that the L-algebra map

$$L[[X]] = R^{\mathrm{univ}}_{L,\rho_L} \xrightarrow{\simeq} \widehat{R} = \kappa((T)) \widehat{\otimes} L,$$

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that, by its very definition, sends $\rho_L^u(\gamma) = 1 + s + X$ to $(\rho^u \otimes_\Lambda L)(\gamma) = 1 + T \otimes 1$, is an isomorphism. Because s on the left is mapped to $1 \otimes s$ on the right, we find that $X \mapsto T \otimes 1 - 1 \otimes s$. This proves the assertion on ψ for $\kappa' = \kappa$.

To complete the proof, it remains to explain the reduction of a general finite extension $\kappa' \supset \kappa$ to the case $\kappa' = \kappa$ just treated. For this observe that $L \cong \kappa'((s)) \cong \kappa((s)) \otimes_\kappa \kappa'$ so that $L \otimes_\kappa L \to L$ can be written as the map

$$\kappa((s)) \otimes_\kappa \kappa((s)) \otimes_\kappa (\kappa' \otimes_\kappa \kappa') \longrightarrow \kappa'((s)), \quad f \otimes g \otimes \alpha \otimes \beta \mapsto fg\alpha\beta.$$

Since $\kappa' \supset \kappa$ is a finite Galois extension, the ring $A := \kappa' \otimes_\kappa \kappa'$ is isomorphic to the product of fields $(\kappa')^{[\kappa':\kappa]}$, and A contains a primitive idempotent corresponding to each factor. Under the multiplication map $A \otimes A \longrightarrow \kappa', \lambda \otimes \mu \mapsto \lambda\mu$, all but one of these map to zero. Hence, all but one of these primitive idempotents lie in $\mathfrak{q}$, and so they vanish under completion at $\mathfrak{q}$. One deduces $L \widehat{\otimes}_\kappa L \cong L \widehat{\otimes}_{\kappa'} L$, and this completes the reduction to $\kappa' = \kappa$. □

The following result is needed in our applications of Theorem 3.3.1. Our focus is on the equicharacteristic case.

Lemma 3.3.5. *Let R be in $\widehat{\mathcal{A}r}_\kappa$ so that* Char $R = p$, *let* $\mathfrak{p} \in \operatorname{Spec} R$ *be a point of dimension* 1 *so that* $\dim R/\mathfrak{p} = 1$. *Let* $\kappa(\mathfrak{p}) = \operatorname{Quot}(R/\mathfrak{p})$, *and consider the homomorphism*

$$\varphi \colon R \otimes_\kappa \kappa(\mathfrak{p}) \longrightarrow \kappa(\mathfrak{p}), \quad r \otimes \alpha \mapsto (r \bmod \mathfrak{p}) \cdot \alpha.$$

Set $\mathfrak{q} := \ker \varphi$, and denote by $\widehat{R}$ the completion of $R \otimes_\kappa \kappa(\mathfrak{p})$ at the maximal ideal $\mathfrak{q}$ and by $\widehat{R}_\mathfrak{p}$ the completion of $R_\mathfrak{p}$ at $R_\mathfrak{p}\mathfrak{p}$. Then the following hold:

(a) *One has an isomorphism $\widehat{R}_\mathfrak{p}[[T]] \cong \widehat{R}$.*
(b) *If $\widehat{R}$ is formally smooth over $\kappa(\mathfrak{p})$ of dimension d, then $R_\mathfrak{p}$ is regular of dimension $d - 1$.*

For a variant of Lemma 3.3.5 in the nonequicharacteristic case, see [BIP21, Lemma 3.36].

Proof. Consider $R \longrightarrow R_\mathfrak{p} \longrightarrow \widehat{R}_\mathfrak{p}$. Tensoring with $\kappa(\mathfrak{p})$ over κ, it yields a diagram

$$\begin{tikzcd}
R \otimes_\kappa \kappa(\mathfrak{p}) \arrow[r] \arrow[d, "\iota"'] & R_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) \arrow[r] \arrow[dl, dashed, "\iota'"'] & \widehat{R}_\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p}) = \left(\varprojlim R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}^n\right) \otimes_\kappa \kappa(\mathfrak{p}) \arrow[dll, dashed, "\iota''"'] \\
\widehat{R} = \varprojlim (R \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n, & &
\end{tikzcd}$$

where ι denotes completion and where the dashed arrows ι' and ι'' will now be constructed. For the existence of ι', we use the universal property of localization. Thus, we need to show that $R \backslash \mathfrak{p} \otimes 1$ is mapped under ι to the units in $\widehat{R}$. The ring $\widehat{R}$ is local with residue map induced from φ, and therefore we need to show that $\varphi \circ \iota(R \backslash \mathfrak{p} \otimes 1)$ lies in $\kappa(\mathfrak{p})^\times$, but this is clear from the definitions and the inclusion $R/\mathfrak{p} \hookrightarrow \kappa(\mathfrak{p})$. Regarding ι'', we first note that $\mathfrak{p} \otimes_\kappa \kappa(\mathfrak{p})$ maps to $\mathfrak{q}$ under ι and hence $\mathfrak{p}^n \otimes_\kappa \kappa(\mathfrak{p})$ to $\mathfrak{q}^n$. Hence, the existence of ι' gives a compatible system of homomorphisms $R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}^n \longrightarrow (R \otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n$, and this provides the construction of ι''.

Let π denote the reduction map $\pi \colon \widehat{R} \longrightarrow \kappa(\mathfrak{p})$, set $\varphi' = \pi \circ \iota'$ and $\varphi'' = \pi \circ \iota''$ and define $\mathfrak{q}' = \ker \varphi'$ and $\mathfrak{q}'' = \ker \varphi''$. Then the arguments just given provide a commutative diagram with canonical isomorphisms in the bottom row

$$\begin{tikzcd}
R\otimes_\kappa \kappa(\mathfrak{p}) \arrow[r] \arrow[d, "\iota"'] & R_{\mathfrak{p}}\otimes_\kappa \kappa(\mathfrak{p}) \arrow[r] \arrow[d, dashed, "\iota'"'] & \widehat{R}_{\mathfrak{p}}\otimes_\kappa \kappa(\mathfrak{p}) = \left(\varprojlim R_{\mathfrak{p}}/R_{\mathfrak{p}}\mathfrak{p}^n\right)\otimes_\kappa \kappa(\mathfrak{p}) \arrow[d, dashed, "\iota''"'] \\
\widehat{R} = \varprojlim (R\otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}^n \arrow[r, "\simeq"] & \widehat{R}' := \varprojlim (R_{\mathfrak{p}}\otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}'^n \arrow[r, "\simeq"] & \widehat{R}'' = \varprojlim (\widehat{R}_{\mathfrak{p}}\otimes_\kappa \kappa(\mathfrak{p}))/\mathfrak{q}''^n,
\end{tikzcd}$$

where by slight abuse of notation we denote the middle and right vertical maps again ι' and ι''. Note that by the Cohen structure theorem in equal characteristic the ring $\widehat{R}_{\mathfrak{p}}$ contains $\kappa(\mathfrak{p})$ as a subfield. Focusing on the right-most arrow and using that $R_{\mathfrak{p}}$ is regular if and only if $\widehat{R}_{\mathfrak{p}}$ is so, it will suffice to prove the following assertion.

Let $\mathcal{R}$ be a complete Noetherian local $\kappa(\mathfrak{p})$-algebra with residue field $\kappa(\mathfrak{p})$ and residue homomorphism $\pi\colon \mathcal{R}\to\kappa(\mathfrak{p})$, let $\psi\colon \mathcal{R}\otimes_\kappa \kappa(\mathfrak{p})\to\kappa(\mathfrak{p})$ be the homomorphism $r\otimes x\mapsto \pi(r)\cdot x$, let $\mathfrak{Q}=\ker\psi$ and let $\widehat{\mathcal{R}}$ be the completion of $\mathcal{R}\otimes_\kappa\kappa(\mathfrak{p})$ at $\mathfrak{Q}$. Then we assert that $\widehat{\mathcal{R}}\cong\mathcal{R}[[t]]$.

To prove the assertion, note first that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are $\kappa(\mathfrak{p})$-algebras with maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ such that $\kappa(\mathfrak{p})$ is in both cases the residue field, then the completion of $\mathcal{S}:=\mathcal{S}_1\otimes_{\kappa(\mathfrak{p})}\mathcal{S}_2$ at the maximal ideal $\mathfrak{m}:=\mathfrak{P}_1\otimes_{\kappa(\mathfrak{p})}\mathcal{S}_2+\mathcal{S}_1\otimes_{\kappa(\mathfrak{p})}\mathfrak{P}_2$ is isomorphic to

$$\varprojlim \mathcal{S}_1/\mathfrak{P}_1^n \widehat{\otimes}_{\kappa(\mathfrak{p})} \varprojlim \mathcal{S}_2/\mathfrak{P}_2^n.$$

If furthermore $\mathcal{S}_1$ is complete with respect to $\mathfrak{P}_1$ and if $\varprojlim \mathcal{S}_2/\mathfrak{P}_2^n\cong\kappa(\mathfrak{p})[[T]]$, then the completion of $\mathcal{S}$ at $\mathfrak{m}$ is $\mathcal{S}_1[[T]]$. We apply this to $\mathcal{S}_1=\mathcal{R}$, $\mathcal{S}_2=\kappa(\mathfrak{p})\otimes_\kappa\kappa(\mathfrak{p})$, $\mathfrak{P}_2=\ker\left(\kappa(\mathfrak{p})\otimes_\kappa\kappa(\mathfrak{p})\to\kappa(\mathfrak{p}), x\otimes y\mapsto xy\right)$. Then by Corollary 3.3.4, we have $\varprojlim \mathcal{S}_2/\mathfrak{P}_2^n\cong\kappa(\mathfrak{p})[[T]]$, and we deduce $\widehat{\mathcal{R}}\cong\mathcal{R}[[T]]$. □

Let us record the following consequence of Theorem 3.3.1 and Lemma 3.3.5.

Corollary 3.3.6. *Suppose E is a local field and $\rho\colon G\to \mathrm{GL}_n(E)$ is a continuous homomorphism. Let κ be the (finite) residue field of E, and let $\bar\rho$ be the semisimple reduction of ρ to κ. Then*

$$\dim_E H^1(G,\mathrm{ad}_\rho)\le\dim_\kappa H^1(G,\mathrm{ad}_{\bar\rho}).$$

Proof. The corollary will follow from the simple fact that the rank of a coherent sheaf cannot decrease under specialization: Let $\mathcal{O}$ be the valuation ring of E. By possibly passing to a finite extension of E, we may assume that E^n contains a $\rho(G)$-stable $\mathcal{O}$-lattice whose reduction is $\bar\rho$. Let $R:=R^{\mathrm{ver}}_{\mathcal{O},\bar\rho}$. We may assume that R is Noetherian, that is, that $\dim_\kappa H^1(G,\mathrm{ad}_{\bar\rho})$ is finite since else there is nothing to show. Let also $\widehat{R}$ and $R':=R^{\mathrm{ver}}_{E,\rho}$ be as in Theorem 3.3.1, and denote by f a map $f\colon R\to\mathcal{O}$ given by the versality of R.

Denote by $\widehat{\Omega}_{R/\mathcal{O}}=\lim_n\Omega_{(R/\mathfrak{m}_R^n)/\mathcal{O}}$ the module of continuous Kähler differentials. Since R is a quotient of a power series ring over $\mathcal{O}$ in finitely many variables, as an R-module $\widehat{\Omega}_{R/\mathcal{O}}$ is finitely generated. By Nakayama's lemma, we have

$$\dim_E\widehat{\Omega}_{R/\mathcal{O}}\otimes_R E\le\dim_\kappa\widehat{\Omega}_{R/\mathcal{O}}\otimes_R\kappa.$$

Let $\mathfrak{p}=(\mathrm{Ker}\, f)$, let $\mathfrak{m}_E=\mathrm{Ker}(R_{\mathfrak{p}}\to E)$, with the map induced from f, and let $\widehat{\mathfrak{m}}_E$ be the maximal ideal of $\widehat{R}$. Then by [BKM21, Lemma 7.3], we have $\widehat{\Omega}_{R/\mathcal{O}}\otimes_R E\cong\mathfrak{p}/\mathfrak{p}^2\otimes_R E\cong\mathfrak{m}_E/\mathfrak{m}_E^2$, and furthermore from Lemma 3.3.5 and Theorem 3.3.1, again combined with [BKM21, Lemma 7.3], we obtain $\mathfrak{m}_E/\mathfrak{m}_E^2\oplus E\cong\widehat{\mathfrak{m}}_E/\widehat{\mathfrak{m}}_E^2\cong\widehat{\Omega}_{\widehat{R}/E}\otimes_{\widehat{R}}E\to\widehat{\Omega}_{R'/E}\otimes_{R'}E$, where the last map is surjective. Hence,

$$\dim_E\widehat{\Omega}_{R'/E}\otimes_{R'}E\le 1+\dim_\kappa\widehat{\Omega}_{R/\mathcal{O}}\otimes_R\kappa.$$

By [Maz97, §17, §21], the dual of $\widehat{\Omega}_{R/\mathcal{O}} \otimes_R \kappa$ is the mod $\mathfrak{m}_{\mathcal{O}}$-tangent space of R at $\mathfrak{m}_R$ and the dual of $\widehat{\Omega}_{R'/E} \otimes_{R'} E$ is the tangent space of R' at $\mathfrak{m}_{R'}$, and the latter can be identified with $H^1(G, \mathrm{ad}_{\bar{\rho}})$ and $H^1(G, \mathrm{ad}_\rho)$, respectively. This proves the corollary. □

3.4. Relative formal smoothness of the determinant functor

A generalized Tate local duality: We recall a generalization of Tate local duality from [Nek06]. Let first G be a profinite group and M a discrete G-module. Then one defines the continuous cohomology $H^i(G, M)$ as $\varinjlim_{U \in \mathfrak{U}} H^i(G/U, M^U)$, where $\mathfrak{U}$ is the set of all normal open subgroups of G; they form a basis of open neighborhoods near the identity of G. This applies for instance if M is a κ-vector space with a continuous G-action and if κ is finite. Suppose, however, that κ is a local field and that M is a finite-dimensional κ-vector space that carries the natural topology induced from κ and a continuous κ-linear G-action. Let $\mathcal{O}$ be the valuation ring of κ with maximal ideal $\mathfrak{m}_{\mathcal{O}}$ and finite residue field $\mathbb{F}$. Because G is compact a standard argument shows that M contains a G-stable $\mathcal{O}$-lattice L. Suppose that G satisfies the finiteness condition (F) that for all open subgroups $U \subset G$ and all $i \geq 0$ one has $\dim_{\mathbb{F}} H^i(U, \mathbb{F}) < \infty$. In this case, one defines continuous cohomology via

$$H^i(G, M) := \varprojlim_n H^i(G, L/\mathfrak{m}_{\mathcal{O}}^n L) \otimes_L \kappa,$$

and one shows that this definition is independent of any choices; it follows from [Nek06, 4.2.2] that this definition agrees with the one used in [Nek06]. Note that one also has $H^i(G, M) = Z^i(G, M)/B^i(G, M)$, where $Z^i(G, M)$ and $B^i(G, M)$ denotes the continuous i-cocycles and i-coboundaries, respectively, and one has $H^0(G, M) = M^G$. For $i = 1$, $Z^1(G, M)$ is the group of continuous maps $c\colon G \to M$ with $c(gh) = gc(h) + c(g)$ for all $g, h \in G$. Note, also, that all 1-coboundaries are continuous by the continuity of the action of G on M; the latter no longer holds for n-coboundaries with $n \geq 2$.

The next result is a generalization of Tate local duality from finite field to local field coefficients. In the form needed, it is due to Nekovář. Let V be a finite-dimensional κ-vector space with the topology induced from κ, and suppose that V carries a continuous κ-linear action by G_K. Write $V^\vee$ for the dual $\mathrm{Hom}_\kappa(V, \kappa)$ of V and $V(1)$ for the twist of V by the cyclotomic character. Set $h^j(K, V) := \dim_\kappa H^j(G_K, V)$.

Theorem 3.4.1 (Tate and Nekovář). *The following assertions hold:*

(a) *One has $h^j(K, V) < \infty$ for $j \in \mathbb{Z}$ and $h^j(K, V) = 0$ for $j \notin \{0, 1, 2\}$;*
(b) *For $j \in \{0, 1, 2\}$ one has natural isomorphisms*

$$H^{2-j}(G_K, V^\vee(1)) \xrightarrow{\simeq} H^j(G_K, V)^\vee;$$

(c) *One has the Euler characteristic formula*

$$\sum_{j \geq 0} (-1)^j h^j(K, V) = -[K : \mathbb{Q}_p] \cdot \dim_E V.$$

Proof. If κ is finite, the above statement is just the usual Tate local duality. If κ is local, let $\mathcal{O}$ be its valuation ring. Because G_K is compact, one can find an $\mathcal{O}$-lattice T in V that is stable under G_K. Let $j \geq 0$. Then [Nek06, Theorem 5.2.6] asserts that each $H^j(G_K, T')$, $T' \in \{T, T^\vee(1)\}$, is a finitely generated $\mathcal{O}$-module and moreover it gives a spectral sequence

$$\mathrm{Ext}^i_{\mathcal{O}}(H^{2-j}(G_K, T^\vee(1)), \mathcal{O}) \Longrightarrow H^{i+j}(G_K, T). \tag{9}$$

Because $\mathcal{O}$ is regular and of dimension 1, the groups $\mathrm{Ext}^1_{\mathcal{O}}(\cdot, \mathcal{O})$ are finitely generated $\mathcal{O}$-torsion modules and $\mathrm{Ext}^i_{\mathcal{O}}(\cdot, \mathcal{O}) = 0$ for $i \geq 2$. After tensoring the spectral sequence (9) with κ over $\mathcal{O}$, Parts (b) and (a) are clear. Part (c) follows from [Nek06, Theorem 4.6.9 and 5.2.11] applied to T, again after tensoring with κ over $\mathcal{O}$. □

The determinant map: The determinant of representations induces a natural transformation

$$\det\colon \mathcal{D}_{\bar\rho} \to \mathcal{D}_{\det\bar\rho} \tag{10}$$

that maps the class of $\rho\colon G \to \mathrm{GL}_n(A)$ to the class of $\det\circ\rho$. The induced map on adjoint representations is the trace map in the short exact sequence

$$0 \longrightarrow \mathrm{ad}^0_{\bar\rho} \longrightarrow \mathrm{ad}_{\bar\rho} \xrightarrow{\mathrm{tr}} \mathrm{ad}_{\det\bar\rho} \cong \kappa \longrightarrow 0. \tag{11}$$

Using that $\mathrm{ad}_{\bar\rho}$ is self-dual it is easy to see that the sequence dual to the sequence (11) is

$$0 \longrightarrow \kappa \xrightarrow{\mathrm{diag}} \mathrm{ad}_{\bar\rho} \longrightarrow \overline{\mathrm{ad}}_{\bar\rho} \longrightarrow 0. \tag{12}$$

We have the following explicit result on det for $G = G_K$ and K a p-adic field with $d = [K : \mathbb{Q}_p]$.

Lemma 3.4.2. *Suppose that $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$. Then $\det\colon \mathcal{D}_{\bar\rho} \to \mathcal{D}_{\det\bar\rho}$ is smooth of relative dimension $d(n^2-1)$. This holds in particular, if $p \not| \, n$, $\kappa \cong \mathrm{End}_{\kappa[G_K]}(\bar\rho)$ and $\zeta_p \in K$.*

Proof. Let $A \to B$ be a small extension in $\mathcal{A}r_\Lambda$. Let I be its kernel so that $I^2 \subset \mathfrak{m}_A I = 0$. For the relative smoothness, we need to show the surjectivity of

$$\mathcal{D}_{\bar\rho}(A) \longrightarrow \mathcal{D}_{\bar\rho}(B) \times_{\mathcal{D}_{\det\bar\rho}(B)} \mathcal{D}_{\det\bar\rho}(A).$$

So suppose we are given deformations $\rho_B \in \mathcal{D}_{\bar\rho}(B)$ and $\tau_A \in \mathcal{D}_{\det\bar\rho}(A)$ with $\det\rho_B = \tau_A \otimes_A B \in \mathcal{D}_{\det\bar\rho}(B)$. We need to find a deformation $\rho_A \in \mathcal{D}_{\bar\rho}(A)$ such that $\rho_A \otimes_A B = \rho_B$ and $\det\rho_A = \tau_A$.

Recall from [Maz89, p. 398] that there is a canonical obstruction class $\mathcal{O}(\rho_B) \in H^2(G_K, \mathrm{ad}_\rho) \otimes_\kappa I$, which vanishes if and only if there exists a deformation of $\bar\rho$ to A that lifts ρ_B. Because of the existence of the deformation τ_A that maps to $\det\rho_B$, the obstruction class $\mathcal{O}(\det\rho_B) \in H^2(G_K, \mathrm{ad}_{\det\bar\rho}) \otimes_\kappa I$ vanishes. By Theorem 3.4.1, the long exact sequence of Galois cohomology arising from the sequence (11) gives the left exact sequence

$$H^2(G_K, \mathrm{ad}^0_{\bar\rho}) \longrightarrow H^2(G_K, \mathrm{ad}_{\bar\rho}) \xrightarrow{H^2(\mathrm{tr})} H^2(G_K, \kappa) \longrightarrow 0$$

By Theorem 3.4.1 the sequence is dual to the right exact sequence

$$0 \longrightarrow H^0(G_K, \kappa(1)) \xrightarrow{H^0(\mathrm{diag}(1))} H^0(G_K, \mathrm{ad}_{\bar\rho}(1)) \longrightarrow H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)),$$

that arises from the sequence (12). By our hypothesis, the map $H^0(\mathrm{diag}(1))$ is an isomorphism, and so by duality the same holds for $H^2(\mathrm{tr})$. By a short explicit computation, one sees that $\mathcal{O}(\rho_B)$ maps to $\mathcal{O}(\det\rho_B) = 0$ under $H^2(\mathrm{tr}) \otimes_\kappa \mathrm{id}_I$, and this implies the vanishing of $\mathcal{O}(\rho_B)$.

We have now proved that there exists $\rho'_A \in \mathcal{D}_{\bar\rho}(A)$ mapping to $\rho_B \in \mathcal{D}_{\bar\rho}(B)$. However, this lift need not satisfy $\det\rho'_A = \tau_A$. At this point, we note that our hypothesis in fact implies that $H^2(G_K, \mathrm{ad}^0_{\bar\rho}) = 0$ so that

$$H^1(G_K, \mathrm{ad}_{\bar\rho}) \longrightarrow H^1(G_K, \mathrm{ad}_{\det\bar\rho}) = H^1(G_K, \kappa) \tag{13}$$

is surjective. Now, $\det\rho'_A$ and τ_A are deformations of τ_B and the space of all such deformations is a principal homogeneous space under $H^1(G_K, \kappa)$, that is, the tangent space of the deformation problem, by [Sch68, Remark 2.15], and likewise the deformations of ρ_B form a principal homogeneous space under $H^1(G_K, \mathrm{ad}_{\bar\rho})$. Since the map (13) is surjective we can thus alter ρ'_A by a class in $H^1(G_K, \mathrm{ad}_{\bar\rho})$ into some other deformation ρ_A of ρ_B that also satisfies $\det\rho_A = \tau_A$. This completes the proof of the

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formal smoothness. Note also that if $p \nmid n$, then the above two sequences are exact on both sides and hence $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$.

By Proposition 3.1.5, it follows that the natural map $R^{\mathrm{univ}}_{\det\bar\rho} \to R^{\mathrm{univ}}_{\bar\rho}$ is formally smooth of relative dimension $h = h^1(K, \mathrm{ad}_{\bar\rho}) - h^1(K, \mathrm{ad}_{\det\bar\rho})$. It remains to identify h with the number in the lemma. Since the map (13) is surjective, by the long exact sequence for $H^*(G_K, \cdot)$ applied to the sequence (11), we deduce

$$h = h^1(K, \mathrm{ad}^0_{\bar\rho}) - h^0(K, \mathrm{ad}_{\det\bar\rho}) + h^0(K, \mathrm{ad}_{\bar\rho}) - h^0(K, \mathrm{ad}^0_{\bar\rho}) = h^1(K, \mathrm{ad}^0_{\bar\rho}) - 1 + 1 - h^0(K, \mathrm{ad}^0_{\bar\rho}).$$

Since $h^2(K, \mathrm{ad}^0_{\bar\rho}) = 0$ by hypothesis and the duality statement of Theorem 3.4.1, the Euler characteristic formula of Theorem 3.4.1 implies $h = d(n^2 - 1)$.

The last assertion is straightforward. If $p \nmid n$, then the sequence (11) splits; the second assumption now yields $H^0(G_K, \mathrm{ad}^0_{\bar\rho}) = 0$. If now κ has characteristic zero, then $0 = H^0(G_K, \kappa(1)) \cong H^0(G_K, \mathrm{ad}_{\bar\rho}(1))$ and we are done. If on the other hand κ has characteristic p and $\zeta_p \in K$, then $\mathrm{ad}^0_{\bar\rho} = \mathrm{ad}^0_{\bar\rho}(1)$ and we are done, as well. □

Let $\bar\rho_1 \colon G_K \to \mathrm{GL}_1(\kappa)$ be a continuous character, and denote by $\bar\rho_0 \colon G_K \to \mathrm{GL}_1(\kappa)$ the trivial character. If κ is finite, denote by $\rho_1 \colon G_K \to \mathrm{GL}_1(\Lambda)$ the Teichmüller lift of $\bar\rho_1$, if κ is a local field, set $\rho_1 := \bar\rho_1$. There is a natural isomorphism $\mathcal{D}_{\bar\rho_0} \to \mathcal{D}_{\bar\rho_1}$ mapping a deformation $\rho \colon G_K \to \mathrm{GL}_1(A)$ to $\rho \otimes \rho_1$. As already observed in [Maz89, §1.4], $\mathcal{D}_{\bar\rho_0}$ is representable by the completed group ring $\Lambda[[G_K^{\mathrm{ab},p}]]$, where $G_K^{\mathrm{ab},p}$ is the completion of the abelianization G_K^{ab} of G_K along normal open subgroups of p-power index; the universal homomorphism

$$G_K \to (\Lambda[[G_K^{\mathrm{ab},p}]])^\times$$

factors via $G_K^{\mathrm{ab},p}$ and sends $g \in G_K^{\mathrm{ab},p}$ to itself as a unit element in $\Lambda[[G_K^{\mathrm{ab},p}]]$. The reciprocity homomorphisms of local class field theory, yields an isomorphism

$$\mathrm{rec}^p \colon K^{\times,p} \to G_K^{\mathrm{ab},p},$$

where $K^{\times,p}$ is the pro-p completion of the multiplicative group $K^\times$. The torsion subgroup of $K^{\times,p}$ is naturally identified with the group $\mu_{p^\infty}(K)$ of p-power roots of unity in K. Combining this with det from diagram (10), we have the following chain of natural ring homomorphisms in $\widehat{\mathcal{A}r}_\Lambda$

$$\Lambda[\mu_{p^\infty}(K)] \longrightarrow \Lambda[[K^{\times,p}]] \xrightarrow{\simeq} R^{\mathrm{ver}}_{\Lambda,\det\bar\rho} \longrightarrow R^{\mathrm{ver}}_{\Lambda,\bar\rho}. \tag{14}$$

Corollary 3.4.3 (Cf. [Nak14, §4]). *Let κ be finite or a local field of characteristic p, and suppose $\Lambda = \kappa$. Suppose that $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho}(1)) = 0$. Then the following hold:*

(a) *Both morphisms in diagram (14) are formally smooth.*
(b) *Both morphism in the following induced diagram are formally smooth:*

$$\Lambda = \Lambda[\mu_{p^\infty}(K)]_{\mathrm{red}} \longrightarrow \Lambda[[K^{\times,p}]]_{\mathrm{red}} \xrightarrow{\simeq} (R^{\mathrm{ver}}_{\Lambda,\det\bar\rho})_{\mathrm{red}} \longrightarrow (R^{\mathrm{ver}}_{\Lambda,\bar\rho})_{\mathrm{red}}.$$

The relative dimensions in both cases are $d + 1$ and $d(n^2 - 1)$, respectively.

Proof. Let $q := \mathrm{ord}\, \mu_{p^\infty}(K)$. By Lemma 3.4.2, the natural map $R^{\mathrm{univ}}_{\det\rho} \longrightarrow R^{\mathrm{univ}}_{\rho}$ is formally smooth of relative dimension $h = d(n^2 - 1)$. From the theory of local fields, one has $K^{\times,p} \cong \mathbb{Z}_p^{[K:\mathbb{Q}_p]+1} \times \mu_{p^\infty}(K)$, where $\mu_{p^\infty}(K)$ is a finite cyclic group of p-power order q. By our hypothesis on the characteristic of κ, the right-hand morphism in diagram (14) can be identified with

$$\kappa[x]/(x^q) \to \kappa[[x_1, \ldots, x_d, x]]/(x^q),$$

and parts (a) and (b) for it are now obvious. To see the second part of (b), note that the kernel of the reduction map $\kappa[x]/(x^q) \longrightarrow \kappa$ is nilpotent. Hence, the kernel of the induced map $\varphi\colon R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \longrightarrow R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \otimes_{\kappa[x]/(x^q)} \kappa$ is nilpotent as well, and the map $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \longrightarrow (R^{\mathrm{ver}}_{\Lambda,\overline{\rho}})_{\mathrm{red}}$ factors via φ. At the same time, formal smoothness is preserved under base change. Hence, $\kappa \longrightarrow R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \otimes_{\kappa[x]/(x^q)} \kappa$ is formally smooth, and therefore $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \otimes_{\kappa[x]/(x^q)} \kappa$ is regular and in particular a domain. We deduce that $R^{\mathrm{ver}}_{\Lambda,\overline{\rho}} \otimes_{\kappa[x]/(x^q)} \kappa \longrightarrow (R^{\mathrm{ver}}_{\Lambda,\overline{\rho}})_{\mathrm{red}}$ is an isomorphism, and this completes (b). □

We end this subsection with a computation of $H^0(G_K, \mathrm{ad}_{\overline{\rho}} \otimes \chi)$ and a variant of Lemma 3.4.2 for certain reducible deformations.

Lemma 3.4.4. *Let E be a finite or local field with its natural topology. Denote by $\rho_i\colon G_K \longrightarrow \mathrm{GL}_{n_i}(E)$ continuous Galois representations for $i = 1, 2$, and let $\rho = \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix}$ be an extension of ρ_1 by ρ_2. Let $\chi\colon G_K \longrightarrow E^\times$ be a continuous character and write 1 for the trivial character. Suppose that*

(a) $\mathrm{Hom}_{G_K}(\rho_1, \rho_2 \otimes \chi) = 0$ *and* $\mathrm{Hom}_{G_K}(\rho_2, \rho_1 \otimes \chi) = 0$.
(b) *For $i = 1, 2$, we have* $\mathrm{End}_{G_K}(\rho_i) \cong E$ *if $\chi = 1$ and* $\mathrm{Hom}_{G_K}(\rho_i, \rho_i \otimes \chi) = 0$ *if $\chi \neq 1$.*
(c) *If $\chi = 1$, then the class $c \in \mathrm{Ext}^1_{G_K}(\rho_2, \rho_1)$ is nontrivial,*

Then $\mathrm{End}_{G_K}(\rho) \cong E$ *if $\chi = 1$ and* $\mathrm{Hom}_{G_K}(\rho, \rho \otimes \chi) = 0$ *if $\chi \neq 1$.*

Proof. To determine $\mathrm{Hom}_{G_K}(\rho, \rho \otimes \chi)$, we consider $A_{ij} \in \mathrm{Mat}_{n_i \times n_j}(E)$ for $1 \le i, j \le 2$ such that

$$\begin{aligned} 0 &\overset{!}{=} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \rho_1 & c \\ 0 & \rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 \otimes \chi & c \otimes \chi \\ 0 & \rho_2 \otimes \chi \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}\rho_1 & A_{11}c + A_{12}\rho_2 \\ A_{21}\rho_1 & A_{21}c + A_{22}\rho_2 \end{pmatrix} - \begin{pmatrix} \rho_1 \otimes \chi \cdot A_{11} + c \otimes \chi \cdot A_{21} & \rho_1 \otimes \chi \cdot A_{12} + c \otimes \chi \cdot A_{22} \\ \rho_2 \otimes \chi \cdot A_{21} & \rho_2 \otimes \chi \cdot A_{22} \end{pmatrix}. \end{aligned}$$

From hypothesis (a) and considering the (2,1)-entry, we deduce $A_{21} = 0$. From hypothesis (b) and considering the (1,1)- and (2,2)-entries, we deduce, depending on χ the following: If $\chi = 1$, then A_{ii} are scalar for $i = 1, 2$, say equal to $\lambda_i \mathbb{1}_{n_i}$ for some $\lambda_i \in E$, respectively; if $\chi \neq 1$, then both $A_{ii} = 0$. Considering the $(1, 2)$-entry, we obtain the relation

$$A_{11}c - c \otimes \chi \cdot A_{22} = \rho_1 \otimes \chi \cdot A_{12} - A_{12}\rho_2.$$

If $\chi \neq 1$, the left-hand side is zero, and from (a) we deduce $A_{12} = 0$ so that the proof in this case is complete. If $\chi = 1$, then we have $(\lambda_1 - \lambda_2)c = \rho_1 A_{12} - A_{12}\rho_2$. Now, $g \mapsto \rho_1(g)A_{12} - A_{12}\rho_2(g)$ is a 1-coboundary with values in $\mathrm{Hom}_{G_K}(\rho_2, \rho_1)$, and so if $\lambda_1 \neq \lambda_2$ the last condition implies that c is the trivial class in $\mathrm{Ext}^1_{G_K}(\rho_2, \rho_1)$ which is excluded by hypothesis (c). This shows $\lambda_1 = \lambda_2$, and $A_{12} \in \mathrm{Hom}_{G_K}(\rho_2, \rho_1)$, and hence $A_{12} = 0$, again by (a). This completes the proof. □

For Proposition 5.3.3, we also need a variant of Corollary 3.4.3 for certain reducible $\overline{\rho}$. So for the remainder of this subsection let $\overline{\rho}_i\colon G_K \longrightarrow \mathrm{GL}_{n_i}(\kappa)$, $i = 1, 2$, be absolutely irreducible, and assume that $\overline{\rho}_1$ is not isomorphic to $\overline{\rho}_2(j)$ for $j \in \{0, \pm 1\}$. Let further $\bar{\rho}$ be a nonsplit extension of dimension $n = n_1 + n_2$ that fits into a short exact sequence $0 \longrightarrow \bar{\rho}_1 \longrightarrow \bar{\rho} \longrightarrow \bar{\rho}_2 \longrightarrow 0$. Define the subfunctor $\mathcal{D}_{\bar{\rho}_1 \subset \bar{\rho}} \subset \mathcal{D}_{\bar{\rho}}$ by mapping A in $\mathcal{A}r_\Lambda$ to the set of $\Gamma_n(A)$-conjugacy classes of liftings ρ_A of $\bar{\rho}$ to A such that ρ_A stabilizes an A-direct summand P_A of A^n of rank n_1 so that the induced representation of ρ_A on P_A is a deformation of $\bar{\rho}_1$; because of the shape of $\overline{\rho}$ the deformations described by $\mathcal{D}_{\bar{\rho}_1 \subset \bar{\rho}}$ are precisely the ‘reducible’ deformations of $\bar{\rho}$. In fact:

Lemma 3.4.5. *$\mathcal{D}_{\bar{\rho}_1 \subset \bar{\rho}} \subset \mathcal{D}_{\bar{\rho}}$ is a relatively representable subfunctor in the sense of [Maz97, §19].*

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Proof. In lack of a reference, we give a proof. Given a diagram $A \to B \leftarrow C$ in $\mathcal{A}r_\Lambda$, one has to show that the induced diagram

$$\begin{tikzcd}
\mathcal{D}_{\bar\rho_1\subset\bar\rho}(A\times_B C) \arrow[r] \arrow[d] & \mathcal{D}_{\bar\rho_1\subset\bar\rho}(A)\times_{\mathcal{D}_{\bar\rho_1\subset\bar\rho}(B)}\mathcal{D}_{\bar\rho_1\subset\bar\rho}(C) \arrow[d] \\
\mathcal{D}_{\bar\rho}(A\times_B C) \arrow[r] & \mathcal{D}_{\bar\rho}(A)\times_{\mathcal{D}_{\bar\rho}(B)}\mathcal{D}_{\bar\rho}(C)
\end{tikzcd}$$

is a pullback in the category of sets. To see this, suppose (ρ_A, ρ_C) is an element in the top right and $\rho_{A\times_B C}$ in the bottom left, both mapping to the same element in the bottom right. Then (ρ_A, ρ_C) gives rise to a pair of subrepresentations (ρ_A^1, ρ_C^1) that are deformations of $\bar\rho_1$ and a pair of quotient representations (ρ_A^2, ρ_C^2) that are deformations of $\bar\rho_2$ and each pair maps to the same element on B, say ρ_B^1 and ρ_B^2, respectively. By our hypotheses on the $\bar\rho_i$ the respective deformation functors $\mathcal{D}_{\bar\rho_i}$ are representable so that each pair patches to a deformations $\rho^1_{A\times_B C}$ of $\bar\rho_1$ and $\rho^2_{A\times_B C}$ of $\bar\rho_2$ on $A\times_B C$, respectively. It now follows from [Urb99, Theorem] that $\rho_{A\times_B C}$ lies in $\mathcal{D}_{\bar\rho_1\subset\bar\rho}(A\times_B C)$ with $\rho^1_{A\times_B C}$ as a sub and $\rho^2_{A\times_B C}$ as a quotient representation. □

Now, by Lemma 3.4.4 we have $\mathrm{End}_{G_K}(\bar\rho)\cong\kappa$ so that $\mathcal{D}_{\bar\rho}$ is representable. By Lemma 3.4.5, it follows that $\mathcal{D}_{\bar\rho_1\subset\bar\rho}$ is a closed subfunctor that is representable by a quotient of $R^{\mathrm{univ}}_{\bar\rho}$ in $\widehat{\mathcal{A}r}_\Lambda$. Let $\rho^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho}\colon G_K \to \mathrm{GL}_n(R^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho})$ be a universal representation for $\mathcal{D}_{\bar\rho_1\subset\bar\rho}$. By the shape of $\overline{\rho}$ and the reducibility of $\rho^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho}$, we can choose a suitable basis for $\rho^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho}$ so that

$$\rho^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho} = \begin{pmatrix} \rho^{\mathrm{univ},1}_{\bar\rho_1\subset\bar\rho} & * \\ 0 & \rho^{\mathrm{univ},2}_{\bar\rho_1\subset\bar\rho}\end{pmatrix}$$

for representations $\rho^{\mathrm{univ},i}_{\bar\rho_1\subset\bar\rho}\colon G_K\to \mathrm{GL}_{n_i}(R^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho})$, and $*$ a suitable nontrivial extension class. Here, the $\rho^{\mathrm{univ},i}_{\bar\rho_1\subset\bar\rho}$ are unique up to conjugation.

We now adapt the setting developed before Lemma 3.4.2 to the present situation. Namely, one has a natural functor

$$\det{}^2\colon \mathcal{D}_{\bar\rho_1\subset\bar\rho}\to\mathcal{D}_{\det\bar\rho_1}\times\mathcal{D}_{\det\bar\rho_2},$$

which for $A\in\widehat{\mathcal{A}r}_\Lambda$ attaches to $\rho_A\in\mathcal{D}_{\bar\rho_1\subset\bar\rho}(A)$ with ρ_A-stable direct summand $P_A(\cong A^{n_1})$ of A^n the pair $(\det(\rho_{A|P_A}),\det(\rho_A \bmod P_A))$ of deformations of $(\det\overline{\rho}_1,\det\overline{\rho}_2)$. Let $\mathrm{ad}_{\bar\rho_1\subset\bar\rho}\subset\mathrm{ad}_{\bar\rho}$ be the block upper triangular subspace of matrices that preserve $\bar\rho_1$. Mapping matrices to the diagonal blocks gives a surjection $\pi\colon \mathrm{ad}_{\bar\rho_1\subset\bar\rho}\subset\mathrm{ad}_{\bar\rho}\to\mathrm{ad}_{\bar\rho_1}\oplus\mathrm{ad}_{\bar\rho_2}$, and we denote by $\mathrm{ad}^{00}_{\bar\rho_1\subset\bar\rho}\subset\mathrm{ad}_{\bar\rho_1\subset\bar\rho}$ the inverse image of $\mathrm{ad}^0_{\bar\rho_1}\oplus\mathrm{ad}^0_{\bar\rho_2}$ under π, and by $\mathrm{rad}_{\bar\rho_1\subset\bar\rho}$ the kernel of π.

Now, as in the proof of Lemma 3.4.2, obstructions to the smoothness of $\det^2$ lie in the group $H^2(G_K,\mathrm{ad}^{00}_{\bar\rho_1\subset\bar\rho})$. Using the short exact sequence $0\to\mathrm{rad}_{\bar\rho_1\subset\bar\rho}\to\mathrm{ad}^{00}_{\bar\rho_1\subset\bar\rho}\to\mathrm{ad}^0_{\bar\rho_1}\times\mathrm{ad}^0_{\bar\rho_2}\to 0$, and the vanishing of $H^2(G_K,\mathrm{rad}_{\bar\rho_1\subset\bar\rho})^*\cong H^0(G_K,\mathrm{Hom}(\bar\rho_1,\bar\rho_2)(1))\cong\mathrm{Hom}_{G_K}(\bar\rho_1,\bar\rho_2(1))$, that relies on local Tate-duality from Theorem 3.4.1 and $\overline{\rho}_1\not\cong\bar\rho_2(1)$, one finds

$$H^2(G_K,\mathrm{ad}^{00}_{\bar\rho_1\subset\bar\rho})\cong H^2(G_K,\mathrm{ad}^0_{\bar\rho_1}\oplus\mathrm{ad}^0_{\bar\rho_2}). \tag{15}$$

Again, from Tate local duality one deduces that $\det^2$ is formally smooth if $H^0(G_K,\overline{\mathrm{ad}}_{\bar\rho_i}(1))=0$ for $i=1,2$. Assuming this, the map of rings $R^{\mathrm{univ}}_{\det\bar\rho_1}\hat\otimes_\Lambda R^{\mathrm{univ}}_{\det\bar\rho_2}\to R^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho}$ is formally smooth and the relative dimension is given by

$$h=\dim\mathrm{im}\left(H^1(G_K,\mathrm{ad}_{\bar\rho_1\subset\bar\rho})\to H^1(G_K,\mathrm{ad}_{\bar\rho})\right)-2\dim H^1(G_K,\kappa).$$

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To compute h, observe first that $H^1(G_K, \mathrm{ad}_{\bar\rho_1\subset\bar\rho}) \to H^1(G_K, \mathrm{ad}_{\bar\rho})$ is injective because $\mathrm{ad}_{\bar\rho}/\mathrm{ad}_{\bar\rho_1\subset\bar\rho} \cong \mathrm{Hom}_\kappa(\bar\rho_1, \bar\rho_2)$ and $\mathrm{Hom}_{G_K}(\bar\rho_1, \bar\rho_2) = 0$. The quantity $\dim H^1(G_K, \kappa)$ was computed in the proof of Corollary 3.4.3 to be $1 + d + \delta_p$, where $\delta_p = 0$ if $\zeta_p \notin K$ and $\delta_p = 1$ if $\zeta_p \in K$. Finally, $\dim H^1(G_K, \mathrm{ad}_{\bar\rho_1\subset\bar\rho})$ is computed by the Euler–Poincaré formula of local Tate duality: Lemma 3.4.4 yields $H^0(G_K, \mathrm{ad}_{\bar\rho_1\subset\bar\rho}) = H^0(G_K, \mathrm{ad}_{\bar\rho}) \cong \kappa$, and in the same way in which we deduced the isomorphism (15), we obtain $H^2(G_K, \mathrm{ad}_{\bar\rho_1\subset\bar\rho}) \cong H^2(G_K, \mathrm{ad}_{\bar\rho_1} \oplus \mathrm{ad}_{\bar\rho_2})$, and by local Tate duality the latter has dimension $2\delta_p$. In total, we find

$$h = (d \cdot \dim \mathrm{ad}_{\bar\rho_1\subset\bar\rho} + 1 + 2\delta_p) - 2(d + 1 + \delta_p) = d \dim \mathrm{ad}^{00}_{\bar\rho_1\subset\bar\rho} - 1.$$

One also easily adapts the proof of Corollary 3.4.3. The following result summarizes the conclusions needed later. For the last assertion, one does not consider the map $\det^2$ but uses a variant of Theorem 3.2.4(e).

Proposition 3.4.6 *Let κ be finite or a local field of characteristic p, let $\Lambda = \kappa$ and suppose that $\bar\rho$ is a nonsplit extension of $\bar\rho_2$ by $\bar\rho_1$ and that $\bar\rho_1$ is not isomorphic to $\bar\rho_2(j)$ for $j \in \{0, \pm 1\}$.*

If $H^0(G_K, \overline{\mathrm{ad}}_{\bar\rho_i}(1)) = 0$ for $i = 1, 2$, then:

(a) $\det^2\colon \mathcal{D}_{\bar\rho_1\subset\bar\rho} \to \mathcal{D}_{\det\bar\rho_1} \times \mathcal{D}_{\det\bar\rho_2}$ *is formally smooth of relative dimension $d(n^2 - n_1 n_2 - 2) - 1$.*
(b) *The ring $(R^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho})_{\mathrm{red}}$ is formally smooth over κ of dimension $d(n^2 - n_1 n_2) + 1$.*

If on the other hand $H^0(G_K, \mathrm{ad}_{\bar\rho_i}(1)) = 0$ for $i = 1, 2$, then $R^{\mathrm{univ}}_{\bar\rho_1\subset\bar\rho}$ is formally smooth over κ of dimension $d(n^2 - n_1 n_2) + 1$.

4. Pseudocharacters and their deformations

In this section, we recall main definitions and results on polynomial laws and pseudocharacters. We assume that the reader is familiar with [Che11; Che14; WE18]. Nevertheless, we will give many reminders. Each subsection gives a short survey over its contents. In Proposition 4.3.9, we prove an analog of the locus of reducibility of [BC09, Proposition 1.5.1] in the context of pseudocharacters. In Subsections 4.5 and 4.6, we introduce twisting and induction as operations on pseudocharacters. In Proposition 4.7.4, we sketch the existence of a universal ring for continuous pseudodeformation where the residue field is a local field and in Proposition 4.7.6 we consider such rings under change of coefficients. These are adaptions of well-known results. Subsection 4.8 presents in detail several results on dimension 1 points in universal pseudodeformation spaces.

Throughout this section, A will be a commutative unital ring with $0 \neq 1$. If A is local, we write $\mathfrak{m}_A$ for its maximal ideal and $\kappa(A)$ for its residue field. We write $\mathcal{A}lg_A$ for the category of A-algebras and $\mathcal{CA}lg_A$ for the full subcategory of commutative A-algebras. By R, S, we always denote objects of $\mathcal{A}lg_A$ and by B an object of $\mathcal{CA}lg_A$. For an A-algebra R, we denote by R^o the A-algebra with the multiplication of R reversed. By G, we denote a group and by $B[G]$ the group algebra over B for any $B \in \mathcal{CA}lg_A$. The letters m, n (also with indices) will denote nonnegative integers. If $\rho\colon G \to \mathrm{GL}_n(B)$ is a representation, then by $\rho^{\mathrm{lin}}\colon B[G] \to \mathrm{Mat}_{n\times n}(B)$ we denote its linearization given by $\sum_i b_i g_i \mapsto \sum_i b_i \rho(g_i)$.

4.1. Pseudocharacters

In this subsection, we introduce pseudocharacters, Azumaya algebras and Cayley–Hamilton A-algebras. Of particular importance is Proposition 4.1.10, which says that a pseudocharacter is determined by its characteristic polynomial coefficients.

For an A-module M, we define the functor $\underline{M}\colon \mathcal{CA}lg_A \to \mathit{Sets},\ B \mapsto M \otimes_A B$.

Definition 4.1.4 [Che14, §1.1]. Let M and N be A-modules.

(a) An *A-polynomial law* $P\colon M \to N$ is a natural transformation $\underline{M} \to \underline{N}$. that is, P is a family of maps $P_B\colon M \otimes_A B \to N \otimes_A B$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ that induce commutative diagrams for every morphism in $\mathcal{CA}lg_A$.

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(b) An A-polynomial law $P\colon M \longrightarrow N$ is called *homogeneous of degree n* if

$$P_B(bx) = b^n P_B(x) \qquad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A),\ b \in B \text{ and } x \in M \otimes_A B.$$

We let $\mathcal{P}_A^n(M, N)$ denote the set of all such.

Let S, S' be objects in $\mathcal{A}lg_A$ so that in particular they are A-modules.

(c) An A-polynomial law $P\colon S \longrightarrow S'$ is called *multiplicative* if

$$P_B(1) = 1 \quad \text{and} \quad P_B(xy) = P_B(x)P_B(y) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A) \text{ and } x, y \in S \otimes_A B.$$

(d) We write $\mathcal{M}_A^n(S, S')$ for the set of multiplicative A-polynomial laws $P\colon S \longrightarrow S'$ that are homogeneous of degree n.
(d) A *pseudocharacter on S of dimension n* is an A-polynomial law $D : S \longrightarrow A$ that is multiplicative and homogeneous of degree n. We let $\mathcal{P}s\mathcal{R}_S^n(A)$ be the set of all such.
(e) If $S = A[G]$ in (d), we call D an *A-valued pseudocharacter on G of dimension n*, and we write $\mathcal{P}s\mathcal{R}_G^n(A)$ for $\mathcal{P}s\mathcal{R}_{A[G]}^n(A)$; occasionally we write $D\colon G \longrightarrow A$ for $D\colon A[G] \longrightarrow A$, and then we explicitly refer to D as a pseudocharacter on G.

Remark 4.1.2 [Che14, after Example 1.2]. A homogeneous polynomial law P of degree n is uniquely determined by $P_{A[T_1,\ldots,T_m]}\colon M[T_1, \ldots, T_m] \to N[T_1, \ldots, T_m]$ for all $m \geq 0$.

Facts 4.1.3. The following facts are easy to verify.

(a) The only multiplicative polynomial law of degree zero is the constant map with value 1.
(b) Multiplicative polynomial laws that are homogeneous of degree 1 are A-algebra homomorphisms and vice versa.
(c) The composition of polynomial laws is a polynomial law; if both are homogeneous, the composition is homogeneous and its degree is the product of the individual degrees.
(d) The composition of multiplicative polynomial laws is multiplicative.
(e) If $D\colon S \longrightarrow A$ is an A-valued pseudocharacter, then for any $B \in \mathcal{CA}lg_A$, the base change $D \otimes_A B\colon S \otimes_A B \longrightarrow B$ is a B-valued pseudocharacter.

Definition 4.1.4 (pseudocharacter of a representation). Let $\rho\colon G \longrightarrow \mathrm{GL}_n(A)$ be a representation. The pseudocharacter D_ρ attached to ρ is the polynomial law that to any $B \in \mathcal{CA}lg_A$ attaches the composition of the determinant $\det\colon \mathrm{Mat}_{n\times n}(B) \longrightarrow B$ with the morphism $(\rho \otimes_A B)^{\mathrm{lin}}$.

Let D be a pseudocharacter on G over A. Because D is multiplicative, we have $D(gh) = D(g)D(h)$ for $g, h \in G$ and $D(1) = 1$. Thus, the map $\varphi\colon G \longrightarrow A^\times, g \mapsto D_A(g)$ is a group homomorphism, and Definition 4.1.4 associates the pseudocharacter D_φ to φ.

Definition 4.1.5 (Determinant of a pseudocharacter). We call $\det(D) := D_\varphi$ *the determinant of D*.

Reminder 4.1.6. From [Mil80, §IV.1 – IV.2], we recall the notion of Azumaya algebra and some of its properties. Let first A be a local ring with residue field κ. An algebra $C \in \mathcal{A}lg_A$ is called an *Azumaya A-algebra* if C is free of finite rank as an A-module and if in $\mathcal{A}lg_A$ the map

$$C \otimes_A C^\circ \longrightarrow \mathrm{End}_A(C), \quad c \otimes c' \longmapsto (x \mapsto cxc'),$$

is an isomorphism; equivalently, there exists a finite étale morphism $A \longrightarrow B$ such that $C \otimes_A B \cong \mathrm{Mat}_{m\times m}(B)$ for some m. One calls m the *degree of* C; it satisfies $\mathrm{rank}_A C = m^2$. Moreover, C carries a *reduced norm map* $\det_C\colon C \longrightarrow A$ characterized by the property that $\det_C \otimes_A B$ is the determinant on $\mathrm{Mat}_{m\times m}(B)$. Its extension to $C[t]$ defines a reduced characteristic polynomial $\chi_c := \det_{C[t]}(t - c) \in A[t]$, monic of degree m, for any $c \in C$. Lastly, $C \otimes \kappa$ is a central simple algebra over κ.

Let now X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{C}$ is called an *Azumaya algebra over X* if $\mathcal{C}$ is coherent as an $\mathcal{O}_X$-module and if for all $x \in X$, the stalk $\mathcal{C}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$; equivalently, there exists

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a Zariski cover $\{U_i\}$ of X and for each i a finite étale surjective cover $U_i' \to U_i$ and an isomorphism $\mathcal{C} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i'} \xrightarrow{\simeq} \mathrm{Mat}_{m_i \times m_i}(\mathcal{O}_{U_i'})$ for suitable $m_i \in \mathbb{N}_{\geq 1}$. In particular, the degree function $\underline{m}\colon X \to \mathbb{N}_{\geq 1}$ such that $\mathrm{rank}_{\mathcal{O}_X} \mathcal{C} = \underline{m}^2$ is locally constant. Also, the reduced norm exists as a map $\det_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{O}_X$. For $X = \operatorname{Spec} A$ affine, one calls $C = \mathcal{C}(X)$ an Azumaya A-algebra.

Example 4.1.7. Let C be an Azumaya A-algebra of degree n with reduced norm $\det_C : C \to A$.

(a) The family of reduced norms $(\det_C \otimes_A B\colon C \otimes_A B \to B)_{B \in \mathcal{CA}lg_A}$ defines a pseudocharacter, also called $\det_C$, of dimension n; see [Che14, §1.5].
(b) If $D\colon C \to A$ is any pseudocharacter of dimension n', then by [Che14, Lemma 2.15], we have $n|n'$ and $D = \det_C^{n'/n}$.

An important notion for pseudocharacters is that of characteristic polynomial.

Lemma 4.1.8 [Che14, §1.10]. *Let $D \in \mathcal{P}s\mathcal{R}^n_S(A)$. Define $\chi_{D,B}(\cdot,t)\colon S \otimes_A B \longrightarrow B[t]$ by the formula $s \mapsto D_{B[t]}(t-s)$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ and $s \in S \otimes_A B$. Then the following hold:*

(a) *$\chi_D(\cdot,t)\colon S \to A[t]$ is a multiplicative homogeneous polynomial law of degree n.*
(b) *There exist unique A-polynomial laws $\Lambda_{D,i}\colon S \to A$ of degree i, $i = 0,\dots,n$ such that*

$$\chi_D(\cdot,t) = \sum_{i=0}^{n} (-1)^i \Lambda_{D,i}(\cdot) t^{n-i}.$$

(c) *$\Lambda_{D,0} = 1$ and $\Lambda_{D,n} = D$.*
(d) *The maps $s \mapsto \sum_{i=0}^n (-1)^i \Lambda_{D,i}(s) s^{n-i}$ for all $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$ and $s \in S \otimes_A B$ define a multiplicative A-polynomial law $\chi_D\colon S \to S$ that is homogeneous of degree n.*

Definition 4.1.9. [Che14, §1.10] Let S, D, $\chi_D(\cdot,t)$ and $\Lambda_{D,i}$ be as in Lemma 4.1.8.

(a) The polynomial law $\chi_D(\cdot,t)$ is called the *characteristic polynomial of* D.
(b) The polynomial law $\Lambda_{D,i}$ is called the i^{th} *characteristic polynomial coefficient of* D.
(c) The A-linear map $\tau_D := \Lambda_{D,1}$ is called the *trace associated with* D.

An important tool to extract properties of multiplicative homogeneous polynomial laws is Amitsur's formula; see [Che14, Formula (1.5)]. It expresses values of such laws in terms of characteristic polynomial coefficients. Using this, one deduces the following result:

Proposition 4.1.10 [Che14, Corollary 1.14], [WE13, 1.1.9.15]. *Let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$.*

(a) *The characteristic polynomial coefficients $(\Lambda_{D,i}\colon G \to A)_{i=1,\dots,n}$ characterize D.*
(b) *Let $C \subset A$ be the subring generated by $\{\Lambda_{D,i}(g) : g \in G, i = 1,\dots,n\}$. Then D factors through a unique C-valued pseudocharacter D_C on G of dimension n.*

A natural operation on pseudocharacters is the formation of direct sums.[2]

Definition 4.1.11 [WE13, §1.1.11]. Let S, S_1, S_2 be $\mathcal{A}lg_A$ and B in $\mathcal{CA}lg_A$.

(a) The *direct sum* of multiplicative homogeneous A-polynomial laws $P_i\colon S_i \to B$ of degree n_i, $i = 1,2$, is the multiplicative homogeneous A-polynomial law of degree $n_1 + n_2$ given by

$$P_1 \oplus P_2\colon S_1 \times S_2 \to B, \quad (x_1,x_2) \mapsto P_1(x_1)P_2(x_2).$$

(b) The *direct sum* of pseudorepresentations $D_i\colon S \to A$ of dimension n_i, $i = 1,2$, is the pseudocharacter of dimension $n_1 + n_2$ given by $D_1 \oplus D_2\colon S \to A, x \mapsto D_1(x)D_2(x)$.

Remark 4.1.12. Note that $\det^{n'/n}$ from Example 4.1.7 could now also be written as $\det^{\oplus(n'/n)}$.

[2]We use the term direct sum in analogy with the case of representations; Chenevier uses the term product.

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Lemma 4.1.13 [WE13, Lemma 1.1.11.7]. *For $i = 1, 2$, let $\rho_i : G \to \mathrm{GL}_{n_i}(A)$ be a representation, and set $\rho := \rho_1 \oplus \rho_2$. Then $D_\rho = D_{\rho_1} \oplus D_{\rho_2}$ for the associated pseudocharacters from Definition 4.1.4.*

Lemma 4.1.14 [Che14, Lemma 2.2]. *Let S_1, S_2 be in $\mathcal{A}lg_A$. Let $B \neq 0$ be in $\mathcal{CA}lg_A$ such that* Spec B *is connected. Let $P : S_1 \times S_2 \longrightarrow B$ be a multiplicative A-polynomial law that is homogeneous of degree n. Then there exist for $i = 1, 2$ unique $n_i \geq 0$ with $n_1 + n_2 = n$ and multiplicative homogeneous A-polynomial laws $P_i : S_i \to B$ of degree n_i such that $P = P_1 \oplus P_2$.*

To any $D \in \mathcal{P}s\mathcal{R}^n_S(A)$, one can naturally assign its kernel $\mathrm{Ker}(D)$.

Definition 4.1.15 [Che14, 1.17]. Let $P : M \to N$ be a polynomial law for A-modules M, N.

(a) The *kernel* $\ker(P)$ *of P* is the A-submodule of M defined as

$$\{x \in M : P(x \otimes b + m) = P(m) \quad \text{for all } B \in \mathrm{Ob}(\mathcal{CA}lg_A),\ b \in B \text{ and } m \in M \otimes_A B\}$$

(b) If $\ker(P) = 0$, then P is called *faithful*.

Proposition 4.1.16 [Che14, 1.19–1.21]. *For $D \in \mathcal{P}s\mathcal{R}^n_S(A)$, the following hold.*

(a) $\ker D$ *is a two-sided ideal of S; there exists a unique $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/\ker D}(A)$ such that $D = \widetilde{D} \circ \pi$ for π the projection $S \to S/\ker D$, and* $\ker D$ *is maximal with this property.*
(b) *If C is an Azumaya A-algebra, then its reduced norm* $\det_C$ *is faithful.*

Over fields, the following is a fundamental result on faithful pseudocharacters.

Theorem 4.1.17 [Che14, Theorem 2.16]. *Let k be a field such that k is perfect, or k has characteristic $p > 0$ and $[k : k^p] < \infty$. Let $D : S \to k$ be a pseudocharacter of dimension n. Then $S/\ker D$ is of finite k-dimension and semisimple as a ring.*

Choose a k-algebra isomorphism $S/\ker D \xrightarrow{\simeq} \prod_{i=1}^s S_i$, where each S_i is a simple k-algebra. Let n_i be the degree of S_i over its center k_i, let $f_i := [k_i \cap k^{\mathrm{sep}} : k]$, and let q_i be the smallest p-power such that $k_i^{q_i} \subset k^{\mathrm{sep}}$; note that all $q_i = 1$ if k is perfect. Then under the above isomorphism one has

$$D = \bigoplus_{i=1}^{s} \det_{S_i}^{\oplus m_i}$$

for some uniquely determined integers $m_i \geq 1$, and one has $n = \sum_i m_i n_i q_i f_i$.

Over algebraically closed field, the following consequence of Theorem 4.1.17 is important.

Theorem 4.1.18 [Che14, Theorem 2.12]. *Suppose that k is an algebraically closed field and S is a k-algebra. If $D : S \to k$ is an n-dimensional pseudocharacter, then there is a semisimple representation $\rho_D : S \to \mathrm{Mat}_{n\times n}(k)$ unique up to isomorphism with associated pseudocharacter D, and one has* $\ker \rho_D^{\mathrm{lin}} = \ker D$.

Definition 4.1.19. Let k be a field, and let $D \in \mathcal{P}s\mathcal{R}^n_G(k)$.

(a) We call $\rho_{D\otimes_k k^{\mathrm{alg}}}$ from Theorem 4.1.18 the *semisimple representation associated to $D \otimes_k k^{\mathrm{alg}}$*.
(b) We call D
  (1) *irreducible* if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is irreducible and reducible otherwise;
  (2) *multiplicity free* if $\rho_{D\otimes_k k^{\mathrm{alg}}}$ is a direct sum of pairwise nonisomorphic irreducible k^{alg}-linear representations of $S \otimes_k k^{\mathrm{alg}}$;
  (3) *split* if $D = D_\rho$ for some representation $\rho : S \to \mathrm{Mat}_{n\times n}(k)$;

Note that if k is finite, then every irreducible representation is split.
We record the following consequence that will be used in the proof of Lemma 12.

Corollary 4.1.20. *Let $\overline{D} : \mathbb{F}[G] \to \mathbb{F}$ be an n-dimensional pseudocharacter. Let $\mathbb{F}'$ be the extension of $\mathbb{F}$ of degree $n!$. Then $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ is a direct sum of irreducible representations.*

Proof. Over finite fields the Brauer group is zero. Thus, by Theorem 4.1.14 we have an isomorphism $\mathbb{F}[G]/\ker D \xrightarrow{\simeq} \prod_{i=1}^{s} \mathrm{Mat}_{d_i\times d_i}(\mathbb{F}_i)$ for integers $d_i \geq 1$ and finite field $\mathbb{F}_i$ over $\mathbb{F}$ such that $n = \sum_i d_i f_i m_i$ for $f_i = [\mathbb{F}_i : \mathbb{F}]$. In particular, all f_i divide $n!$ and hence $\mathbb{F}' \supset \mathbb{F}_i$ for all i. Over perfect fields semisimple rings are absolutely semisimple (see Definition A.2.1 and Remark A.2.2) and thus $\mathbb{F}'[G]/\ker(D \otimes_{\mathbb{F}} \mathbb{F}') \xrightarrow{\simeq} \prod_{i=1}^{s}\prod_{j=1}^{f_i} \mathrm{Mat}_{d_i\times d_i}(\mathbb{F}')$. We conclude using Lemma 4.1.14, Example 4.1.7(b) and Remark 4.1.12. □

Next, we recall the concept of the Cayley–Hamilton property for pseudocharacters.

Definition 4.1.21 [Che14, 1.17]. Let S be an A-algebra, and let D be in $\mathcal{P}s\mathcal{R}^n_S(A)$.

(a) The *Cayley–Hamilton ideal* $\mathrm{CH}(D)$ *of* D is the two-sided ideal of S generated by the coefficients of the polynomials[3]

$$\chi_{D,A[t_1,\dots,t_m]}(s) \in S[t_1,\dots,t_m],$$

where m ranges over all positive integers and s over all elements of $S[t_1,\dots,t_m]$.[4]

(b) One calls D *Cayley–Hamilton* if $\mathrm{CH}(D) = 0$, or, equivalently, if χ_D is identically zero.

Proposition 4.1.22 [Che14,1.20f.], [WE13,1.1.8.6]. *For $D \in \mathcal{P}s\mathcal{R}^n_S(A)$, the following hold.*

(a) $\ker(D) \supset \mathrm{CH}(D)$*, and hence D factors via some $\widetilde{D} \in \mathcal{P}s\mathcal{R}^n_{S/\mathrm{CH}(D)}(A)$.*
(b) *If D is Cayley–Hamilton and $S' \subset S$ is any A-subalgebra, then $D|_{S'}$ is Cayley–Hamilton.*
(c) *For any morphism $S \to S'$ in $\mathcal{A}lg_A$, one has $S'/\mathrm{CH}(D \otimes_S S') \cong (S/\mathrm{CH}(D)) \otimes_S S'$.*

Definition 4.1.23 [Che14, 1.17]. Let S be an A-algebra, and let D be in $\mathcal{P}s\mathcal{R}^n_S(A)$.

(a) One calls $S^{\mathrm{CH}}_D := S/\mathrm{CH}(D)$ the *Cayley–Hamilton quotient of S with respect to D.*
(b) One calls the induced A-algebra homomorphism $\rho^{\mathrm{CH}}_D : S \to S/\mathrm{CH}(D)$ D the *Cayley–Hamilton representation attached to D.*

Any pseudocharacter $D \in \mathcal{P}s\mathcal{R}^n_S(A)$ possesses a factorization

$$S \xrightarrow{\rho^{\mathrm{CH}}_D} S^{\mathrm{CH}}_D \xrightarrow{\widetilde{D}} A \tag{16}$$

with $\widetilde{D}$ from Proposition 4.1.22(a). In the special case $S = A[G]$, the factorization is a composition of a group homomorphism $G \to (S^{\mathrm{CH}}_D)^\times$ with $\widetilde{D}$, that is, $D = \widetilde{D} \circ \rho^{\mathrm{CH}}_D : A[G] \to A$. Because of the following result and the good behavior of $\mathrm{CH}(\cdot)$ under base change, one might think of ρ^{CH}_D as a natural substitute of a representation ρ with $D = D_\rho$ when such a representation does not exist. First, we need one more piece of notation.

Definition 4.1.24. Let $B \in \mathcal{CA}lg_A$, $D \in \mathcal{P}s\mathcal{R}^n_G(B)$ and $X := \operatorname{Spec} B$. Let $x \in X$ with residue homomorphism $\pi_x : B \to \kappa(x)$, and let $\overline{x}$ be a geometric point of X above x so that $\kappa(x) \hookrightarrow \kappa(\overline{x})$.

(a) We call $D_x := \pi_x \circ D$ the *pseudocharacter of D at x* and set $D_{\bar{x}} := D_x \otimes_{\kappa(x)} \kappa(\overline{x})$.
(b) We call $\rho_{\overline{x}} := \rho_{D_{\overline{x}}} : G \to \mathrm{GL}_n(\kappa(\overline{x}))$ the *(semisimple) representation at $\bar{x}$*.[5]
(c) We say that x has a property if $D_{\overline{x}}$ (or $\rho_{\overline{x}}$) satisfies this property.

If B is a universal ring for some space of pseudocharacters and $x \in \operatorname{Spec} B$, then by writing D_x it will be implicitly understood that D refers to the corresponding universal pseudocharacter.

The following is a significant generalization of Theorem 4.1.17 to families.

Proposition 4.1.25 (Cf. [Che14, Corollary 2.23]). *Let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be such that D_x is irreducible for all $x \in \operatorname{Spec} A$. Then $C := A[G]^{\mathrm{CH}}_D$ is an A-Azumaya algebra of degree n, and $D = \det_C \circ \rho^{\mathrm{CH}}_D$ for $\rho^{\mathrm{CH}}_D : A[G] \to C$ the Cayley–Hamilton representation restricted to G.*

[3]Observe that $\chi_{D,A[t_1,\dots,t_m]}$ is from Lemma 4.1.8(d).
[4]It suffices to let the s range over all elements of the form $\sum_{j=1}^{m} s_j t_j$ with $s_j \in S$.
[5]We sometimes ignore the subtlety of geometric points and simply write ρ_x.

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Remark 4.1.26. We shall use the notation ρ_D for $D \in \mathcal{PsR}^n_G(A)$ in two situations: Either A is an algebraically closed field and then it is the semisimple representation ρ_D from Theorem 4.1.18. Or A is arbitrary and D_x is irreducible for all $x \in \operatorname{Spec} A$, and then it is an abbreviation for ρ^{CH}_D. Because of Proposition 4.1.25, this assignment is well-defined.

4.2. Universal rings of pseudocharacters

Here, we recall the existence of a universal pseudodeformation ring and that irreducible points form an open subscheme. Moreover, we introduce morphisms related to the addition of pseudocharacters.

Proposition 4.2.1 [Che14, Proposition 1.6, Example 1.7]. *The functor $\mathcal{PsR}^n_S(\,\cdot\,)\colon \mathcal{CAlg}_A \to Sets$ is representable for any S in $\mathcal{Alg}_A$ by some ring $R^{\mathrm{univ}}_{S,n} \in \mathcal{CAlg}_A$. Moreover, for any $B \in \mathcal{CAlg}_A$, the natural map $B \otimes_A R^{\mathrm{univ}}_{S,n} \to R^{\mathrm{univ}}_{B\otimes_A S,n}$ is an isomorphism.*

The above means that there is a natural isomorphism $\operatorname{Hom}_{\mathcal{CAlg}}(R^{\mathrm{univ}}_{S,n},\,\cdot\,) \to \mathcal{PsR}^n_S(\,\cdot\,)$. Let the pseudocharacter corresponding to $\mathrm{id}_{R^{\mathrm{univ}}_{S,n}}$ be

$$D^{\mathrm{univ}}_{S,n}\colon S \otimes_A R^{\mathrm{univ}}_{S,n} \longrightarrow R^{\mathrm{univ}}_{S,n}.$$

Definition 4.2.2. The commutative A-algebra $R^{\mathrm{univ}}_{S,n}$ and the A-scheme $X^{\mathrm{univ}}_{S,n} := \operatorname{Spec} R^{\mathrm{univ}}_{S,n}$ are called the *n-dimensional universal pseudocharacter ring* and *space*, respectively, and $D^{\mathrm{univ}}_{S,n}$ is called the *n-dimensional universal pseudocharacter*.

For $S = \mathbb{Z}[G]$, we abbreviate $R^{\mathrm{univ}}_{G,n} := R^{\mathrm{univ}}_{S,n}$, $D^{\mathrm{univ}}_{G,n} := D^{\mathrm{univ}}_{S,n}$ and $X^{\mathrm{univ}}_{G,n} := X^{\mathrm{univ}}_{S,n}$.

Remark 4.2.3.

(a) In [Che14], the ring $R^{\mathrm{univ}}_{S,n}$ is denoted by $\Gamma^n_A(S)^{\mathrm{ab}}$; in our notation A is implicit in the structural map of S as an A-algebra.
(b) For A-schemes X there is an obvious notion of *$\mathcal{O}(X)$-valued pseudocharacter $S \to \mathcal{O}(X)$ of dimension n*. The space $X^{\mathrm{univ}}_{S,n}$ represents the resulting functor of pseudocharacters on the category of A-schemes.

Example 4.2.4. Recall the determinant $\det(D) \in \mathcal{PsR}^1_G(A)$ of any $D \in \mathcal{PsR}^n_G(A)$ from Definition 4.1.5. If we apply this to $D^{\mathrm{univ}}_{A[G],n}$, we obtain

$$\det(D^{\mathrm{univ}}_{A[G],n}) \in \mathcal{PsR}^1_{S\otimes_A R^{\mathrm{univ}}_{A[G],n}}(R^{\mathrm{univ}}_{A[G],n}).$$

The last assertion in Proposition 4.2.1 and the universality of $R^{\mathrm{univ}}_{A[G],1}$ now yields a homomorphism in $\mathcal{CAlg}_A$

$$\det\colon R^{\mathrm{univ}}_{A[G],1} \to R^{\mathrm{univ}}_{A[G],n}$$

and an induced morphism of schemes $\det\colon X^{\mathrm{univ}}_{A[G],n} \to X^{\mathrm{univ}}_{A[G],1}$, both of which we denote by det.

Lemma 4.2.5 [Rob63, Théorème III.4]. *The following assertions hold:*

(a) *The canonical map $R^{\mathrm{univ}}_{S_1\times S_2,n} \to \bigoplus_{i=0}^n R^{\mathrm{univ}}_{S_1,i} \otimes R^{\mathrm{univ}}_{S_2,n-i}$, induced from the universal property of these rings, is an isomorphism in $\mathcal{CAlg}_A$.*
(b) *Let $B \neq 0$ be in $\mathcal{CAlg}_A$ such that* $\operatorname{Spec} B$ *is connected. Then any A-algebra homomorphism $R^{\mathrm{univ}}_{S_1\times S_2,n} \to B$ corresponding to P factors via some summand $R^{\mathrm{univ}}_{S_1,i} \otimes R^{\mathrm{univ}}_{S_2,n-i}$ in Part (a).*

Corollary 4.2.6 [WE13, Lemma 1.1.11.7]. *Suppose $n_1 + n_2 = n$ for $n_i \geq 0$. Then the map*

$$\iota_{n_1,n_2}\colon X^{\mathrm{univ}}_{S,n_1} \times_A X^{\mathrm{univ}}_{S,n_2} \longrightarrow X^{\mathrm{univ}}_{S,n},\, (D_1, D_2) \mapsto D_1 \oplus D_2$$

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is a morphism of affine A-schemes that corresponds to the ring homomorphism

$$R^{\mathrm{univ}}_{S,n} \xrightarrow{\Delta} R^{\mathrm{univ}}_{S\times S,n} \xrightarrow{4.2.5(a)} R^{\mathrm{univ}}_{S,n_1} \otimes R^{\mathrm{univ}}_{S,n_2},$$

where Δ is induced from the diagonal map $S \to S \times S$ and the universality of the rings.

4.3. Generalized matrix algebras

Generalized matrix algebras are important in the study of Cayley–Hamilton pseudocharacters over Henselian local rings and were introduced for that purpose in [BC09, §1.3] in the context of Taylor-pseudocharacters. This subsection recalls some basic result. In Proposition 4.3.9, we shall generalize [BC09, Proposition 1.5.1], in Proposition 4.3.9 on the ideal of total reducibility to pseudocharacters.

Definition 4.3.1 (Cf. [BC09, Definition 1.3.1]). A *generalized matrix algebra (or simply GMA) over A (of type $(n_1, \dots, n_r)$)* is an A-algebra S together with

(i) a set of orthogonal idempotents $e_1, \dots, e_r \in S$ with $\sum_{i=1}^r e_i = 1_S$, and

(ii) a set of A-algebra isomorphisms $\psi_i : e_i S e_i \xrightarrow{\sim} \mathrm{Mat}_{n_i\times n_i}(A)$ for $i = 1, \dots, r$

such that the associated trace map $\tau : S \to A, x \mapsto \sum_{i=1} \mathrm{tr}(\psi_i(e_i x e_i))$ is central, that is, it satisfies $\tau(xy) = \tau(yx)$ for all $x, y \in S$. The tuple $\mathcal{E} := \{e_i, \psi_i\}_{i=1,\dots,r}$ is called the *data of idempotents of* S. If we wish to emphasize the entire structure of a GMA, we write $(S, \mathcal{E})$ instead of S. The dimension of S will be $\sum_i n_i$.

Notation 4.3.2. Let S be an GMA over A of type $(n_1, \dots, n_r)$. For $1 \le i \le r$ and $1 \le k, l \le n_i$, we denote by $E_i^{k,l}$ the unique element in $e_i S e_i$ that maps under ψ_i to the matrix in $\mathrm{Mat}_{n_i\times n_i}(A)$ that has 1 in the (k, l)-entry and 0 everywhere else. For later use, we also introduce elements $E^j := E_{i+1}^{i',i'}$ for $j = 1, \dots, n$, where $i, i' \ge 1$ are unique such that $j = n_1 + \ldots + n_i + i'$ with $1 \le i' \le n_{i+1}$. We write $\mathcal{A}^j$ for $E^j S E^j$ and φ^j for the isomorphism $\mathcal{A}^j \to A$ induced from τ.

The following result explains why GMA are generalizations of matrix algebras.

Lemma 4.3.3 (Structure of a GMA [BC09, p. 21ff.]). *The following assertions hold:*

(a) *Let $(S, \mathcal{E})$ be a GMA over A of type $(n_1, \dots, n_r)$, and define the following data:*
   (1) *A-modules $\mathcal{A}_{i,j} := E_i^{1,1} S E_j^{1,1}$ for $1 \le i, j \le r$,*
   (2) *isomorphisms $\mathcal{A}_{i,i} \cong A$ under τ for $i = 1, \dots, r$,*
   (3) *A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ induced from the product in S.*
   Then they satisfy the following conditions:
   (UNIT) *For $1 \le i, j \le r$, we have $\mathcal{A}_{i,i} = A$ and both $\varphi_{i,i,j}$ and $\varphi_{i,j,j}$ agree with the A-module structure on $A_{i,j}$.*
   (ASSO) *For $1 \le i, j, k, l \le r$ and $x \otimes y \otimes z \in \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \otimes_A \mathcal{A}_{k,l}$, we have*

$$\varphi_{i,k,l}\big(\varphi_{i,j,k}(x \otimes y) \otimes z\big) = \varphi_{i,j,l}\big(x \otimes \varphi_{j,k,l}(y \otimes z)\big) \quad \text{in } \mathcal{A}_{i,l}.$$

   (COMM) *For $1 \le i, j \le r$, $x \in \mathcal{A}_{i,j}$ and $y \in \mathcal{A}_{j,i}$, we have $\varphi_{i,j,i}(x \otimes y) = \varphi_{j,i,j}(y \otimes x)$.*
   Then the structures in (1)–(3) induce an A-algebra structure on

$$\begin{pmatrix} \mathrm{Mat}_{n_1\times n_1}(\mathcal{A}_{1,1}) & \cdots & \mathrm{Mat}_{n_1\times n_r}(\mathcal{A}_{1,r}) \\ \vdots & \ddots & \vdots \\ \mathrm{Mat}_{n_r\times n_1}(\mathcal{A}_{r,1}) & \cdots & \mathrm{Mat}_{n_r\times n_r}(\mathcal{A}_{r,r}) \end{pmatrix} \tag{17}$$

   and the latter is isomorphic to S.

(b) *Conversely, suppose we are given a family $(\mathcal{A}_{i,j})_{1\le i,j\le r}$ of A-modules together with A-linear maps $\varphi_{i,j,k} : \mathcal{A}_{i,j} \otimes_A \mathcal{A}_{j,k} \to \mathcal{A}_{i,k}$ for $1 \le i, j, k \le r$ satisfying the above conditions (UNIT), (ASSO)*

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and (COMM). Then there is a unique structure of a GMA of type $(n_1, \dots, n_r)$ on the A-module $S := \oplus_{i,j=1}^r \mathrm{Mat}_{n_i \times n_j}(\mathcal{A}_{i,j})$.

Next, we provide some technical lemmas:

Lemma 4.3.4. *Let S be a GMA over A of type $(n_1, \dots, n_r)$ over A, and $B \in \mathrm{Ob}(\mathcal{CA}lg_A)$. Then $S \otimes_A B$ is a GMA over B of type $(n_1, \dots, n_r)$.*

The proof of Lemma 4.3.4 is straightforward and left as an exercise.

Proposition 4.3.5 [WE18, Proposition 2.23]. *Given a GMA $(S, \mathcal{E})$ over A of dimension n, there exists a natural n-dimensional Cayley–Hamilton pseudocharacter $\det_{(S,\mathcal{E})} : S \to A$, called the determinant of the GMA $(S, \mathcal{E})$, and given, for any B in $\mathcal{CA}lg_A$, by the formula*

$$\det_{(S.\mathcal{E})}(x) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{\text{cycles } \gamma \text{ of } \sigma} \varphi^{l_0}\Big(\prod_{l \in \gamma} E^l x E^{\sigma(l)} \Big) \tag{18}$$

for any $x \in R \otimes_A B$. Here, the product is first over the cycles γ in the cycle decomposition of σ and then over the elements l of the cycle γ taken in the order that they appear in the cycle, where l_0 is a choice of initial element of γ, and φ^{l_0} is from Notation 4.3.2. We also have $\tau = \Lambda^1_{D_\mathcal{E}}$.

The next results are auxiliary for Proposition 4.3.9 on the locus of reducibility of a GMA.

Lemma 4.3.6 [Che14, Lemma 1.12(i)]. *One has $D(1 + ss') = D(1 + s's)$ for all $s, s' \in S$.*

Lemma 4.3.7. *Let $(S, \mathcal{E})$ be a GMA, and let $D \colon S \to A$ a pseudocharacter. Then for any $x \in \mathrm{Mat}_{n_i \times n_j}(\mathcal{A}_{i,j})$ for some $1 \le i, j \le r$ with $i \ne j$, we have $D(1 + e_i x e_j) = 1$.*

Proof. By Lemma 4.3.6 we have $D(1 + e_i x e_j) = D(1 + e_j e_i x) = D(1) = 1$. □

Lemma 4.3.8 [Che14, Lemma 2.4]. *Let S be an A-algebra, $e \in S$ be an idempotent, and $D \colon S \to A$ be a pseudocharacter of dimension n. Suppose that* Spec(A) *is connected.*

(a) *The polynomial law $D_e \colon eSe \to A$, $s \mapsto D(s + 1 - e)$, is a pseudocharacter; its dimension $r(e)$ satisfies $r(e) \le n$ and one has $r(1 - e) + r(e) = n$.*
(b) *The restriction of D to the A-subalgebra $eSe \oplus (1 - e)S(1 - e)$ is the sum $D_e \oplus D_{1-e}$. It is a pseudocharacter of dimension n.*
(c) *If D is faithful or Cayley–Hamilton, then D_e is faithful or Cayley–Hamilton, respectively.*
(d) *Suppose that D is Cayley–Hamilton. Then $e = 1$ if and only if $D(e) = 1$, and $e = 0$ if and only if $r(e) = 0$. If $e_1, \dots, e_s$ is a family of nonzero orthogonal idempotents of S, then $s \le n$ and $\sum_{i=1}^s r(e_i) \le n$. Further, $\sum_{i=1}^s r(e_i) = n$ if and only if $\sum_{i=1}^s e_i = 1$.*

The next result is the adaption of [BC09, Proposition 1.5.1] to pseudocharacters.

Proposition 4.3.9. *Let $(S, \mathcal{E})$ be a GMA over A, and let $\mathcal{A}_{i,j}$ and $\varphi_{i,j,k}$ be as in Lemma 4.3.3. Define $I = \sum_{i \ne j} \mathcal{A}_{i,j}\mathcal{A}_{j,i}$ as the ideal of total reducibility in A.*

(a) (1) *If $I = 0$, then the map $\pi \colon S \to \sum_i e_i S e_i \subset S, x \mapsto \sum_i e_i x e_i$ is a ring homomorphism.*
(2) *Denoting by D_i the map $e_i S e_i \xrightarrow{\psi_i} \mathrm{Mat}_{n_i \times n_i}(A) \xrightarrow{\det} A$ for $i = 1, \dots, r$, one has*

$$\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D_i \circ \pi \bmod I.$$

(b) *Suppose that there exist m_i-dimensional pseudocharacters $D'_i \colon S \to A$ with $m_i > 0$ for $i \in \{1, \dots, r\}$ such that one has $\det_{(S,\mathcal{E})} = \oplus_{i=1}^r D'_i$. Then $I = 0$ and for a unique permutation $\sigma \in \mathfrak{S}_r$ we have $D'_{\sigma(i)} = D_i \circ \pi$ with D_i and π from (i).*

Proof. Part (1) of (a) is a straightforward matrix calculation using $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \neq j$ from $\{1,\dots,r\}$. To see Part (2) of (a), observe that equation (18) for $r = 1$ is simply the Leibniz formula for matrix determinants. Hence, by our definitions we have the explicit formula

$$D_i \bmod I\colon e_iSe_i \longrightarrow A/I, \quad x \longmapsto \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{\text{cycles } \gamma_i \text{ of } \sigma_i} \varphi^l\Big(\prod_{l\in\gamma_i} E^l x E^{\sigma(l)}\Big) \bmod I,$$

and using distributivity for $x \in S$

$$\begin{aligned}\prod_{i=1}^r (D_i \circ \pi)(x) \bmod I &= \prod_{i=1}^r \sum_{\sigma_i \in \mathfrak{S}_{n_i}} \operatorname{sgn}(\sigma_i) \prod_{\text{cycles } \gamma_i \text{ of } \sigma_i} \varphi^l\Big(\prod_{l\in\gamma_i} E^l x E^{\sigma(l)}\Big) \bmod I\\ &= \sum_{\sigma_1 \in \mathfrak{S}_{n_1}} \dots \sum_{\sigma_r \in \mathfrak{S}_{n_r}} \prod_{i=1}^r \operatorname{sgn}(\sigma_i) \prod_{\text{cycles } \gamma_i \text{ of } \sigma_i} \varphi^l\Big(\prod_{l\in\gamma_i} E^l x E^{\sigma(l)}\Big) \bmod I.\end{aligned}$$

We have to compare the latter expression to

$$\det_{(S.\mathcal{E})}(x) \bmod I = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{\text{cycles } \gamma \text{ of } \sigma} \varphi^{l_0}\Big(\prod_{l\in\gamma} E^l x E^{\sigma(l)}\Big) \bmod I.$$

Now, in the last expression, the term $\varphi^{l_0}\Big(\prod_{l\in\gamma} E^l x E^{\sigma(l)}\Big) \bmod I$ vanishes unless γ is contained in a single factor under the inclusion $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_r} \hookrightarrow \mathfrak{S}_n$, by the definition of I and using Lemma 4.3.3. This shows that

$$\det_{(S.\mathcal{E})}(x) \bmod I = \sum_{\sigma=(\sigma_1,\dots,\sigma_r)\in\mathfrak{S}_{n_1}\times\dots\times\mathfrak{S}_{n_r}} \prod_{i=1}^r \operatorname{sgn}(\sigma_i) \prod_{\text{cycles } \gamma \text{ of } \sigma_i} \varphi^{l_0}\Big(\prod_{l\in\gamma} E^l x E^{\sigma(l)}\Big) \bmod I,$$

and it completes the proof of (a).

We now prove (b). We begin by proving the following **Claim:** There is a unique permutation $\sigma \in \mathfrak{S}_r$ such that $D_i = (D'_{\sigma(i)})_{e_i}$ and $(D'_{i'})_{e_i} = 1$ for $i' \neq \sigma(i)$. For this, we restrict $\oplus_{i'=1}^r D'_{i'}$ to e_iSe_i so that

$$D_i = (\det_{(S,\mathcal{E})})_{e_i} = \oplus_{i'}(D'_{i'})_{e_i}.$$

By Lemma 4.3.8 the $(D'_{i'})_{e_i}$ are pseudocharacters of dimension $m_{i',i} := \dim(D'_{i'})_{e_i} \leq m_{i'}$. Now, under addition in the sense of Corollary 4.2.6 dimensions are added, and it follows that

$$n_i = \sum\nolimits_{i'=1}^r m_{i',i}.$$

But because $e_iSe_i = \operatorname{Mat}_{n_i\times n_i}(A)$, it follows from Example 4.1.7(b) that each $m_{i',i}$ is divisible by n_i. Hence, there is a unique map $\sigma\colon \{1,\dots,r\} \to \{1,\dots,r\}$ such that $m_{\sigma(i),i} = n_i$ and $m_{i',i} = 0$ for $i' \neq \sigma(i)$, and moreover $D_i = (D'_{\sigma(i)})_{e_i}$. It remains to show that σ is bijective. It will suffice to show that σ is surjective.

For this, let $S'_{i'} := \oplus_{i\in\sigma^{-1}(i')} e_iSe_i$ so that $S = \oplus_{i'} S'_{i'}$. The restriction of $D'_{i''}$ to $S'_{i'}$ is zero if $i'' \neq i'$, and the restriction of $D'_{i'}$ to $S'_{i'}$ is a pseudocharacter with

$$m_{i'} \overset{4.3.8}{\geq} \dim D'_{i'}|_{S'_{i'}} = \dim \oplus_{i''=1}^r D'_{i''}|_{S'_{i'}} = \dim \det_{(S,\mathcal{E})}|_{S'_{i'}} = \sum_{i\in\sigma^{-1}(i')} n_i.$$

Summing over all i' in the image of σ implies $\sum_{i'\in\sigma(\{1,\dots,r\})} m_{i'} \geq n$. However, all $m_{i'}$ are strictly positive and $\sum_{i'=1}^r m_{i'} = n$, and this implies that σ is surjective, and hence the claim is proved.

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For simplicity of notation, we assume from here on, without loss of generality, that $\sigma = \mathrm{id}$. We now show that $I = 0$. For this, it suffices to show that $\mathcal{A}_{i,j}\mathcal{A}_{j,i} = 0$ for all $i \neq j$. By restricting to the subalgebra $S' = e_iSe_i + e_jSe_j + e_iSe_j + e_jSe_i$ with $\mathcal{E}' = (e_i, \psi_i, e_j, \psi_j,)$, that is, by considering $D_{e_i+e_j}$, and using $\det_{(S,\mathcal{E})}|_{S'} = \det_{(S',\mathcal{E}')}$, we may assume $r = 2$ for the proof of $I = 0$.

Let b be in $\mathcal{A}_{1,2}$ and c in $\mathcal{A}_{2,1}$, and write x for $e_1E_1^{1,1}bE_2^{1,1}e_2$ and y for $e_2E_2^{1,1}ce : 1^{1,1}e_1$ with $E_i^{k,l}$ from Notation 4.3.2. Using the description of GMAs from Lemma 4.3.3, one easily verifies that

$$1 + xy = 1 + E_1^{1,1}bc \in e_1Se_1 + (1 - e_1),\ \ 1 + yx = 1 + E_2^{1,1}bc \in (1 - e_2) + e_2Se_2.$$

Note moreover that by Lemma 4.3.6 we have $D(1 + xy) = D(1 + yx)$ for every pseudocharacter $D : S \longrightarrow A$. If we apply this to D_i' and our earlier observations on $(D_i')_{e_{i'}}$, we find that

$$D_1'(1 + xy) = D_1'(1 + yx) = (D_1')_{e_1}(1 - e_2) \cdot (D_1')_{e_2}(e_2 + E_2^{1,1}bc) = 1 \cdot 1 = 1,$$

and similarly $D_2'(1 + xy) = 1$, and hence from hypothesis (2) that $\det_{(S,\mathcal{E})}(1 + E_1^{1,1}bc) = 1$. From the formula for $\det_{(S,\mathcal{E})}$ on $e_1Se_1 + e_2Se_2 \cong \mathrm{Mat}_{n_1\times n_1}(A) \times \mathrm{Mat}_{n_2\times n_2}(A)$, we deduce that

$$\det_{(S,\mathcal{E})}(1 + E_1^{1,1}bc) = 1 + bc,$$

and hence that $bc = 0$, as was to be shown.

For the second assertion, observe that by Lemma 4.3.7 we have $D_i'(1 + e_ixe_j) = 1$ for any $i \neq j$ and $x \in \mathrm{Mat}_{n_i\times n_j}(\mathcal{A}_{i,j})$. It follows that $D_i'(1 + u) = 1$ for any u in the kernel of π. And now the second assertion follows from knowing the restriction of D_i' to $\sum_i e_iSe_i$ given in the first claim of the proof of (b). ◻

The following result of Chenevier gives an application of GMAs to pseudocharacters.

Theorem 4.3.10 [Che14, Theorem 2.22], [WE13, Theorem 2.27]. *Assume that A is a Henselian local ring with maximal ideal m_A and residue field $\kappa(A)$. Let S be an A-algebra and suppose that $D \in \mathcal{P}s\mathcal{R}_S^n(A)$ is Cayley–Hamilton. Denote by $\overline{D} = D \otimes_A \kappa(A) : S/m_AS \longrightarrow \kappa(A)$ the residual pseudocharacter of D. Suppose that $\overline{D}$ is split (see Definition 4.1.19). Then the following hold:*

(a) *If $\overline{D}$ is irreducible, then $D = \det \circ \rho$ for some A-algebra isomorphism $\rho : S \xrightarrow{\sim} \mathrm{Mat}_{n\times n}(A)$.*
(b) *If $\overline{D}$ is multiplicity free, then S is a generalized matrix algebra $(S, \mathcal{E})$ and $D = \det_{(S,\mathcal{E})}$. If $\overline{D} = \bigoplus_{i=1}^{l} \overline{D}_i$ for irreducible $\overline{D}_i$, then the type of S is $(n_1, \ldots, n_l)$ for n_i the degree of $\overline{D}_i$.*

4.4. Continuous pseudocharacters

In our application, mainly continuous pseudocharacters (of a profinite group G) will play a role. In this subsection, we will recall this concept and some of its properties. We denote throughout this subsection by G a profinite group. Let us refer to [Gro60, Chapter 0 §7, Chapter 1 §10] for a more thorough introduction to topological rings and formal schemes.

We introduce in Definition 4.4.2 a category of admissible κ-algebras that is perhaps not standard. In Lemma 4.4.7, we prove a finiteness statement for continuous pseudocharacters on G_K with K p-adic and values in a finite field of characteristic p.

Definition 4.4.1 (Cf. [Che14, §2.30]). Let A be a commutative topological ring. Then $D \in \mathcal{P}s\mathcal{R}_G^n(A)$ is called *continuous* if and only if the characteristic polynomial functions (restricted to G) $\Lambda_{D,i} : G \longrightarrow A$ are continuous for $i = 1, \ldots, n$.

We shall study continuity only for two types of commutative rings A that we now describe. Consider a directed set J with minimal element 0 and an inverse system A_λ, $\lambda \in J$, of topological commutative rings with continuous transition maps and such that $A_\lambda \longrightarrow A_0$ is surjective with nilpotent kernel for any

$\lambda \in J$. Then the inverse limit

$$\lim_{\lambda \in J} A_\lambda \tag{19}$$

is a topological ring with respect to the weakest topology for which the projections to all A_λ are continuous.

Definition 4.4.2. Let κ be a local or a finite field with its natural topology.

(a) We say that a commutative topological ring A is *κ-admissible* if there is an inverse system $(A_\lambda)_{\lambda \in J}$ as above expression (19) and an isomorphism of topological rings $A \cong \lim_{\lambda \in J} A_\lambda$ such that each A_λ is a finite-dimensional topological κ-algebra with the natural topology of a finite-dimensional κ-vector space.

(b) We denote by $\mathcal{A}dm_\kappa$ the category whose objects are κ-admissible commutative topological rings and whose morphisms are continuous κ-algebra homomorphisms.

Note that $\widehat{\mathcal{A}r}_\kappa$ is a full subcategory of $\mathcal{A}dm_\kappa$, but objects in $\mathcal{A}dm_\kappa$ are in general only semilocal and with residue field of finite κ-dimension.

Definition 4.4.3 [Che14, §3.9]. Let $W(\mathbb{F})$ be the topological ring of Witt vectors over $\mathbb{F}$.

(a) A commutative topological ring A is *admissible* if there is an inverse system $(A_\lambda)_{\lambda \in J}$ as above expression (19) and an isomorphism of topological rings $A \cong \lim_{\lambda \in J} A_\lambda$ such that each A_λ carries the discrete topology.

(b) We denote by $\mathcal{A}dm_{W(\mathbb{F})}$ the category whose objects are admissible commutative topological rings A together with a continuous homomorphism $W(\mathbb{F}) \to A$ and whose morphisms are continuous $W(\mathbb{F})$-algebra homomorphisms.

Remark 4.4.4. Suppose A is admissible or κ-admissible, and suppose that $A = \lim_{\lambda \in J} A_\lambda$ for an inverse system $(A_\lambda)_{\lambda \in J}$ as in the above definitions. Then one can form the completed group ring as the inverse limit

$$A[[G]] := \lim_{\lambda, H} A_\lambda[G/H],$$

where H ranges over all open normal subgroups of G; it contains $A[G]$ and is in fact the completion of $A[G]$ with respect to the topology of $A[G]$ inherited from $A[[G]]$.

Using Amitsur's formula, one can verify that the above definition of continuity is equivalent to the condition that for every commutative topological A-algebra B, with $B \in \mathcal{A}dm$ or $\mathcal{A}dm_\kappa$, respectively, the map $D_B \colon B[G] \to B$ is continuous; see [WE13, Definition 3.1.0.10]. This allows one also to extend D_B to a (continuous) pseudocharacter $B[[G]] \to B$.

The following is the basic result on continuity if A is discrete.

Lemma 4.4.5 [Che14, Lemma 2.33]. *Let A be a discrete, and let $D \colon A[G] \to A$ be a pseudocharacter. Then D is continuous if and only if* $\ker(D)$ *is contained in the kernel of the canonical map $A[G] \to A[G/H]$ for some normal open subgroup $H \subset G$. In this case, the natural representation $G \longrightarrow (B[G]/\ker(D))^\times$ factors through G/H.*

We record the following consequence:

Corollary 4.4.6 [Che14, Example 2.34]. *Let k be a discrete field, and let $D \in \mathcal{P}s\mathcal{R}^n_G(k^{\mathrm{alg}})$ be continuous. Then the representation $\rho_D \colon G_K \to \mathrm{GL}_n(k^{\mathrm{alg}})$ associated by Theorem 4.1.18 is continuous, its image is finite and it is defined over a finite extension of k.*

Proof. We provide a proof, expanding on [Che14, Example 2.34]: Because D is continuous, we know by Lemma 4.4.5 that $\ker D$ contains the kernel of $k[G] \to k[G/H]$ for some open subgroup H of G. By Theorem 4.1.18, the kernels of ρ_D^{lin} and of D are the same, and hence ρ_D is continuous since it is trivial on the open subgroup H. Since G/H is finite, this also shows that $\rho_D(G) \subset \mathrm{GL}_n(k^{\mathrm{alg}})$ is finite. It follows that the entries of the matrices in the image of ρ_D lie in a finite extension of k, and this proves the last assertion. □

When combined with earlier results, we deduce the following finiteness statement:

Proposition 4.4.7. *Let $\mathbb{F}$ be a finite field of characteristic p, and let $n \geq 1$ be an integer. Then there exist only finitely many continuous pseudocharacters $\overline{D}\colon G_K \to \mathbb{F}$ on G_K of dimension n.*

Denote by $\mathbb{F}' \supset \mathbb{F}$ the unique field extension of degree $n!$. Then for any $\overline{D}$ as above $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ is a direct sum of split irreducible pseudocharacters $\overline{D}_i\colon G_K \to \mathbb{F}'$ on G_K.

Proof. The second part is immediate from Corollary 4.1.20. Hence, it suffices to prove the first part for split irreducible $\overline{D}$. It moreover suffices to assume that $\mathbb{F}$ contains the unique extension of the residue field of K of degree $n!$. The result follows from Lemma A.3.1. □

The next result shows the existence of a minimal ring of definition for any continuous pseudocharacter, and it gives an important result on their structure.

Lemma 4.4.8 [Che14, Lemma 3.10]. *Let A be in $\mathcal{A}dm_{W(\mathbb{F})}$, let $D\colon A[G] \to A$ be a continuous pseudocharacter and let $C \subset A$ be the closure of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D,i}(g)$ for $g \in G$ and $i \geq 1$.*

(a) *The ring C is an admissible profinite subring of A. In particular, $C = \varprojlim_i C_i$ is a finite product of local $W(\mathbb{F})$-algebras with finite residue fields.*
(b) *If further $\iota\colon A \longrightarrow A'$ is a continuous $W(\mathbb{F})$-algebra homomorphism, $D'\colon A'[G] \to A'$ is the induced pseudocharacter and $C' \subset A'$ is the closure $C' \subset A'$ of the $W(\mathbb{F})$-algebra generated by the characteristic polynomial coefficients $\Lambda_{D',i}(g)$ for $g \in G$ and $i \geq 1$, then ι induces a surjection $C \to C'$ in $\mathcal{A}dm_{W(\mathbb{F})}$.*

We use the Lemma 4.4.8 to make the following useful definitions.

Definition 4.4.9 [Che14, Definition 3.11]. For a finite field $\mathbb{F}$, one defines

$$|G(n)| := \{z \in \mathrm{Spec}(R^{\mathrm{univ}}_{W(\mathbb{F})[G],n}) : z \text{ is closed and } \kappa(z) \text{ is finite}\}.$$

Definition 4.4.10 [Che14, Definition 3.12]. Let A be in $\mathcal{A}dm_{W(\mathbb{F})}$, let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be continuous, let $C \subset A$ be the ring from Lemma 4.4.8 and let $D_C\colon C[G] \to C$ be the pseudocharacter from Proposition 4.1.10.

(a) We call C *the ring of definition of D over $W(\mathbb{F})$.*
(b) If C is local so that $\kappa(C)$ is finite, one calls D *residually constant*.
(c) One calls D *residually equal* to D_z for some $z \in |G(n)|$, if C is local and $D_z \cong D_C \otimes_C \kappa(C)$.

4.5. *Twisting of pseudocharacters*

In this subsection, we introduce a twisting operation for pseudocharacters that is the analog of the twist of a representation by a character, and we state some of its basic properties. Our approach require us to recall a number of results on the universal pseudocharacter that go back to Roby. Our main construction is only carried out for pseudocharacter of a topological group G. Our exposition of background material follows [WE13, Section 1.1].

Definition 4.4.11. Let M be an A-module. The divided power algebra of M relative to A is the commutative A-algebra $\Gamma_A(M)$ that is the quotient algebra of the polynomial algebra generated by the symbols $m^{[i]}$, $m \in M$, $i \in \mathbb{N}$, subject to the relations

(i) $m^{[0]} = 1$ for all $m \in M$,
(ii) $(am)^{[i]} = a^i m^{[i]}$ for $a \in A$, $m \in M$, $i \in \mathbb{N}$,
(iii) $m^{[i]}m^{[j]} = \binom{i+j}{i} m^{[i+j]}$ for $m \in M$, $i, j \in \mathbb{N}$ and
(iv) $(m+n)^{[i]} = \sum_{j=0}^{i} m^{[j]} n^{[i-j]}$ for $m, n \in M$ and $i \in \mathbb{N}$,

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The ring $\Gamma_A(M)$ is a graded A-algebra $\Gamma_A(M) = \bigoplus_{i\geq 0}\Gamma_A^i(M)$ with its i-th graded piece $\Gamma_A^i(M)$ being the A-module generated by the element $m^{[i]}$, $m \in M$. The construction $M \mapsto \Gamma_A(M)$ defines a functor from A-modules to graded A-algebras. If $\varphi\colon M \to N$ is an A-module homomorphism, the induced map $\Gamma_A(\varphi)\colon \Gamma_A(M) \to \Gamma_A(N)$ is characterized by $m^{[i]} \mapsto (\varphi(m))^{[i]}$, $m \in M$, $i \in \mathbb{N}$. One has compatibility with base change, that is, natural isomorphisms $\Gamma_A^d(M) \otimes_A B \cong \Gamma_B^d(M \otimes_A B)$.

Definition 4.5.2. The universal degree d homogenous polynomial law $L_M^d \in \mathcal{P}_A^d(M, \Gamma_A^d(M))$ is defined by the maps

$$L_{M,B}^d\colon M \otimes_A B \longrightarrow \Gamma_A^d(M) \otimes_A B \cong \Gamma_B(M \otimes_A B), m \otimes b \mapsto (bm)^{[i]}, \quad m \in M, b \in B.$$

The universality of L^d is expressed by the following result:

Theorem 4.5.3 [Rob63, Théorème IV.1]. *Let M, N be two A-modules, and let d be in $\mathbb{N}$. There is a canonical isomorphism*

$$\mathrm{Hom}_A(\Gamma_A^d(M), N) \xrightarrow{\simeq} \mathcal{P}_A^d(M, N), f \longmapsto f \circ L_M^d.$$

To describe the map in the converse direction, let $P \in \mathcal{P}_A^d(M, N)$. Define the index set $I_d := \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \mid \sum \alpha_i = d\}$. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in I_d$ set $T^\alpha = \prod_{j=1}^d T_j^{\alpha_j}$ for indeterminates $(T_1, \dots, T_d)$, and set $m^{[\alpha]} = \prod_{j=1}^d m_i^{[\alpha_i]} \in \Gamma_A^d(M)$ for $m = (m_1, \dots, m_d) \in M^d$. Define now for all $\alpha \in I_d$ simultaneously maps $P^{[\alpha]}\colon M^d \to N$ by

$$P_{A[T_1,\dots,T_d]}(T_1 m_1 + \ldots + T_d m_d) = \sum_{\alpha \in I_d} P^{[\alpha]}(m) T^\alpha$$

for $m = (m_1, \dots, m_d) \in M^d$. In the proof of Theorem 4.5.3 by Roby, it is shown that given any $P \in \mathcal{P}_A^d(M, N)$, there exists an A-module homomorphism $f\colon \Gamma_A^d(M) \to N$ such that

$$f\left(m^{[\alpha]}\right) = P^{[\alpha]}(m), \quad \forall \alpha \in I_d \text{ and } m \in M^d, \tag{20}$$

and that $f \circ L_M^d = P$.

If M is a free A-module, the A-module $\Gamma_A^d(M)$ has the following explicit description.

Theorem 4.5.4 [Rob63, Théorème IV.2]. *Suppose that M is a free A-module with basis $(e_i)_{i\in I}$. Then for $d \in \mathbb{N}$, the A-module $\Gamma_A^d(M)$ is free with basis*

$$\left\{e_{i_1}^{[k_1]} \cdot \ldots \cdot e_{i_j}^{[k_h]} \mid h \in \mathbb{N}, (i_1, \dots, i_h) \in I^h, (k_1, \dots, k_h) \in \mathbb{N}_{\geq 1}, \sum_{j=1}^h k_h = d\right\}.$$

If M is an A-algebra R, then [Rob80] defines an A-algebra structure on each $\Gamma_A^d(R)$, different from that on $\Gamma_A(R)$, by defining a multiplication $\Gamma_A^d(R) \otimes_A \Gamma_A^d(R) \to \Gamma_A^d(R)$, that we now recall. The multiplication map is defined as the composition of two maps. The first map exists for any A-module M, the second is built from the ring structure of R. Let first M be an arbitrary A-module. Then the map $\beta_M : M \oplus M \to M \otimes_A M, (m, m') \mapsto m \otimes m'$ is a homogeneous polynomial law of degree 2, and thus $L_{M\otimes M}^d \circ \beta_M$ lies in $\mathcal{P}_A^{2d}(M \oplus M, M \otimes M)$. By Theorem 4.5.3, we have $L_{M\otimes M}^d \circ \beta_M = \eta_M \circ L_{M\oplus M}^{2d}$ for a unique A-linear map

$$\eta_M : \Gamma_A^{2d}(M \oplus M) \to \Gamma_A^d(M \otimes_A M).$$

[Rob63, Théorème III.4] gives an isomorphism $\bigoplus_{i=0}^e \Gamma_A^i(M) \otimes \Gamma_A^{e-i}(M) \to \Gamma_A^e(M)$ for any $e \in \mathbb{N}$. It is further shown in [Rob80, p. 869] that the maps $\Gamma_A^i(M) \otimes \Gamma_A^{2d-i}(M) \to \Gamma_A^d(M \otimes M)$ induced from

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η_M are zero for $i \neq d$, and that the induced map $\widetilde{\eta}_M : \Gamma^d_A(M) \otimes \Gamma^d_A(M) \to \Gamma^d_A(M \otimes M)$ is given by the explicit formula

$$\widetilde{\eta}_M \left(m^{[\alpha]} \otimes n^{[\beta]}\right) = \sum_{\gamma \in \mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})} \prod_{(i,j)\in\{1,\ldots,d\}^2} (m_i \otimes n_j)^{[\gamma_{ij}]}, \tag{21}$$

for $m, n \in M^d$, $\alpha, \beta \in I_d$, and where $\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})$ denotes the set of all matrices $\gamma = (\gamma_{ij})$ in $\mathrm{Mat}_{d\times d}(\mathbb{N})$ whose rows sum to β and whose columns sum to α. Let now $M = R$ be an A-algebra. Then the multiplication map $\mu_R : R \otimes_A R \to R$ is A-linear, and thus it induces a graded map $\Gamma_A(\mu_R)$ whose d-th graded piece is a homomorphism $\Gamma^d_A(\mu_R) : \Gamma^d_A(R \otimes R) \to \Gamma^d_A(R)$. Roby defines

$$\mu^d_R := \Gamma^d_A(\mu_R) \circ \widetilde{\eta}_R : \Gamma^d_A(R) \otimes_A \Gamma^d_A(R) \to \Gamma^d_A(R)$$

It is shown in [Rob80, p. 870] that if R is unital, associative or commutative, respectively, then the same property holds for $\Gamma^d_A(R)$ with the multiplication μ^d_R, for any $d \in \mathbb{N}$. It turns out that L^d_R is multiplicative with respect to this multiplication on $\Gamma^d_A(R)$. The key result is the following description of multiplicative polynomial laws:

Theorem 4.5.5 [Rob80, Théorème]. *For A-algebras S, S', the following map is a bijection*

$$\mathrm{Hom}_{A\text{-}\mathcal{A}lg}(\Gamma^d_A(S), S') \to \mathcal{M}^d_A(S, S'), f \mapsto f \circ L^d_R.$$

Suppose now that $R = A[G]$ for a group G. Note that the elements of G form an A-basis of $A[G]$, and hence an A-basis of $\Gamma^n_A(A[G])$ is described in Theorem 4.5.4. Let $D : A[G] \to A$ be a pseudocharacter of dimension d. From Theorems 4.5.3 and 4.5.5 and using equation (20), we deduce:

Proposition 4.5.6. *There exists a unique homomorphism $f_D : \Gamma^d_A(A[G]) \to A$ such that*

$$f_D\left(g^{[\alpha]}\right) = D^{[\alpha]}(g), \quad \forall \alpha \in I_d \text{ and } g \in G^d.$$

It is multiplicative for the product on $\Gamma^d_A(A[G])$ given by $\mu^d_{A[G]}$.

Let now $\chi : G \to A^\times$ be a group homomorphism. Define for $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $g = (g_1, \ldots, g_d) \in G^d$ the notation $\chi(g^{[\alpha]})$ to by $\chi(g^{[\alpha]}) := \prod_{i=1}^d \chi(g_i)^{\alpha_i}$. Because $\{g^{[\alpha]} \mid \alpha \in I_d, g \in G^d\}$ is a basis of $\Gamma^d_A(A[G])$ we have a unique A-linear map $f_{D,\chi} : \Gamma^d_A(A[G]) \to A$ such that

$$f_{D,\chi}\left(g^{[\alpha]}\right) = D^{[\alpha]}(g) \cdot \chi(g^{[\alpha]}), \quad \forall \alpha \in I_d \text{ and } g \in G^d.$$

Proposition 4.5.7. *Suppose that $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ and that $\chi : G \to A^\times$ is a group homomorphism. Then the following hold:*

(a) *The map $f_{D,\chi}$ defined above is multiplicative.*

Define the d-dimensional pseudocharacter $D \otimes \chi$ to be $f_{D,\chi} \circ L^d_{A[G]}$.

(b) *The characteristic polynomial coefficients of $D \otimes \chi$ satisfy the identities*

$$\Lambda_{D\otimes\chi,i}(g) = \Lambda_{D,i}(g) \cdot \chi(g)^i \quad \textit{for all } i \textit{ and all } g \in G.$$

(c) *If D and χ are continuous, then so is $D \otimes \chi$.*
(d) *If $D = D_\rho$ for a representation ρ of G, then $D \otimes \chi = D_{\rho\otimes\chi}$.*

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Proof. To see Part (a), we need to show that $f_{D,\chi}(g\cdot h) = f_{D,\chi}(g)f_{D,\chi}(h)$ for $g=(g_1,\ldots,g_d), h=(h_1,\ldots,h_d)\in G^d$ and for $\cdot$ the multiplication given by $\mu^d_{A[G]}$. Using equation (21), we compute

$$g^{[\alpha]}\cdot h^{[\beta]} = \mu^d_{A[G]}\Big(\widetilde{\eta}_M\left(g^{[\alpha]}\otimes h^{[\beta]}\right)\Big) = \mu^d_{A[G]}\Big(\sum_{\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})}\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_i\otimes h_j)^{[\gamma_{ij}]}\Big)$$
$$= \sum_{\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})}\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_ih_j)^{[\gamma_{ij}]}.$$

Observe that $\sum_{i,j}\gamma_{ij}=d$ for $\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})$ and that index pairs (i,j) with $\gamma_{ij}=0$ can be ignored. We write $\underline{\gamma}$ for the flattening of γ truncated to length d, that is, we first regard γ as a d^2-tuple in one index and then omit the highest d^2-d indices, where $\gamma_{ij}=0$. Using in $(*)$ the definition of $\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})$, we find

$$\begin{aligned}
f_{D,\chi}(g^{[\alpha]}\cdot h^{[\beta]}) &= \sum_{\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})} f_{D,\chi}\Big(\prod_{(i,j)\in\{1,\ldots,d\}^2}(g_ih_j)^{[\gamma_{ij}]}\Big)\\
&= \sum_{\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})} D^{[\underline{\gamma}]}((g_ih_j)_{(i,j)\in\underline{\gamma}})\Big(\prod_{(i,j)\in\{1,\ldots,d\}^2}\chi(g_ih_j)^{\gamma_{ij}}\Big)\\
&\overset{(*)}{=} \sum_{\gamma\in\mathrm{Mat}^{\alpha,\beta}_{d\times d}(\mathbb{N})} D^{[\underline{\gamma}]}((g_ih_j)_{(i,j)\in\underline{\gamma}})\chi(g^{[\alpha]})\chi(h^{[\beta]})\\
&= \chi(g^{[\alpha]})\chi(h^{[\beta]})f_D(g^{[\alpha]}\cdot h^{[\beta]})\\
&\overset{f_D\text{ multipl.}}{=} \chi(g^{[\alpha]})\chi(h^{[\beta]})f_D(g^{[\alpha]})f_D(h^{[\beta]}) = f_{D,\chi}(g^{[\alpha]})f_{D,\chi}(h^{[\beta]}).
\end{aligned}$$

Concerning (b), note that

$$D_{A[T']}(1-Tg') = D_{A[T_1,\ldots,T_d]}\Big(\sum_{i=1}^d T_ig_i\Big)\Big|_{g=(e,\ldots,e,g'),T=(1,0,\ldots,0,T')}$$

so that

$$\begin{aligned}
\Lambda_{D\otimes\chi,i}(g') &= (-1)^i(D\otimes\chi)^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g')\\
&= (-1)^iD^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g')\chi((e,\ldots,e,g')^{[d-i,0,\ldots,0,i]})\\
&= (-1)^iD^{[d-i,0,\ldots,0,i]}(e,\ldots,e,g')\chi(g)^i = \Lambda_{D,i}(g')\chi(g)^i.
\end{aligned}$$

Part (c) follows from (b) and Definition 4.4.1, Part (d) follows from (b) and the theorem of Brauer–Nesbitt. □

Definition 4.5.8 (Twist of pseudocharacters). We call the multiplicative polynomial law $D\otimes\chi\in\mathcal{P}s\mathcal{R}^n_G(A)$ from Proposition 4.5.7 the *twist of D by χ*.

Remark 4.5.9. It should be interesting to define the tensor product of two pseudocharacters of any dimensions n,n'.

Lemma 4.5.10. *Let D,D' be in $\mathcal{P}s\mathcal{R}^n_G(A)$, and let $\chi\colon G\to A^\times$ be a group homomorphism. Then $D'=D\otimes\chi$ if and only if $\Lambda_{D',i}(g)=\Lambda_{D,i}(g)\cdot\chi(g)^i$ for all i and all $g\in G$.*

Proof. Proposition 4.5.7(b) shows that the condition given is necessary. That it is also sufficient follows from Proposition 4.1.10(a), which says that a pseudocharacter is determined by its characteristic polynomial coefficients. □

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Corollary 4.5.11. *Let D be in $\mathcal{P}s\mathcal{R}^n_G(A)$, and let $\chi\colon G \to A^\times$ be a character of finite order. Suppose that $\chi(g) - 1$ lies in $A^\times$ whenever $g \in G\backslash\ker\chi$. Then the following hold:*

(a) *$D = D \otimes \chi$ if and only if*

$$\forall g \in G, \forall i = 0, \dots, n : \Lambda_{D,i}(g) = 0 \text{ or } \operatorname{ord}\chi(g) \text{ divides } i.$$

(b) *Let I be the ideal of A generated by the set*

$$\{\Lambda_{D,i}(g) : (g,i) \in G \times \{1, \dots, n\} \text{ such that } \operatorname{ord}\chi(g) \nmid i\}.$$

Then the locus of Spec A *on which $D = D \otimes \chi$ is the closed subscheme* Spec A/I.

Proof. To see Part (a), note that by Lemma 4.5.10 we have $D = D \otimes \chi$ if and only if

$$\Lambda_{D,i}(g) = \Lambda_{D,i}(g) \cdot \chi^i(g) \qquad \text{for all } i \text{ and all } g \in G.$$

Since $1-\chi^i(g)$ is a unit in $A^\times$ whenever $\operatorname{ord}\chi(g) \nmid i$ and is zero otherwise, the latter is clearly equivalent to the condition given in the corollary.

By Part (a), we have for any ideal J of A

$$(D \otimes_A A/J) \otimes \chi = D \otimes_A A/J \quad \Longleftrightarrow \quad I \subset J,$$

and this implies Part (b). □

4.6. Induction for pseudocharacters

In this subsection, we introduce the operation of inducing a pseudocharacter from a finite index subgroup. The main result is Theorem 4.6.7. Following a suggestion of the referee, we describe a construction that works in all cases.[6] The idea is a pullback to a universal situation. For this, we use Theorem A.4.4 which is a variant of an important result of Vaccarino. The uniqueness of the construction, that is, its characterizing property, is guaranteed by explicit formulas for the characteristic polynomial of the induction. The present subsection begins by recalling the construction of induction of a representation and then analyzes it to give in Lemma 4.6.6 a formula for the resulting characteristic polynomial. This is then used in the main result, Theorem 4.6.7.

We fix a group G and a subgroup $H \subset G$ of finite index m. As in Section 2, we set $N := \bigcap_{g\in G/H} H^g$. It is of finite index and normal in G, and the largest such subgroup contained in H. If G is a profinite group, we require H to be open, and then $N \subset G$ is open, as well.

Lemma 4.6.1. *Let C be an Azumaya A-algebra. Consider a representation $\rho\colon H \to C^\times$. There exists a representation $\rho^*\colon G \to \mathrm{GL}_m(C)^\times$ such that for any étale extension $A \to A'$ that splits C, there is an isomorphism $\rho^* \otimes_A A' \cong \mathrm{Ind}_H^G(\rho \otimes_A A')$ of G-representations over A.*

The linearization $(\rho^*)^{\mathrm{lin}}\colon A[G] \to \mathrm{Mat}_{m\times m}(C)$ of ρ^* takes values in the Azumaya algebra $\mathrm{Mat}_{m\times m}(C)$*, and by Example 4.1.7 the associated pseudocharacter D_{ρ^*} takes values in A.*

Proof. To prove the lemma, we adapt the description of the induced matrix representation from [CR81, pp. 227-230] to the setting of Azumaya algebras. Let $g_1, \dots, g_m$ be a set of representatives of left cosets of G/H so that $G = \bigsqcup_{i=1}^m g_i H$. We extend ρ from H to G by defining

$$\widetilde{\rho}\colon G \longrightarrow C, \quad g \longmapsto \begin{cases} \rho(g) & \text{if } g \in H, \\ 0 & \text{if } g \in G\backslash H. \end{cases}$$

[6] In an earlier version, using a different argument, our construction had some technical limitations but was nevertheless sufficient for the main result of this work.

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Consider the map

$$\rho^*\colon G \longrightarrow \mathrm{Mat}_{m\times m}(C), \quad g \longmapsto \begin{pmatrix} \widetilde{\rho}(g_1^{-1}gg_1) & \cdots & \widetilde{\rho}(g_1^{-1}gg_m) \\ \vdots & \ddots & \vdots \\ \widetilde{\rho}(g_m^{-1}gg_1) & \cdots & \widetilde{\rho}(g_m^{-1}gg_m) \end{pmatrix}.$$

Define for $g \in G$ and $j \in \{1,\ldots,m\}$ the element $i_j \in \{1,\ldots,m\}$ by the condition $gg_jH = g_{i_j}H$ so that the map $j \mapsto i_j$ is a permutation of $\{1,\ldots,m\}$. This shows that $\rho^*(g)$ is a monomial matrix over the skew field C which in each column j has a unique nonzero entry $\rho(g_{i_j}^{-1}gg_j) \in C^\times$ in row i_j. In particular, this also shows that $\rho^*(g)$ lies in $\mathrm{GL}_m(C)$.

We claim that ρ^* has the properties asserted in the lemma. Let $A \to A'$ be finite étale so that $C \otimes_A A' = \mathrm{Mat}_{r\times r}(A')$ for a suitable $r \in \mathbb{N}_{\geq 1}$. Then by our construction, that follows [CR81], $\rho^* \otimes_A A'$ is the matrix representation of the induced representation of

$$\rho \otimes_A A'\colon H \longrightarrow \mathrm{GL}_r(A').$$

This implies the multiplicativity of the map ρ^*, that is, that it is a homomorphism. Moreover, it shows that $\rho^* \otimes_A A'$ is the usual induced representation of $\rho \otimes_A A'$. □

Remark 4.6.2. It can be shown that $\rho \mapsto \rho^*$ in Lemma 4.6.1 is uniquely characterized as the right adjoint of the restriction homomorphism from G-representations to H-representations on Azumaya algebras. In particular, up to isomorphism ρ^* is independent of the chosen representatives $g_1,\ldots,g_m$ of G/H.

Definition 4.6.3. We call ρ^* from Lemma 4.6.1 the *representation induced from ρ under $H \subset G$* and denote it by $\mathrm{Ind}_H^G \rho$.

The reason for introducing induction with Azumaya algebra coefficients is to be able to formulate Theorem 4.6.7(e); see Remark 4.6.9.

Example 4.6.4. Since we have just seen an explicit form of ρ^*, for later use we consider the following example: Let $H \subset G$ be a normal subgroup of index p. Fix $g_0 \in G\backslash H$, and set $g_i := g_0^i$ for $i = 1,\ldots,p$ so that $G = \bigsqcup_{i=1}^p g_iH$ and the map

$$\lambda\colon G/H \to \mathbb{Z}/(p), g_0^iH \mapsto i (\mathrm{mod}\ p)$$

is a group isomorphism. Let C be an Azumaya E-algebra for a field E of characteristic p, and let $\rho\colon H \to C^\times$ be a representation. Define the induced representation $\rho^*\colon G \to \mathrm{Mat}_{p\times p}(C)^\times$ as in the above proof. Let $A \in \mathrm{Mat}_{p\times p}(C)$ be the diagonal matrix with diagonal $(i\cdot 1_C)_{i=0,\ldots,p-1}$. Then we claim that one has for all $g \in G$ the relation

$$\rho^*(g)A\rho^*(g^{-1}) - A = -\lambda(g)1_{\mathrm{Mat}_{p\times p}(C)}.$$

The reader is advised to compare this with Lemma 2.3.2. The claim asserts that λ defines a nontrivial class in $H^0(G, C/E)$ with G acting on C/E via the adjoint representation of ρ.

To prove the claim, let $g \in G$. We shall verify $\rho^*(g)A - A\rho^*(g) = \lambda(g)\rho^*(g)$. Observe that $\widetilde{\rho}(g_i^{-1}gg_j) = \widetilde{\rho}(g_0^{-i}gg_0^j) = 0$ unless $gH = g_0^{i-j}H$, that is, unless $\lambda(g) = i - j$. In the following, we write a lower subscript $_{i,j}$ to indicate the (i,j)-entry of a matrix in $\mathrm{Mat}_{p\times p}(C)$. Then

$$\begin{aligned}(\rho^*(g)A - A\rho^*(g))_{i,j} = \rho^*(g)_{i,j}\cdot j - i\cdot\rho^*(g)_{i,j} &= (j-i)\cdot\widetilde{\rho}(g_0^{-i}gg_0^j) \\ &\overset{\text{observ.}}{=} -\lambda(g)\cdot\rho^*(g)_{i,j},\end{aligned}$$

and this completes the proof of our assertion, and ends our example.

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Presumably formulas for the characteristic polynomial of $\mathrm{Ind}_H^G\rho$ are well known. But we could not locate suitable references. So we develop this from scratch. We need to fix some notation: Let C be an Azumaya A-algebra of degree n. Recall from Reminder 4.1.6 that elements $c \in C$ have a reduced characteristic polynomial χ_c; we define its coefficients $\Lambda_{c,i}$ by $\chi_c(t) = \sum_{i=0}^n (-1)^i \Lambda_{c,i}(c) t^{n-i}$. We write χ_c^m for the reduced characteristic polynomial (of degree nm) of an element $c \in \mathrm{Mat}_{m\times m}(C)$.

Lemma 4.6.5. *Let $c = (c_{i,j})$ be in $\mathrm{Mat}_{m\times m}(C)$. Suppose that there is a permutation $\sigma \in \mathfrak{S}_m$ such that $c_{i,j} = 0$ for $i \neq \sigma(j)$ and such that $c_{\sigma(j),j}$ lies in $C^\times$ for all j. Then χ_c^m has the following description:*

Write σ in its cycle decomposition $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_v$, where the σ_l are disjoint cycles of length m_l such that $\sum_{l=1}^v m_l = m$, and let j_l be in the support of σ_l such that $\sigma_l = (j_l, \sigma(j_l), \ldots, \sigma^{m_l-1}(j_l))$. Then

$$\chi_c^m(t) = \prod_{l=1}^{v} \chi_{c(l)}(t^{m_l}) \qquad \text{with } c(l) := c_{j_l,\sigma^{m_l-1}(j_l)} c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} \cdot \ldots \cdot c_{\sigma(j_l),j_l}.$$

Proof. Let $s_l = m_1 + \ldots + m_{l-1}$ for $l = 1, \ldots, v$, with $s_1 = m_0 = 0$, and let $\tau \in \mathfrak{S}_m$ be the permutation whose inverse is given by

$$\begin{pmatrix} s_1+1 & s_1+2 & \cdots & s_1+m_1 \\ j_1 & \sigma(j_1) & \ldots & \sigma^{m_1-1}(j_i) \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} s_v+1 & s_v+2 & \cdots & s_v+m_v \\ j_v & \sigma(j_v) & \ldots & \sigma^{m_v-1}(j_v) \end{pmatrix},$$

and let $p = p_\tau$ in $\mathrm{Mat}_{m\times m}(C)$ be the permutation matrix attached to τ, that is, with $p_{i,j} = 0$ for $i \neq \tau(j)$ and $p_{\tau(j),j} = 1_C$ for all j. Then one verifies that $p_\tau c p_\tau^{-1}$ is a block diagonal matrix in $\mathrm{Mat}_{m\times m}(C)$ with v blocks on the diagonal, the l^{th} block lies in $\mathrm{Mat}_{m_l\times m_l}(C)$ and is of the form

$$B_l = \begin{pmatrix} 0 & 0 & & \ldots & c_{j_l,\sigma^{m_l-1}(j_l)} \\ c_{\sigma(j_l),j_l} & 0 & \ddots & & 0 \\ 0 & c_{\sigma^2(j_l),\sigma(j_l)} & \ddots & & \ddots \\ \vdots & \ddots & & & \\ 0 & \ldots & 0 \quad c_{\sigma^{m_l-1}(j_l),\sigma^{m_l-2}(j_l)} & & 0 \end{pmatrix}.$$

By conjugating B_l with the block diagonal matrix with entry at spot (i,i) the block given by $c_{\sigma^{i-1}(j_l),\sigma^{i-2}(j_l)} \cdot c_{\sigma^{i-2}(j_l),\sigma^{i-1}(j_l)} \cdot \ldots \cdot c_{\sigma(j_l),j_l}$ so that at $(1,1)$ the entry is 1, the matrix B_l is transferred to the block companion matrix

$$B_l' = \begin{pmatrix} 0 & 0 & & \ldots & c(l) \\ 1 & 0 & \ddots & & 0 \\ 0 & 1 & \ddots & & \ddots \\ \vdots & \ddots & & & \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

It suffices by a genericity argument to assume that A is an algebraically closed field so that C is split over A. Then the claim that B_l has characteristic polynomial $\chi_{c(l)}(t^{m_l})$ is a simple linear algebra calculation. In the case when $c(l)$ is a diagonal matrix, after an obvious change of basis B_l' becomes the direct sum of standard companion matrices each with the same type of last row as B_l' and hence with characteristic polynomial $t^{m_l} - c(l)_{i,i}$ so that one finds $\chi_{B_l}(t) = \chi_{c(l)}(t^{m_l})$. In the general case, one uses that semisimple matrices are dense open in the set of all matrices, or a devissage argument once $c(l)$ is in Jordan form.

A more heuristic argument for $\chi_{B_l}(t) = \chi_{c(l)}(t^{m_l})$ runs as follows. The matrix $B_l^{m_l}$ is block scalar with diagonal scalar factor $c(l) \in C^\times$. So by the Cayley–Hamilton theorem $B_l^{m_l}$ is annihilated by

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$\chi_{c(l)}(t)$, and hence B_l is annihilated by $\chi_{c(l)}(t^{m_l})$, and for a generic $c(l)$ one would expect $\chi_{c(l)}(t^{m_l})$ to be the minimal polynomial of B_l. □

Lemma 4.6.6. *Let the hypotheses be as in Lemma 4.6.1 with $N \subset H \subset G$, and let $g \in G$. Let $\gamma_{g,l} \in G$, $l = 1, \ldots, v_g$, form a set of representatives for the double coset space $\langle g\rangle\backslash G/H$ so that $G = \bigsqcup_{i=1,\ldots,v_g} \langle g\rangle\gamma_{g,l}H$, and let $m_l = [\langle g\rangle\gamma_{g,l}H : H]$ so that $m_l > 0$ is minimal such that $g^{m_l}\gamma_{g,l}H = \gamma_{g,l}H$. Then one has*

$$\chi_{\mathrm{Ind}_H^G\rho(g)}(t) = \prod_{l=1,\ldots,v_g} \chi_{\rho(\gamma_{g,l}^{-1}\cdot g^{m_l}\cdot\gamma_{g,l})}(t^{m_l}).$$

Moreover, on any left coset $g'N$, $g' \in G$, the map $g \longrightarrow v_g$ is constant, and the double cosets space $\langle g\rangle\backslash G/H$ is independent of g, and hence so are the m_l, and also one can choose uniform representatives $\gamma_{g,l}$ for $\langle g\rangle\backslash G/H$, independently of $g \in g'N$.

If H is normal in G, the double coset space $\langle g\rangle\backslash G/H$ is in bijection with the right coset space $\langle g\rangle H\backslash G$, and so $v_g = [G : \langle g\rangle H]$ and $m_l = m' := \mathrm{ord}_{G/H}(gH)$ is independent of l.

Proof. We first recall the construction of $\mathrm{Ind}_H^G\rho$ from the proof of Lemma 4.6.1. As a convenient set of coset representatives for G/H, we take $\bigsqcup_{l=1,\ldots,v_g}\{g^j\gamma_{g,l} : j = 0, \ldots, m_l - 1\}$. With this choice, $\rho^*(g)$ is already in block diagonal form with v_g blocks and the size of the l-th block is m_l. Moreover, by explicit computation, one finds that the l-th block has nonzero entries only at index pairs $(j, j-1)$, $j = 2, \ldots, m_l$, where the entry is $1 \in C$, and at $(1, m_l)$, where the entry is $\rho(\gamma_{g,l}^{-1}g^{m_l}\gamma_{g,l})$. The asserted formula is now an immediate consequence of the formula in Lemma 4.6.5.

Concerning the constancy statement for all g in a fixed coset $g'N$, observe that all assertions follow from the following observation: Let $\tilde{g} = gn$ for some $n \in N$, and let $\langle g\rangle\gamma H$ be a double coset for some $\gamma \in G$. Then by normality of N in G and using $N \subset H$, we have

$$\tilde{g}^i\gamma H = \tilde{g}^i\gamma NH = \tilde{g}^i N\gamma H = g^i N\gamma H = g^i\gamma H,$$

and thus we have equality of double cosets $\langle g'\rangle\gamma H = \langle g\rangle\gamma H$. The remaining assertions when H is normal in G we leave as exercises to the reader. □

Let now

$$D_H : B[H] \longrightarrow B$$

be a pseudocharacter of dimension n with values in a commutative ring B. If G is a profinite group, we assume that D_H is continuous. The following result establishes the existence of the induction of D_H.

Theorem 4.6.7. *There exists a unique pseudocharacter $D_G : B[G] \to B$ whose characteristic polynomial for each $g \in G$ is given as follows. Let $\gamma_{g,l}$, $l = 1, \ldots, v_g$, be elements of G that form a set of representatives for the double coset space $\langle g\rangle\backslash G/H$, and define $m_l = [\langle g\rangle\gamma_{g,l}H : H]$. Then*

$$\chi_{D_G,B}(g,t) = \prod_{l=1,\ldots,v_g} \chi_{D_H,B}(\gamma_{g,l}^{-1}\cdot g^{m_l}\cdot\gamma_{g,l}, t^{m_l}). \tag{22}$$

The pseudocharacter D_G has the following properties.

(a) *One has*

$$\mathrm{Res}_N^G D_G \cong \bigoplus_{g\in G/H} (\mathrm{Res}_N^H D_H)^g.$$

(b) *For any left coset $g'N$, $g' \in G$, in formula (22) the value of v_g and the elements $\gamma_{g,l}$ can be taken independent of $g \in g'N$. Hence, if G is profinite and D_H is continuous, then so is D_G.*

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(c) *The formation of* D_G *commutes with base change, that is, the following holds. Let* $B \to B'$ *be any homomorphism. Set* $D'_H := D_H \otimes_B B'$ *and* $D'_G := D_G \otimes_B B'$*. Then equation (22) holds with* D_H *and* D_G *replaced by* D'_H *and* D'_G*, respectively.*
(d) *For any geometric point* $\overline{x} \to \operatorname{Spec} B$ *the representations* $\rho_{D_{G,\overline{x}}}$ *is isomorphic to the semisimplification of* $\operatorname{Ind}_H^G \rho_{D_{H,\overline{x}}}$*.*
(e) *Suppose* $U = \operatorname{Spec} B' \subset \operatorname{Spec} B$ *is affine open such that* $D_{H,x}$ *is irreducible for all* $x \in U$ *and set* $D'_H = D_H \otimes_B B'$ *and* $C := B'[G]/\mathrm{CH}(D'_H)$ *so that by Proposition 4.1.25,* C *is an Azumaya* B*-algebra and* $\psi = \rho_D^{\mathrm{CH}} : G \to C^\times$ *is a representation such that* $\det_C \circ \psi = D'_H$*. Then for* $\operatorname{Ind}_H^G \psi \colon G \to \mathrm{GL}_m(C)^\times$ *from Definition 4.6.3, we have*

$$D_G \otimes_B B' = D_{\operatorname{Ind}_H^G \psi}.$$

(f) *One has* $D_G = D_G \otimes \chi$ *for any character* $\chi \colon G \to B^\times$ *whose kernel contains* H*.*
(g) *Suppose* H *is normal in* G*. Then* $\chi_{D_G,B}(g,t)$ *is a polynomial in* $t^{\mathrm{ord}_{G/H}(gH)}$*; and in particular its coefficients satisfy* $\Lambda_{D_G,i}(g) = 0$ *whenever* $\mathrm{ord}_{G/H}(gH) \nmid i$*.*

Proof. Let us first consider the situation irrespective of any topology, that is, G is an abstract group and $D_H : B[H] \to B$ is a pseudocharacter on H of a finite index subgroup. Let $\pi \colon \mathrm{FG}(X) \to G$ be a surjective group homomorphism from the free group on a suitable set of symbols X. By the Nielsen–Schreier Theorem, any subgroup of $\mathrm{FG}(X)$ is free again, and so let $X_H \subset \mathrm{FG}(X)$ be a subset of free generators of $\pi^{-1}(H)$, and write $\pi_H : \mathrm{FG}(X_H) \to H$ for the restriction of π to $\pi^{-1}(H)$.

Recall from Theorem A.4.4 the following description of the universal pseudocharacter of the group ring $\mathbb{Z}\{X_H^\pm\} = \mathbb{Z}[\mathrm{FG}(X_H)]$. Let $F_{X_H^\pm}(n) = \mathbb{Z}[\xi_{x,i,j} : x \in X_H, 1 \le i,j \le n][\frac{1}{\det(\xi_x)} : x \in X_H]$ be the commutative $\mathbb{Z}$-algebra of the coefficients of the generic invertible $n\times n$-matrices $\xi_x = (\xi_{x,i,j})_{i,j=1,\dots,n} \in \mathrm{Mat}_{n\times n}(F_{X_H^\pm}(n))$ of all $x \in X_H$. Define the representation

$$\rho_{X_H^\pm} : \mathbb{Z}\{X_H^\pm\} \longrightarrow \mathrm{Mat}_{n\times n}(F_{X_H^\pm}(n)), \quad x \longmapsto \xi_x,$$

and let $E_{X_H^\pm}(n)$ be the subring of $F_{X_H^\pm}(n)$ generated by $\det \circ \rho_{X_H^\pm}\big(\mathbb{Z}\{X_H^\pm\}\big)$. Then by Theorem A.4.4

$$D_{X_H^\pm} := \det \circ \rho_{X_H^\pm} : \mathbb{Z}\{X_H^\pm\} \longrightarrow E_{X_H^\pm}(n)$$

is the universal n-dimensional pseudocharacter of $\mathbb{Z}\{X_H^\pm\}$ (up to isomorphism).

Let $\alpha \colon E_{X_H^\pm}(n) \to B$ be the unique ring homomorphism (by universality of $D_{X_H^\pm}$) such that $\alpha \circ D_{X_H^\pm} = D_H \circ \pi_H : \mathbb{Z}\{X_H^\pm\} \to B$. Define

$$\operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm} := \det \circ \operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} \rho_{X_H^\pm} : \mathbb{Z}\{X^\pm\} \longrightarrow F_{X_H^\pm}(n)$$

with $\operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)}$ as in Lemma 4.6.1. Now, Lemma 4.6.6 shows that all characteristic polynomials of elements in $\mathrm{FG}(X)$ lie in $E_{X_H^\pm}(n)$, and it follows from [Che14, Corollary 1.14] (a consequence of Amitsur's formula) that $\operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm}$ has ring of definition contained in $E_{X_H^\pm}(n)$, and we regard it as a pseudocharacter

$$\operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm} : \mathbb{Z}\{X^\pm\} \longrightarrow E_{X_H^\pm}(n).$$

We now claim that the composition

$$\alpha \circ \operatorname{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm} : \mathbb{Z}\{X^\pm\} \longrightarrow B$$

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has kernel containing $K = \mathrm{Ker}(\pi\colon \mathrm{FG}(X) \to G)$. For this, note first that by Lemma 2.1.4(b) we have

$$\mathrm{Res}^{\mathrm{FG}(X)}_{\pi^{-1}(N)} \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} \rho_{X_H^\pm} = \bigoplus_{g\in G/H} \left(\mathrm{Res}^{\mathrm{FG}(X_H)}_{\pi^{-1}(N)} \rho_{X_H^\pm}\right)^{\tilde g},$$

for $\tilde g \in \mathrm{FG}(X_H)$ a preimage of g under π. If we compose the equation with det and α, this gives

$$\mathrm{Res}^{\mathrm{FG}(X)}_{\pi^{-1}(N)} \left(\alpha \circ \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}\right) = \bigoplus_{g\in G/H} \left(\mathrm{Res}^{H}_{N} D_H\right)^{g} \circ \pi_H. \tag{23}$$

The kernel of the right-hand side clearly contains K, and this proves the claim. As a consequence $\alpha \circ \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}$ factors via $\pi\colon \mathrm{FG}(X) \to G$, and using Proposition 4.1.16 we define

$$D_G \colon G \longrightarrow B$$

as the unique pseudocharacter such that $D_G \circ \pi = \alpha \circ \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}$. Note that by its very construction and by Lemma 4.6.6, $\mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}$ satisfies formula (22) and the formula is preserved under composition with α, and under passage via π from $\mathrm{FG}(X)$ to G. This implies formula (22) for D_G. Since by Proposition 4.1.10, the characteristic polynomial $\chi_{D_G}(\cdot, t)$ completely characterizes D_G, the required uniqueness of D_G is also shown.

We now prove Parts (a) to (f). Part (a) follows from equation (23) and our definition of D_G. The first part of (b) follows from the construction of D_G and Lemma 4.6.6. To prove the continuity assertion in (b), we need to show that the characteristic polynomial coefficients of D_G are continuous. But this follows from the first part of (b), equation (22) and the continuity hypothesis on D_H.

Part (c) is immediate from our construction which is based via pullback to the induction $\mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}$ in a free group setting. To see (d), note first that by Part (c) formula (22) is preserved under base change to $\kappa(\overline{x})$, that is, the formula holds if we replace simultaneously D_G by $D_{G,\overline{x}}$ and D_H by $D_{H,\overline{x}}$. By its definition, $\rho_{D_{H,\overline{x}}}$ has characteristic polynomial $\chi_{D_{H,\overline{x}}}$, and by Lemma 4.6.6, the right-hand side of equation (22) over $\kappa(\overline{x})$ is equal to $\chi_{\mathrm{Ind}_H^G \rho_{\overline{x}}}$. This proves (d). For Part (e), note that over B' we have equality of characteristic polynomials $\chi_\psi = \chi_{D'_H}$ from Proposition 4.1.25. Using Lemma 4.6.6 for $\mathrm{Ind}_H^G \psi$ and formula (22) and Part (c) for D'_H, we deduce $\chi_{D_G \otimes_B B', B'} = \chi_{\mathrm{Ind}_H^G \psi}$ and hence Part (e).

The proof of (f) will follow after pullback to $\mathrm{FG}(X)$. To carry this out, let d be the order of χ and $\mathbb{Z}[\chi]$ the extension of $\mathbb{Z}$ obtained by adjoining a primitive d-th root of unity. In an analogous way, we define $F_{X^\pm}(n)[\chi]$ and $E_{X^\pm}(n)[\chi]$ and we extend $\alpha\colon E_{X^\pm}(n) \to B$ to a homomorphism $\alpha\colon E_{X^\pm}(n)[\chi] \to B$. Let now $\chi_E \colon G \to E_{X^\pm}(n)$ be the unique character such that $\alpha \circ \chi_E = \chi$. By surjectivity of π (and by applying α), it will suffice to show that

$$\chi_E \otimes \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm} = \mathrm{Ind}^{\mathrm{FG}(X)}_{\mathrm{FG}(X_H)} D_{X_H^\pm}$$

But by construction of $D_{X_H^\pm}$ and Proposition 4.5.7(d) this reduces to the same formula with $\rho_{X_H^\pm}$, and this formula holds by Mackey's tensor product theorem formulated in Lemma 2.1.3.

Finally, Part (g) follows from the last part of Lemma 4.6.6: The normality of H in G implies that $m_l = \mathrm{ord}_{G/H}(gH)$ for all $l = 1, \ldots, \nu_g$, and so the formula for $\chi_{\mathrm{Ind}_H^G \rho(g)}(t)$ in that lemma is a polynomial in $t^{\mathrm{ord}_{G/H}(gH)}$. □

Definition 4.6.8. We call the pseudocharacter D_G from Theorem 4.6.7 the *induced pseudocharacter of* D_H *under* $H \subset G$ and write $\mathrm{Ind}_H^G D_H$ for it.

Remark 4.6.9. Our construction of $\mathrm{Ind}_H^G D_H$ does not need the generality of Azumaya algebra coefficients in Definition 4.6.3. However, over the absolutely irreducible locus of D_H, one has an elementary construction of induction indicated in Theorem 4.6.7(e). In fact, if B is for instance reduced and Noethe-

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rian and if the set U in Theorem 4.6.7(e) is dense in Spec B, one can uniquely reconstruct $\mathrm{Ind}_H^G D_H$ from this elementary construction. This approach had been pursued in an earlier version of this work.

Remark 4.6.10. In our approach to Theorem 4.6.7, we strongly rely on the explicit but somewhat technical formulas from Lemma 4.6.6. We use them to uniquely characterize the induction, once existence is shown. We also use these formulas in the existence part to show that $\mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm}$ takes values in $E_{X_H^\pm}(n)$; initially the corresponding representation is only known to be defined over $F_{X_H^\pm}(n)$. V. Paškūnas suggested in personal communication that perhaps one could avoid using Lemma 4.6.6, at least for the unique characterization of $\mathrm{Ind}_H^G D_H$. His remarks led to the following discussion, which sketches an alternative proof of Theorem 4.6.7:

Suppose first that the target B of $D_H : H \to B$ is a domain, and let η be the generic point of Spec B. Then $\mathrm{Ind}_H^G D_H$ is uniquely characterized by Theorem 4.6.7(d) for the geometric point $\overline{x}$: Spec $\kappa(\eta)^{\mathrm{alg}} \to$ Spec B because it states that $(\mathrm{Ind}_H^G D_H)_{\overline{x}}$ arises from $\mathrm{Ind}_H^G \rho_{D_{H,\overline{x}}}$.

To handle the case of general B in a similar way, one can regard induction as a functorial type construction in the sense that for any group epimorphism $\varphi\colon G' \to G$, any surjection of rings $\alpha\colon B' \to B$ and any pseudocharacter $D'_{H'}\colon H' \to B'$ with $H' := \varphi^{-1}(H)$ such that $\alpha \circ D'_H = D_H \circ \varphi$, one requires that $\alpha \circ (\mathrm{Ind}_{H'}^{G'} D'_H) = (\mathrm{Ind}_H^G D_H) \circ \varphi$; this compatibility can be shown for our construction. Assuming this compatibility, the uniqueness of $\mathrm{Ind}_H^G D_H$ follows from that of $\mathrm{Ind}_{H'}^{G'} D'_{H'}$ for a suitable $D'_{H'}$. And now one can apply the observation of the previous paragraph to the universal situation from the proof of Theorem 4.6.7, where $D'_{H'} = D_{X_H^\pm}$ and where $B' = E_{X_H^\pm}(n)$ is a domain. This shows that uniqueness follows from the stated functoriality and Theorem 4.6.7(d).

Lastly, we indicate how to deduce that $E_{X_H^\pm}(n)$ is the ring of definition of $\mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm}$ without the use of Lemma 4.6.6. For this, it suffices to assume that X is finite: To see this, let $Y \subset X$ be finite such that $Y \cup X_H$ generates $\mathrm{FG}(X)$. Then one verifies that it suffices to consider the restriction of $D_{X_H^\pm}$ to $\mathrm{FG}(Y')$ for all finite subsets $Y' \subset Y \cup X_H$ that contain Y. Assume now that X is finite and, by possibly adding generators, that X_H contains at least $n+4$ elements. We also assume that D_H is of degree $n > 1$, and we set $m = \#X_H$ so that $m \geq n+4$.

Consider the morphism $\pi\colon \mathrm{Spec}\, F_{X_H^\pm}(n) \to \mathrm{Spec}\, E_{X_H^\pm}(n)$. The ring $E_{X_H^\pm}(n)$ is a normal domain because it is the ring of invariants under the connected reductive group GL_n of the normal domain $F_{X_H^\pm}(n)$. Let $U \subset \mathrm{Spec}\, F_{X_H^\pm}(n)$ be the open subscheme over which the generic matrix representation $\rho_{X_H^\pm} : \mathbb{Z}\{X_H^\pm\} \to \mathrm{Mat}_{n\times n}(F_{X_H^\pm}(n)), x \mapsto \xi_x$ is irreducible, cf. [Che14, Example 2.20]. It is known that U is dense in $\mathrm{Spec}\, F_{X_H^\pm}(n)$, and we will give a much stronger result in the next paragraph. It is also known that the induced map $U \to V := \pi(U)$ is a PGL_n-torsor, and V is open nonempty and hence also dense in the integral scheme $\mathrm{Spec}\, E_{X_H^\pm}(n)$, see [Nak00, §3 and Corollary 6.5]. It follows that $\dim E_{X_H^\pm}(n) = \dim V = 1 + mn^2 - (n^2-1) = (m-1)n^2 + 2$ because clearly $\dim F_{X_H^\pm}(n) = 1 + mn^2$.

We claim that V contains all points of codimension at most 1 of $\mathrm{Spec}\, E_{X_H^\pm}(n)$. For this, we shall show that the reducible locus $Z := \mathrm{Spec}\, F_{X_H^\pm}(n) \backslash U$ has dimension at most $(m-1)n^2$ from which it follows that $\pi(Z) = \mathrm{Spec}\, E_{X_H^\pm}(n) \backslash V$ has codimension at least 2. Because Z is of finite type over $\mathbb{Z}$, it suffices to analyze the dimensions after base change from $\mathbb{Z}$ to an algebraically closed field k. Then $\rho_{X_H^\pm}$ is reducible at a closed point if and only if there is a proper parabolic subgroup P of GL_n that contains the set of matrices $\rho_{X_H^\pm}(X_H)$, that is the set can by simultaneously conjugated by GL_n to a standard parabolic P of GL_n. The stabilizer of this conjugation action is P itself and the dimension of P is at most $n^2 - n + 1$, and there are only finitely many such standard P once a maximal torus and a Borel are chosen for GL_n. It follows that the dimension over k of the set of reducible points is at most $m \dim P + (n^2 - \dim P) = (m-1)\dim P + n^2 = (m-1)n^2 - ((m-1)(n-1) - n^2)$ and the claim on $\dim Z$ follows from our hypothesis $m \geq n+4$.

We now give an argument independent of Lemma 4.6.6 that show that $\mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)} D_{X_H^\pm}$ takes values in $E_{X_H^\pm}(n)$: Let $\mathrm{Spec}\, B \subset V$ be any affine open subset. Then by Proposition 4.1.25 the pseudocharacter $D_{X_H^\pm} : \mathrm{FG}(X_H) \to E_{X_H^\pm}(n) \to B$ factors as a representation $\rho_B\colon \mathrm{FG}(X_H) \to C^\times$ for C an Azumaya B-algebra of degree n followed by the pseudocharacter associated to C in Example 4.1.7. By change

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of coefficients to an algebraic closure $\mathbb{K}$ of the generic point of $F_{X_H^\pm}$, it follows that $\mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)}\rho_{X_H^\pm}$ and $\mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)}\rho_B$, from Definition 4.6.3, are isomorphic over $\mathbb{K}$. Hence, $\det \circ \mathrm{Ind}_{\mathrm{FG}(X_H)}^{\mathrm{FG}(X)}\rho_{X_H^\pm}$ takes values in B. But because V contains all points of codimension at most 1 of the integral normal scheme $\operatorname{Spec} E_{X_H^\pm}(n)$ it follows that the intersection of all such rings B (inside $\mathrm{Frac}(E_{X_H^\pm}(n))$) is equal to $E_{X_H^\pm}(n)$, and we are done.

For later use, we formulate the following simple finiteness result related to induction.

Lemma 4.6.11. *Let k be a field, let $\chi\colon G \to k^\times$ be a character of finite order m with kernel $H := \ker\chi$ and let D be in $\mathcal{P}s\mathcal{R}_G^n(k^{\mathrm{alg}})$. Define*

$$\mathcal{S}_D := \{D' \in \mathcal{P}s\mathcal{R}_H^{n/m}(k^{\mathrm{alg}}) : \mathrm{Ind}_H^G D' = D\}.$$

Then the following hold:

(a) *$\mathcal{S}_D$ is finite.*
(b) *$\mathcal{S}_D$ is nonempty if and only if $D = D \otimes \chi$.*

If moreover G is profinite, k^{alg} carries the discrete topology and D is continuous, then there is a finite extension of k in k^{alg} over which all $D' \in \mathcal{S}_D$ are defined and split.

Proof. By Theorem 4.1.18 and Corollary 4.4.6, the map $\rho \to D_\rho$ from semisimple representations of G over k^{alg} to pseudocharacters of G over k^{alg} is a bijection, and the same holds over H. We also have $D_\rho \otimes \chi = D_{\rho\otimes\chi}$ by Proposition 4.5.7(d). Thus, (a) and (b) are really assertions on semisimple representations. Now, if ρ is a representation and if $\rho = \mathrm{Ind}_H^G \rho'$ for some representation ρ', then ρ' is a direct summand of the semisimple representation $\rho|_H$ by Lemma 2.1.4. Since up to isomorphism there are only finitely many such summands and since these are unique up to permutation, Part (a) follows. Part (b) is now immediate from Corollary 2.2.2. The last assertion is a consequence of Corollary 4.4.6 since $\mathcal{S}_D$ is finite. □

4.7. Pseudodeformations and their universal rings

This subsection recalls in Proposition 4.7.4 the main object of our interest, the universal pseudodeformation ring of a residual pseudocharacter $\overline{D}$. Here, continuity plays a major role. We state basic results relevant to the present work. In addition to the usual treatment, we also give some special attention to functors $\widehat{\mathcal{A}r}_\kappa \to \mathcal{S}ets$ where κ is a local field. The subsection also contains some results on deformations over formal schemes and on the locus of irreducibility.

We let $\mathbb{F}$ be either a finite or a local field; in the former case, Λ is a complete Noetherian local commutative $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$. In the latter case, $\Lambda = \mathbb{F}$. Recall the categories $\mathcal{A}r_\Lambda$ and $\widehat{\mathcal{A}r}_\Lambda$ from Subsection 3.1 and the topological conditions we impose on there objects and morphisms. By A, we denote a ring in $\widehat{\mathcal{A}r}_\mathbb{F}$; its maximal ideal is $\mathfrak{m}_A$ and it comes with a natural reduction map $\pi_A\colon A \to A/\mathfrak{m}_A = \mathbb{F}$. We let G be a profinite group and we denote by $\overline{D}\colon \mathbb{F}[G] \to \mathbb{F}$ a continuous pseudocharacter of dimension n.

Definition 4.7.1 [WE13, §3.1.4.3].

(a) A *pseudodeformation of $\overline{D}$ to A* is a continuous pseudocharacter $D\colon A[G] \to A$ such that $D\otimes_A \mathbb{F} = \pi_A \circ D\colon \mathbb{F}[G] \to \mathbb{F}$ is equal to $\overline{D}$.
(b) The functor

$$\mathcal{P}s D_{\overline{D}}\colon \widehat{\mathcal{A}r}_\Lambda \to \mathcal{S}ets, \quad A \longmapsto \{D\colon G \longrightarrow A \text{ is a pseudodeformation of } \overline{D}\},$$

is called the *pseudodeformation functor* of the *residual pseudocharacter $\overline{D}$*.

Note that unlike in parts of [WE13] for us all pseudodeformations will be continuous.

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Definition 4.7.2. Let $\pi\colon B \to \mathbb{F}$ be a morphism in $\mathcal{CA}lg_\Lambda$, and let $D\colon B[G] \to B$ be a pseudocharacter such that $D \otimes_B \mathbb{F} = \overline{D}$.

An ideal I of B is called *D-open* if the following conditions hold:

(a) The map π factors via B/I and B/I is a local Artin ring.
(b) $D_I := D \otimes_B B/I$ is continuous if we equip B/I with the topology of an object in $\mathcal{A}r_\Lambda$.

Lemma 4.7.3. *With the notation from Definition 4.7.2, the D-open ideals form a basis of a topology on B.*

Proof. (Cf. [WE13, Theorem 3.1.4.6]) One has to show that if I, I' are D-open ideals, then so is $I \cap I'$. Consider the injective homomorphism

$$\iota\colon B/(I\cap I') \longrightarrow B/I \times B/I'.$$

For both Λ that we consider, it is straightforward to see that ι is a topological isomorphism onto its image. Now, a pseudocharacter is continuous if and only if this holds for its characteristic polynomial functions; cf. Definition 4.4.1. Since both I and I' are D-open, it is now immediate that $I \cap I'$ is D-open. □

The following result is proved in [Che14, Proposition 3.3] for $\Lambda = W(\mathbb{F})$ and in [WE13, Theorem 3.1.4.6] for $\Lambda \in \widehat{\mathcal{A}r}_{W(\mathbb{F})}$.

Proposition 4.7.4. *The pseudodeformation functor $\mathcal{P}sD_{\overline{D}}$ is prorepresentable by a topological Λ-algebra $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ that is a filtered inverse limit of objects in $\mathcal{A}r_\Lambda$, together with a universal pseudodeformation*

$$D^{\mathrm{univ}}_{\Lambda,\overline{D}}\colon G \longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}.$$

Proof. We recall a sketch of the proof from [WE13, Theorem 3.1.4.6] to indicate that it also applies to the case when $\Lambda = \kappa$ is a local field. Consider the universal ring $R^{\mathrm{univ}}_{\Lambda[G],n}$ from Definition 4.2.2 with its universal pseudocharacter $D^{\mathrm{univ}}_{\Lambda[G]}\colon G \longrightarrow R^{\mathrm{univ}}_{\Lambda[G],n}$ on G. By definition, $R^{\mathrm{univ}}_{\Lambda[G],n}$ is a Λ-algebra. The map $\overline{D}$ induces a Λ-algebra homomorphism $\pi\colon R^{\mathrm{univ}}_{\Lambda[G],n} \to \mathbb{F}$. By Lemma 4.7.3, the D-open ideals of $R^{\mathrm{univ}}_{\Lambda[G],n}$ form the basis of a topology on $R^{\mathrm{univ}}_{\Lambda[G],n}$, and one defines $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ as the completion of $R^{\mathrm{univ}}_{\Lambda[G],n}$ with respect to this topology. It is then straightforward to establish the asserted properties for $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ together with the pseudocharacter $D^{\mathrm{univ}}_{\Lambda,\overline{D}} := D^{\mathrm{univ}}_{\Lambda[G]} \otimes_{R^{\mathrm{univ}}_{\Lambda[G,]}n} R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ by verifying it for the restriction of $\mathcal{P}sD_{\overline{D}}$ to $\mathcal{A}r_\Lambda$. □

Definition 4.7.5. The ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ from Proposition 4.7.4 is called the *universal (Λ-)pseudodeformation ring of $\overline{D}$*, the pseudocharacter $D^{\mathrm{univ}}_{\Lambda,\overline{D}}\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}}[G] \to R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ the *universal* (Λ-)*-pseudodeformation of $\overline{D}$* and the space $X^{\mathrm{univ}}_{\Lambda,\overline{D}} := \operatorname{Spec} R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ the *universal (Λ-)pseudodeformation space of $\overline{D}$*; we write $R^{\mathrm{univ}}_{G,\Lambda,\overline{D}}$ if there is a need to indicate G; we often drop the index Λ if it is clear from context.

The ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ behaves well under change of the coefficient ring Λ.

Proposition 4.7.6 (Cf. [Wil95, p. 457]). *Let $\bar{f}\colon \kappa \to \kappa'$ be a homomorphism between either two finite or two local fields, and let $f\colon \Lambda \to \Lambda'$ be a local homomorphisms of complete local Noetherian commutative rings that reduces on residue fields to $\bar{f}$. Define $\overline{D}' := \overline{D} \otimes_\kappa \kappa'\colon \kappa'[G] \to \kappa'$. Then one has a natural isomorphism*

$$R^{\mathrm{univ}}_{\Lambda',\overline{D}'} \longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}} \hat{\otimes}_\Lambda \Lambda'.$$

Proof. The proof is as in [Wil95, p. 457] for deformation rings: If $\bar{f}$ is the identity, one can proceed as follows. Any $A \in \widehat{\mathcal{A}r}_{\Lambda'}$ can be regarded as a ring in $\widehat{\mathcal{A}r}_\Lambda$ via the action induced from f; the residue fields of A, Λ and Λ' are the same. Then the assertion follows rapidly by using the isomorphism $\mathrm{Hom}_{\widehat{\mathcal{A}r}_\Lambda}(A,B) \cong \mathrm{Hom}_{\Lambda'}(A\otimes_\Lambda \Lambda', B)$ for $A\in\widehat{\mathcal{A}r}_\Lambda$ and $B\in\widehat{\mathcal{A}r}_{\Lambda'}$ together with the universal properties of $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}$ and $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$.

In the general case, define for any $B'\in\widehat{\mathcal{A}r}_{\Lambda'}$ the ring B'' as the subring of B' of elements whose reduction to κ' lies in the subfield κ so that $B''\in\widehat{\mathcal{A}r}_{\Lambda''}$. The argument just given applies to $\Lambda\longrightarrow\Lambda''$. For $\Lambda''\longrightarrow\Lambda'$ note first that any $D'\in\mathcal{P}sD_{\Lambda',\overline{D}'}(B')$ takes values in B'' because $\overline{D}'$ takes values in κ so that D' defines a $D''\in\mathcal{P}sD_{\Lambda'',\overline{D}}(B'')$. Conversely, if such a D'' is given, we may form $D''\otimes_{\Lambda''}\Lambda'$ and compose it with the natural Λ'-homomorphism $B''\otimes_{\Lambda''}\Lambda'\to B'$ to get back to D'. This yields the following chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\Lambda'}(R^{\mathrm{univ}}_{\Lambda',\overline{D}'},B') \cong \mathcal{P}sD_{\Lambda',\overline{D}'}(B') &\cong \mathcal{P}sD_{\Lambda'',\overline{D}}(B'')\\ &\cong \mathrm{Hom}_{\Lambda''}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}},B'') \cong \mathrm{Hom}_{\Lambda''}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}},B')\\ &\cong \mathrm{Hom}_{\Lambda'}(R^{\mathrm{univ}}_{\Lambda'',\overline{D}}\otimes_{\Lambda''}\Lambda',B')\end{aligned}$$

We deduce $R^{\mathrm{univ}}_{\Lambda',\overline{D}'}\cong R^{\mathrm{univ}}_{\Lambda'',\overline{D}}\otimes_{\Lambda''}\Lambda'$ because any $B'\in\widehat{\mathcal{A}r}_{\Lambda'}$ can occur as test objects. □

The previous proposition justifies the following definition.

Definition 4.7.7. If $\mathbb{F}$ is finite, we call $\overline{R}^{\mathrm{univ}}_{\overline{D}} := R^{\mathrm{univ}}_{\mathbb{F},\overline{D}}$ the *universal mod p pseudodeformation ring of* $\overline{D}$ and we call $\overline{X}^{\mathrm{univ}}_{\overline{D}} := X^{\mathrm{univ}}_{\mathbb{F},\overline{D}}$ the *special fiber* of the universal pseudodeformation space of $\overline{D}$.

We shall also need to consider Cayley–Hamilton quotients. Recall from Remark 4.4.4 that $D^{\mathrm{univ}}_{\Lambda,\overline{D}}$ induces a continuous pseudorepresentation (for which we shall use the same name)

$$D^{\mathrm{univ}}_{\Lambda,\overline{D}}\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]]\longrightarrow R^{\mathrm{univ}}_{\Lambda,\overline{D}}.$$

Let the following be the diagram induced from diagram (16)

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \xrightarrow{\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}} S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} := (R^{\mathrm{univ}}_{G,\overline{D}}[[G]])^{\mathrm{CH}}_{D^{\mathrm{univ}}_{\Lambda,\overline{D}}} \xrightarrow{D^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}} R^{\mathrm{univ}}_{\Lambda,\overline{D}}. \tag{24}$$

Definition 4.7.8. For 'object' the algebra $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$, the CH-representation $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$, or the pseudocharacter $D^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$, respectively, we use the term *universal Cayley–Hamilton object attached to* $\overline{D}$.

Remark 4.7.9. As explained in [Che14, Proposition 1.23], the factorization in diagram (24) has indeed a universal property.

Definition 4.7.10 (Cf. [WE13, 3.1.5]). Suppose $\mathbb{F}$ is finite. Then we define condition $\Phi_{\overline{D}}$ to be condition $\Phi_{\rho_{\overline{D}\otimes_{\mathbb{F}}\mathbb{F}^{\mathrm{alg}}}}$ from Definition 3.2.2.

We recall a criterion for $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ and $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$ to be Noetherian.

Proposition 4.7.11 [WE18, Propositions 3.2 and 3.6]. *The following hold if* $\mathbb{F}$ *is finite and* $\Phi_{\overline{D}}$ *holds:*

(a) *The topological* Λ*-algebra* $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ *lies in* $\widehat{\mathcal{A}r}_\Lambda$.
(b) *The CH-representation* $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$ *is a continuous homomorphism.*
(c) *The ring* $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$ *is module-finite as an* $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$*-algebra, and therefore Noetherian.*
(d) *On* $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$ *the profinite topology, the* $\mathfrak{m}_{\overline{D}}$*-adic topology, and the quotient topology from the surjection* $\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}$ *are equivalent.*

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Remark 4.7.12. Suppose $G = G_K$ for K a p-adic field. Then by Propositions 3.2.3 and 4.7.11, the ring $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ is Noetherian.

Corollary 4.7.13. *Suppose $\mathbb{F}$ is finite and $\Phi_{\overline{D}}$ holds. Let A be a quotient of $R := R^{\mathrm{univ}}_{W(\mathbb{F}),\overline{D}}$. Then A is the ring of definition of $D_A = D^{\mathrm{univ}}_{\overline{D}} \otimes_R A$ over $W(\mathbb{F})$ in the sense of Definition 4.4.10.*

Proof. Let $C \subset A$ be the ring of definition of D_A over $W(\mathbb{F})$, and let D_C be the pseudocharacter over C such that $D_C \otimes_C A = D_A$. By the universality of R we have a unique $W(\mathbb{F})$-algebra homomorphism $R \to C$ such that $D_C = D^{\mathrm{univ}}_{\overline{D}} \otimes_R C$. We deduce that the composition $R \to C \hookrightarrow A$ is equal to the initially given quotient map. Hence, $C \hookrightarrow A$ must be the identity. □

We shall also need the following result which in parts can be traced back to the proof of Proposition 4.7.11 in [WE13].

Proposition 4.7.14. *If $\mathbb{F}$ is finite and condition $\Phi_{\overline{D}}$ is satisfied, then the following hold:*

(a) *For any $\varphi\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to A$ in $\widehat{\mathcal{A}r}_\Lambda$ giving rise to the pseudocharacter D_A, the induced maps*

$$(\Lambda[[G]] \otimes_\Lambda A)^{\mathrm{CH}}_{D_A} \to (R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A} \to (A[[G]])^{\mathrm{CH}}_{D_A} \to S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A$$

are isomorphisms.

(b) *The $\mathbb{F}$-algebra $(\mathbb{F}[[G]])^{\mathrm{CH}}_{\overline{D}}$ is finite-dimensional as an $\mathbb{F}$-vector space.*

Proof. For (a) consider the maps in

$$\Lambda[[G]] \otimes_\Lambda A \to R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A \to A[[G]]. \tag{25}$$

They are injective with dense image. By the definition of the Cayley–Hamilton ideal, this still holds after passing to Cayley–Hamilton quotients. By [WE13, Corollary 1.2.2.9 and Proposition 3.2.2.1] the A-algebra $(A[[G]])^{\mathrm{CH}}_{D_A}$ is a finitely generated A-module and hence Noetherian. It follows that its subrings $(\Lambda[[G]] \otimes_\Lambda A)^{\mathrm{CH}}_{D_A} \subset (R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A}$ are also finite A-modules. By completeness of A and their density in $(A[[G]])^{\mathrm{CH}}_{D_A}$, the inclusions must be equalities. By Proposition 4.1.22(c), we also know that the formation of the Cayley–Hamilton quotient commutes with base change. Hence, $(R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A)^{\mathrm{CH}}_{D_A} \to S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} A$ is an isomorphism, and this completes the proof of (a). Part (b) follows from [WE13, Theorem 1.3.3.2]; it is also a consequence of Part (a) and Proposition 4.7.11. □

The next result concerns the reducible locus for multiplicity free $\overline{D}$.

Corollary 4.7.15. *Suppose $\overline{D}$ is split and multiplicity free over $\mathbb{F}$ and equal to $\overline{D}_1 \oplus \overline{D}_2$. Then the morphism $\iota_{\overline{D}_1,\overline{D}_2}\colon X^{\mathrm{univ}}_{\overline{D}_1} \widehat{\times} X^{\mathrm{univ}}_{\overline{D}_1} \to X^{\mathrm{univ}}_{\overline{D}}, (D_1, D_2) \mapsto D_1 \oplus D_2$ is a closed immersion.*

Proof. We need to show that the ring homomorphism

$$R^{\mathrm{univ}}_{\overline{D}} \longrightarrow R^{\mathrm{univ}}_{\overline{D}_1} \hat{\otimes}_{\mathbb{F}} R^{\mathrm{univ}}_{\overline{D}_2}$$

corresponding to $\iota_{\overline{D}_1,\overline{D}_2}$ is surjective. Since both sides are complete Noetherian local rings with isomorphic residue field, it suffices to show the surjectivity for the induced map of the duals of their tangent spaces; that is, the injectivity of

$$\mathcal{P}sD_{\overline{D}_1}(\mathbb{F}[\varepsilon]) \times \mathcal{P}sD_{\overline{D}_2}(\mathbb{F}[\varepsilon]) \longrightarrow \mathcal{P}sD_{\overline{D}}(\mathbb{F}[\varepsilon]), \quad (D_1, D_2) \longmapsto D_1 \oplus D_2. \tag{26}$$

Consider n_i-dimensional pseudodeformations $D_i, D_i' \in \mathcal{P}sD_{\overline{D}_i}(\mathbb{F}[\varepsilon])$ for $i = 1, 2$ such that $D := D_1 \oplus D_2 = D_1' \oplus D_2'$. We need to show $D_i = D_i'$ for $i = 1, 2$.

Let $A = \mathbb{F}[\varepsilon]$ and let S be the Cayley–Hamilton algebra $A[G]/\mathrm{CH}(D)$. Observe that $\mathrm{CH}(D)$ is contained in both, $\mathrm{CH}(D_i)$ and $\mathrm{CH}(D_i')$: We explain this for D_i. Recall $s \mapsto \chi_{D,B}(\cdot) = \chi_{D,B}(t,s)|_{t=s}$ from Lemma 4.1.8(b). The equality $D = D_1 \oplus D_2$ implies

$$\chi_{D,A[t_1,\dots,t_n]}(\textstyle\sum_i s_i t_i) = \chi_{D_1,A[t_1,\dots,t_n]}(\sum_i s_i t_i) \cdot \chi_{D_2,A[t_1,\dots,t_n]}(\sum_i s_i t_i),$$

for $n \in \mathbb{N}$, $s_1,\dots,s_n \in S$. The ideal $\mathrm{CH}(D_i)$ is generated by the coefficients of all polynomials $\chi_{D_i,A[t_1,\dots,t_n]}(\sum_i s_i t_i)$. It follows from the displayed formula, that $\mathrm{CH}(D)$ is zero modulo $\mathrm{CH}(D_i)$, and this gives $\mathrm{CH}(D_i) \supset \mathrm{CH}(D)$. As a consequence, we find that D_i and D_i' factor via S.

By Theorem 4.3.10(b), the Cayley–Hamilton algebra is a GMA over A with $D = \det_{(S,\mathcal{E})}$ for a datum of idempotents $\mathcal{E} = \{e_j, \psi_j\}_{j=1,\dots,r}$. The proof in [Che14, Theorem 2.22] shows that the idempotents e_j correspond bijectively to the irreducible summands of $\overline{D}$, and so we write $\overline{D} = \oplus_j \overline{D}_{e_j}$ in the notation of Lemma 4.3.8. Write $J = \{1,\dots,r\}$ as a disjoint union $J = J_1 \cup J_2$ such that $\overline{D}_i = \oplus_{j\in J_i} \overline{D}_{e_j}$, using that $\overline{D}$ is mulitplicity free.

Because S is a GMA, the algebra $e_j S e_j$ is isomorphic to $\mathrm{Mat}_{n_j\times n_j}(A)$ for some $n_j \in \mathbb{N}$, and where $\sum_j n_j = n$. It follows from Example 4.1.7 in particular that $(D_i)_{e_j}$ is $n_j f_{i,j}$-dimensional for some $f_{i,j} \in \mathbb{N}_0$. Using $(D_i)_{e_j} \bmod (\varepsilon) = (\overline{D}_i)_{e_j}$, we find $f_{i,j} = 1$ for $j \in J_i$ and $f_{i,j} = 0$ for $j \in J_{3-i}$.

Let $E_i = \sum_{j\in J_i} e_j$. Then by Lemma 4.3.8, we have $\dim((D_i)_{E_i}) = \dim(D_i)$ and $\dim((D_i)_{E_{3-i}}) = 0$, and thus $D_i = (D_i)_{E_i} = (D_i)_{E_i} \oplus (D_{3-i})_{E_i} = D_{E_i}$. But the idempotent E_i only depends on $\overline{D}_i$, and so arguing in the same way for the D_i', we find $D_i' = D_{E_i} = D_i'$, which concludes the proof. □

The locus of irreducible points shall be of special importance.

Definition 4.7.16. The *irreducible locus of* $X^{\mathrm{univ}}_{\overline{D}}$ is defined as

$$(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} := \{x \in X^{\mathrm{univ}}_{\overline{D}} : (D^{\mathrm{univ}}_{\overline{D}})_x \text{ is irreducible}\}$$

and its *reducible locus* $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ as the topological space $X^{\mathrm{univ}}_{\overline{D}} \backslash (X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}}$. We overline the notation for the corresponding subsets of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$.

The argument in [Che14, Example 2.20] also proves.

Proposition 4.7.17. *The subsets* $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset X^{\mathrm{univ}}_{\overline{D}}$ *and* $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *are Zariski open.*

By Proposition 4.7.11(c), we can associate to $S^{\text{CH-univ}}_{\Lambda,\overline{D}}$ a sheaf of coherent $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\overline{D}}}$-algebras $\mathcal{S}^{\text{CH-univ}}_{\Lambda,\overline{D}}$ under the finiteness condition $\Phi_{\overline{D}}$. The next result is not stated verbatim in [Che14]; however, its proof is that of [Che14, Corollary 2.23], with a continuity requirement added.

Proposition 4.7.18. *Over* $(X^{\mathrm{univ}}_{\Lambda,\overline{D}})^{\mathrm{irr}}$*, the sheaf* $\mathcal{S}^{\text{CH-univ}}_{\Lambda,\overline{D}}$ *is an Azumaya* $\mathcal{O}_{X^{\mathrm{univ}}_{\Lambda,\overline{D}}}$*-algebra of rank* n^2 *equipped with its reduced norm.*

Over affine open subsets of $(X^{\mathrm{univ}}_{\Lambda,\overline{D}})^{\mathrm{irr}}$, Proposition 4.7.18 is a variant of Proposition 4.1.25 under some continuity constraints.

4.8. Pseudodeformations over local fields

In this subsection, we develop some results analogous to Subsection 3.3 for continuous pseudodeformations of a fixed one $D\colon \kappa[G] \to \kappa$, where κ is a local field. Also, continuity is an important theme; for instance, to deduce under weak hypotheses from the continuity of a pseudocharacter that of its associated representation.

Lemma 4.8.1. *Let* κ *be a local field with valuation ring* $\mathcal{O}_\kappa$*, and let* $D\colon \kappa[G] \to \kappa$ *be a continuous* n*-dimensional pseudocharacter. Then the following hold:*

(a) *There exists $D_{\mathcal{O}} \in \mathcal{P}s\mathcal{R}^n_G(\mathcal{O}_\kappa)$ such that $D_{\mathcal{O}} \otimes_{\mathcal{O}_\kappa} \kappa = D$.*

Let $C \subset \mathcal{O}_\kappa$ be the admissible profinite subring of $\mathcal{O}_\kappa$ from Lemma 4.4.8, and let $D_C \in \mathcal{P}s\mathcal{R}^n_G(C)$ be such that $D_C \otimes_C \mathcal{O}_\kappa = D_{\mathcal{O}}$. Then furthermore:

(b) *C is local, its residue field $\kappa(C)$ is finite, either C is a finite field, or κ is a finite extension of the fraction field of C, $\overline{D} := D_C \otimes_C \kappa(C)$ is equal to D_z for some $z \in |G(n)|$ and $D_{\mathcal{O}}$ is residually equal to D_z in the sense of Definition 4.4.10.*

Proof. Let $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}$ be the representation from Theorem 4.1.18. For (a) observe first that the characteristic polynomial coefficients $\Lambda_{D,i}$ of $\chi_D(g,\,\cdot\,)$ are continuous for $1 \le i \le n$, and hence the sets $\Lambda_{D,i}(G)$ are compact in κ. Assume that for some $g \in G$, $\Lambda_{D,i}(g)$ does not lie in $\mathcal{O}_\kappa$. Then at least one eigenvalue of $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}(g)$ has valuation different from 0, and, since we can pass to g^{-1}, we may assume that this valuation is negative. Let $\lambda_1, \ldots, \lambda_n \in \kappa^{\mathrm{alg}}$ denote the eigenvalues of $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}(g)$ and index them so that $\lambda_1, \ldots, \lambda_j$ are precisely those with negative valuation. Then for $n > 0$, the valuation of $\Lambda_{D,j}(g^n)$ is the valuation of $(\lambda_1 \cdot \ldots \cdot \lambda_j)^n$. The latter valuations are unbounded. This contradicts the compactness of $\Lambda_{D,j}(G)$ and thus proves (a).

We now prove (b). By Lemma 4.4.8, the ring C is a finite product $\prod_i C_i$ of local admissible profinite $W(\mathbb{F})$-algebras C_i and the residue field of each C_i is finite. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_\kappa$. Then $C \cap \mathfrak{m}$ is topologically nilpotent and $C/(C \cap \mathfrak{m})$ is a finite field that surjects onto the product of the residue fields of the C_i. It follows that C is local with finite residue field $\kappa(C)$.

It remains to show the assertion on the fraction field of C, since the last part of (b) follows from Lemma 4.4.8. For this, we may assume that C is infinite. Let κ' be the fraction field of C. Because C is infinite and $\kappa(C)$ is finite, we find $f \in C\backslash\kappa(C)$ so that f has strictly positive valuation. Then $\kappa' \supseteq \kappa(C)((f))$ is a nontrivially valued locally compact subfield of the locally compact field κ. It now follows from [Wei67, I.§2. Corollary 2 of Theorem 3, p. 6] that $[\kappa : \kappa']$ is finite. □

The following result is a generalization of Corollary 4.4.6.

Corollary 4.8.2. *Let κ be a local field, let A be in $\mathcal{A}r_\kappa$ and let $D \in \mathcal{P}s\mathcal{R}^n_G(A)$ be continuous. Define $\overline{D}$ as in Lemma 4.8.1, and assume that condition $\Phi_{\overline{D}}$ holds. Then the following hold:*

(a) *If $A = \kappa$, then $\rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}$ is continuous.*
(b) *If D is split and irreducible, then $\rho_D = \rho^{\mathrm{CH}}_D$ from Proposition 4.1.25 is a continuous representation to* $\mathrm{Mat}_{n\times n}(A)$.

Proof. We first prove (a), and so here we assume $A = \kappa$. Set $\Lambda := \mathcal{O}_\kappa$, and consider the diagram

$$R^{\mathrm{univ}}_{\Lambda,\overline{D}}[[G]] \xrightarrow{\rho^{\mathrm{CH}}_{\Lambda,\overline{D}}} S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \xrightarrow{\mathrm{id}\otimes\varphi} S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \mathcal{O}_\kappa \xrightarrow{\mathrm{id}\otimes\iota} S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \kappa^{\mathrm{alg}},$$

where $\varphi\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to \mathcal{O}_\kappa$ is the map induced from the universal property of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$, and where $\iota\colon \mathcal{O}_\kappa \to \kappa^{\mathrm{alg}}$ is the natural inclusion. The first map is continuous by Proposition 4.7.11(b), the second by Proposition 4.7.11(d), which says that $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}}$ carries the $\mathfrak{m}_{\overline{D}}$-adic topology, By Proposition 4.7.11(d), the ring $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \mathcal{O}_\kappa$ is finitely generated as an $\mathcal{O}_\kappa$-module, and hence the $\mathfrak{m}_{\overline{D}}$-topology also coincides with the topology inherited from $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \kappa \in \mathcal{A}r_\kappa$; it follows that also the last map is continuous and that $S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \kappa^{\mathrm{alg}}$ has finite κ^{alg}-dimension. But then also the map

$$S^{\mathrm{CH\text{-}univ}}_{\Lambda,\overline{D}} \otimes_{R^{\mathrm{univ}}_{\Lambda,\overline{D}}} \kappa^{\mathrm{alg}} \xrightarrow[4.7.14]{\simeq} (\kappa^{\mathrm{alg}}[[G]])^{\mathrm{CH}}_D \longrightarrow \kappa^{\mathrm{alg}}[[G]]/\ker(D)$$

is continuous. Hence, in the factorization of $D\colon \kappa[G] \to \kappa^{\mathrm{alg}}$ via $\kappa^{\mathrm{alg}}[[G]]/\ker(D)$ given in Proposition 4.1.16, the first map is continuous. From Theorem 4.1.17, we know that $\kappa^{\mathrm{alg}}[[G]]/\ker(D)$ is semisimple and finite-dimensional over κ^{alg} and that its determinants are given by determinants of the simple matrix algebra factors of $\kappa^{\mathrm{alg}}[[G]]/\ker(D)$. Hence, the second map in the factorization given in Proposition 4.1.16 is continuous and thus so is the composition ρ.

The proof of (b) is analogous. One has to replace κ by A in most places and substitute Theorem 4.1.17 by Theorem 4.3.10. □

Remark 4.8.3. In an abstract setting, $\Phi_{\overline{D}}$ in Corollary 4.8.2 seems hard to check. In our concrete applications, we know $\overline{D}$ because of Proposition 3.2.3, so then the formulation is useful. A more natural condition to require would be Φ_D; we do suspect that this condition also suffices. Also, we wonder if the conclusion of Corollary 4.8.2 might hold without assuming $\Phi_{\overline{D}}$, and without invoking Lemma 4.8.1 just because $A \in \mathcal{A}r_\kappa$.

Corollary 4.8.4. *Let κ be a local field, and let $D \in \mathcal{P}s\mathcal{R}^n_G(\kappa)$ be continuous. Define $\overline{D}$ as in Lemma 4.8.1, and assume that condition $\Phi_{\overline{D}}$ holds. Then there exists a finite extension κ' of κ and split irreducible continuous $D_i \in \mathcal{P}s\mathcal{R}^{n_i}_G(\kappa')$, $i = 1, \ldots, r$ such that*

$$D \otimes_\kappa \kappa' = D_1 \oplus \ldots \oplus D_r \tag{27}$$

Moreover, $D \in \mathcal{P}s\mathcal{R}^n_G(\mathcal{O}_\kappa)$ and $D_i \in \mathcal{P}s\mathcal{R}^n_G(\mathcal{O}_{\kappa'})$ for $i = 1, \ldots, r$.

Proof. By Theorem 4.1.17, the κ-algebra $S := \kappa[[G]]/\ker(D \otimes_{\mathcal{O}_\kappa} \kappa)$ has finite κ-dimension. Hence, Lemma A.2.3 allows us to find a finite extension κ' of κ such that $S \otimes_\kappa \kappa'/\mathrm{Rad}(S \otimes_\kappa \kappa')$ is a product of matrix rings over κ'. It follows that $\kappa'[[G]]/\ker(D \otimes_{\mathcal{O}_\kappa} \kappa')$ is a product of matrix algebras over κ'. Hence, we have $D \otimes_\kappa \kappa' = \oplus_{i=1}^r D_i$ for split irreducible $D_i \in \mathcal{P}s\mathcal{R}^{n_i}_G(\kappa')$. We find that $(\oplus_{i=1}^r \rho_{D_i}) \otimes_{\kappa'} \kappa^{\mathrm{alg}} \cong \rho_{D \otimes_{\mathcal{O}_\kappa} \kappa^{\mathrm{alg}}}$. Since the latter is continuous by Corollary 4.8.2, so are the ρ_{D_i}. Finally, Lemma 4.8.1 shows that the D_i can be defined over $\mathcal{O}_{\kappa'}$ and D over $\mathcal{O}_\kappa$. □

To prove a more general result than Corollary 4.8.4, we need some preparations.

Lemma 4.8.5. *Let $\mathbb{F}$ be a finite field, let $A \in \widehat{\mathcal{A}r}_{\mathbb{F}}$ be a domain and let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime of dimension 1, and consider the completion $\widehat{A}_{\mathfrak{p}}$ as a topological ring in $\widehat{\mathcal{A}r}_{\kappa(\mathfrak{p})}$. Then the canonical map $\iota\colon A \to \widehat{A}_{\mathfrak{p}}$ is continuous and injective, and $A \to \iota(A)$ is a homeomorphism if $\iota(A)$ is equipped with the subspace topology.*

Proof. The injectivity of ι is clear, since $A \to A_{\mathfrak{p}}$ is injective, as A is a domain, and completion is injective since $A_{\mathfrak{p}}$ is Noetherian.

Recall that A carries the $\mathfrak{m}_A$-adic topology and that the topology on $\widehat{A}_{\mathfrak{p}}$ is the weakest topology such that the canonical maps $\widehat{A}_{\mathfrak{p}} \to R_n := A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} \cong \widehat{A}_{\mathfrak{p}}/\mathfrak{p}^n \widehat{A}_{\mathfrak{p}}$ are continuous for all n, with R_n carrying the unique topology as a finite-dimensional vector space with a continuous action of the local field $\kappa(\mathfrak{p})$. Let $\iota_n\colon A \to \widehat{A}_{\mathfrak{p}} \to R_n$ be the canonical map. Because A and $\widehat{A}_{\mathfrak{p}}$ are topological modules, it remains to prove continuity near 0, that is, we have to show the following two assertions: (i) For $n \in \mathbb{N}$ and $U \subset R_n$ an open neighborhood of 0, there exists $m \in \mathbb{N}$ such that $\iota_n(\mathfrak{m}_A^m) \subset U$. (ii) For $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $U \subset R_n$ open such that $\mathfrak{m}_A^m \supset \iota_n^{-1}(U)$.

Before we tackle (i) and (ii), we show the following assertion (iii): There exists $\omega \in \mathfrak{m}_A$ with nonzero image in $A/\mathfrak{p}$ such that for each $n \geq 1$ there exists a coefficient field K_n for R_n such that $K_n \supset \mathbb{F}[[\iota_n(\omega)]]$ – we also gather further properties of $\mathbb{F}[[\iota_n(\omega)]]$; recall that being a coefficient field means that $K_n \subset R_n$ is a subfield that under the reduction map $R_n \to \kappa(\mathfrak{p})$ maps onto $\kappa(\mathfrak{p})$.

For the proof of (iii), let $\mathcal{O}$ be the ring of integers of the local field $K_1 = \kappa(\mathfrak{p})$. The ring $\mathcal{O}$ is also the integral closure of $A/\mathfrak{p} = \iota_1(A)$ in $\kappa(\mathfrak{p})$. By [Wei67, I.§4. Proposition 6, p. 22], for any $\omega \in \mathfrak{m}_A$ with $\iota_1(a) \neq 0$ the ring $\mathcal{O}$ is finite over $\mathbb{F}_p[[\iota_1(\omega)]]$, and hence a finite free $\mathbb{F}_p[[\iota_1(\omega)]]$-module. Because $A/\mathfrak{p}$ and $\mathcal{O}$ have the same quotient field, the field $A/\mathfrak{p}$ is also a full $\mathbb{F}_p[[\iota_1(\omega)]]$-sublattice of $\kappa(\mathfrak{p})$ and it follows that there exists $j > 0$ such that $A/\mathfrak{p} \supset \iota_1(\omega)^j \mathcal{O}$. Thus, for a uniformizer $t \in \mathcal{O}$, all sufficiently large powers of t lie in $A/\mathfrak{p}$. We now choose (a new!) $\omega \in A$ such that $\iota_1(\omega) = t^e$ for some $e > 0$

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coprime to p. One easily verifies that $\iota_1(\omega)$ is a p-basis of $\kappa(\mathfrak{p})$. It follows from [Hoc14, Theorem 12], that $K_n := \bigcap R_n^{p^n}[\iota_n(\omega)]$ is a coefficient field for R_n. Because K_n is complete, it contains $\mathbb{F}[[\iota_n(\omega)]]$. Let $\mathcal{O}_n$ be the ring of integers of K_n. Then $\mathcal{O}_n$ is finite free over $\mathbb{F}[[\iota_n(\omega)]]$. Also, $A/\mathfrak{p}\subset K_1$ is finite free over $\mathbb{F}[[\iota_1(\omega)]]$, and $A/\mathfrak{p}[1/\iota_1(\omega)] =\kappa(\mathfrak{p})$.

We now prove (i). Let $\mathfrak{p}^{(n)} = \iota_n^{-1}(0) \supset \mathfrak{p}^n$. Then $A/\mathfrak{p}^n \to A/\mathfrak{p}^{(n)}A$ is surjective, and the induced map $A/\mathfrak{p}^{(n)}A \to R_n$ is injective. Because the $A/\mathfrak{p}$-modules $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ are finitely generated over $A/\mathfrak{p}$ and thus over $\mathbb{F}[[\iota_1(\omega)]]$, as a module over $\mathbb{F}[[\iota_n(\omega)]]$ the ring $A/\mathfrak{p}^{(n)}A$ is finite free. Moreover, it is an $\mathbb{F}[[\iota_n(\omega)]]$-lattice in the $\mathbb{F}((\iota_n(\omega)))$-vector space $(A/\mathfrak{p}^{(n)}A)[1/\iota_n(\omega)] \subset R_n$. By induction on n one also sees that $(A/\mathfrak{p}^{(n)}A)[1/\iota_n(\omega)] = R_n$: This is clearly true for $n = 1$ by the previous paragraph. In the induction step, we know that under reduction $(A/\mathfrak{p}^{(n)}A)[1/\iota_n(\omega)]$ maps onto R_{n-1}. Moreover, $\mathfrak{p}^{n-1}/\mathfrak{p}^n[1/\iota_n(\omega)]$ is a $\kappa(\mathfrak{p}) = A/\mathfrak{p}[1/\iota_1(\omega)]$-vector space, and it follows that $(A/\mathfrak{p}^{(n)}A)[1/\iota_n(\omega)]$ is a $\kappa(\mathfrak{p})$-vector space and thus equal to R_n. Now, the topology of R_n as a K_n or as a $\mathbb{F}((\iota_1(\omega)))$-vector space is the same, and it follows that $\iota_n(A)$ is a compact open neighborhood of 0 in R_n, and hence there exists $j > 0$ such that $\iota_n((\omega)^jA) \subset U$, and also $\iota_n(A)/\iota_n(\omega^jA)$ is finite. It follows that $A/(\mathfrak{p}^{(n)} + \omega^jA)$ is a local Artin ring, and so there exists $m > 0$ such that $\mathfrak{m}_A^m \subset \mathfrak{p}^{(n)} + \omega^jA$, and also $\iota(\mathfrak{m}_A^n) \subset U$. This proves (i).

For (ii), we show first that there exists $n \in \mathbb{N}$ such that $\mathfrak{m}_A^m \supset \mathfrak{p}^{(n)}$: To see this, note that the ring maps $A \to A_\mathfrak{p} \to \widehat{A}_\mathfrak{p}$ are injective and the $\mathfrak{p}$-adic topology on $\widehat{A}_\mathfrak{p}$ is separated, that is, we have $\bigcap_n \mathfrak{p}^{(n)} = 0$. The existence of n now follows from Chevalley's lemma, [Che43, Lemma 7], which asserts that the topology on A generated by the ideals $(\mathfrak{p}^{(n)})_{n\geq 0}$ is finer then the $\mathfrak{m}_A$-adic topology.

Now, by the choice of ω we have $\omega A \subset \mathfrak{m}_A$. Therefore, $\omega^m A + \mathfrak{p}^{(n)} \subset \mathfrak{m}_A^m$. It follows that $U = \iota_n(\omega^m A)$ is an open neighborhood of 0 such that $\iota_n^{-1}(\iota_n(A) \cap U) \subset \mathfrak{m}_A^m$. ◻

Proposition 4.8.6. *Suppose $\Phi_{\overline{D}}$ holds. Let $R_{\overline{D}}^{\mathrm{univ}} \to A$ be a morphism in $\widehat{\mathcal{A}r}_\mathbb{F}$, and let D_A be the corresponding pseudodeformation. Suppose A is a domain with fraction field $\mathbb{K}$ and that $D_\mathbb{K} := D_A \otimes_A \mathbb{K}$ is multiplicity free. Then there exist*

(i) *a finite extension $\mathbb{K}'$ of $\mathbb{K}$ with integral closure A' of A in $\mathbb{K}'$, and*
(ii) *continuous irreducible pseudocharacters $D_i'\colon A'[G] \to A'$,*

such that $D_A \otimes_A A' = \oplus_i D_i'$. The ring A' lies in $\widehat{\mathcal{A}r}_{\mathbb{F}'}$ for some finite field $\mathbb{F}' \supset \mathbb{F}$. If $\overline{D}$ is split, the ring of definition $A_i \subset A'$ of each D_i' lies in $\mathcal{A}dm_\mathbb{F}$ and one has $\overline{D} = \oplus_i(D_i' \otimes_{A_i} \kappa(A_i))$ over $\mathbb{F}$.

Proof. Define the rings

$$S_A := S_{\Lambda,\overline{D}}^{\text{CH-univ}} \otimes_{R_{\Lambda,\overline{D}}^{\mathrm{univ}}} A \text{ and } S_\mathbb{K} := S_A \otimes_A \mathbb{K}.$$

Then by Proposition 4.7.11, the A-algebra S_A is finitely generated as an A-module, and the induced homomorphism $G \to S_A^\times$ is continuous if S_A is equipped with the $\mathfrak{m}_A$-adic topology as an A-module. In particular, the $\mathbb{K}$-algebra $S_\mathbb{K}$ is of finite $\mathbb{K}$-dimension.

Now, Lemma A.2.3 gives a finite extension $\mathbb{K}'$ of $\mathbb{K}$ so that $S' := S_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}'/\mathrm{Rad}(S_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}')$ is a product of matrix algebras $S' = \prod_i \mathrm{Mat}_{n_i\times n_i}(\mathbb{K}')$. Since $D_{\mathbb{K}'} := D_\mathbb{K} \otimes_\mathbb{K} \mathbb{K}'$ factors via S' and is multiplicity free, it is the composition of $G \to S_A^\times \to (S')^\times$ with $\prod_i \det_{n_i}$ where $\det_{n_i}$ is the determinant of $\mathrm{Mat}_{n_i\times n_i}(\mathbb{K}')$. Write $D_{\mathbb{K}'} = \bigoplus_i D_i$ with D_i corresponding to $\det_{n_i}$ on the i-th factor of S'.

Let A' be the integral closure of A in $\mathbb{K}'$. Because A is Nagata by Lemma A.1.1(a), the ring A' is finite over A and hence lies in $\mathcal{A}dm_\mathbb{F}$. Because A is complete Noetherian, so is A', and because A' must be semilocal, it is a product of local rings. But A' is also a subring of the field $\mathbb{K}'$, and thus A' lies in $\widehat{\mathcal{A}r}_{\mathbb{F}'}$ for some finite field extension $\mathbb{F}' \supset \mathbb{F}$. Let $S_{A'} := S_A \otimes_A A'$, and write $(S_A)'$ for the image of $S_{A'}$ in S'.

Because $D_{\mathbb{K}'} = \bigoplus_i D_i$, the attached characteristic polynomials satisfy

$$\prod_i \chi_{D_i'}(\cdot, t) = \chi_{D_{\mathbb{K}'}}(\cdot, t) \in \mathbb{K}'[t].$$

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By hypothesis, the coefficients of $D_{\mathbb{K}'}$ lie in $A \subset A' \subset \mathbb{K}'$. Now, because A' is integrally closed, by [Mat89, Theorem 9.2] the coefficients of the $\chi_{D'_i}(\cdot, t)$ lie in A'. Hence, by Amitsur's formula in the form Proposition 4.1.10, the D'_i take values in A'.

Now, note that the proposition is trivial if A has Krull dimension 0, and so we assume it to be strictly positive from now on so that Spec $A' \backslash \{\mathfrak{m}_{A'}\}$ is a nonempty Jacobson scheme and thus the points of dimension 1 of Spec A' are very dense in it; see Definition A.1.10 and Proposition A.1.11. Moreover, the locus of irreducibility of each D_i is open in Spec A' by Proposition 4.7.17 and contains the generic point of Spec A' by construction. Also, by Lemma A.1.1 the complete Noetherian local ring A' is a Nagata ring and so the regular locus is open in Spec A'; it is nonempty because A' is a domain. Hence, there exists a point $\mathfrak{p} \in$ Spec A' of dimension 1 at which all D_i are simultaneously irreducible and such that $A'_{\mathfrak{p}}$ is regular local. The former condition on $\mathfrak{p}$ implies by Proposition 4.7.18 that each D_i when considered as a pseudocharacter $A'_{\mathfrak{p}}[G] \to A'_{\mathfrak{p}}$ is equal to the reduced norm composed with a representation $G \to (C_i)^\times$ for C_i an $A'_{\mathfrak{p}}$-Azumaya algebra. The latter condition implies that one has an inclusion of Brauer groups $\mathrm{Br}(A'_{\mathfrak{p}}) \hookrightarrow \mathrm{Br}(\mathbb{K}')$ by [AG60, Theorem 7.2], and hence that all C_i have trivial Brauer class by the choice of $\mathbb{K}'$, that is $C_i \cong \mathrm{Mat}_{n_i \times n_i}(A_{\mathfrak{p}})$ for suitable $n_i > 0$. It follows that over $A'_{\mathfrak{p}}$ we have that D' is the determinant of a direct sum of representations

$$G \to \prod_i \mathrm{Mat}_{n_i \times n_i}(\widehat{A'}_{\mathfrak{p}})^\times.$$

By our hypothesis, $G \to S_A^\times$ is continuous as is the induced pseudocharacter $D\colon S_A \to A$. Let $S_{\mathfrak{p}}$ be $S_A \otimes_A \widehat{A'}_{\mathfrak{p}}/\mathrm{CH}(D')$. Then by Lemma 4.8.5 also $G \to S_{\mathfrak{p}}^\times$ is continuous, as is the induced pseudocharacter $S_{\mathfrak{p}} \to A'$. From the above and Proposition 4.3.9, it follows that $S_{\mathfrak{p}}$ is a GMA with trivial ideal of total reducibility. Now, the continuity of $G \to S_{\mathfrak{p}}^\times$ implies that of $D_i\colon G \to \mathrm{Mat}_{n_i \times n_i}(\widehat{A'}_{\mathfrak{p}})^\times \xrightarrow{\det} \widehat{A'}_{\mathfrak{p}}$ obtained by applying the i-the projection and the determinant, and again from Lemma 4.8.5 we deduce that $D_i\colon A'[G] \to A'$ is continuous.

It remains to prove the last assertion, assuming that $\overline{D}$ is split: Let $A_i \subset A'$ be the ring of definition of D'_i, denote by $D_i\colon A_i[G] \to A_i$ the corresponding pseudocharacter and let $\overline{D}_i := D_i \otimes_{A_i} \kappa(A_i)$. Note that the $\kappa(A_i)$ are the rings of definition of $\overline{D}_i$. Let $\mathbb{F}''$ be the smallest extension of $\mathbb{F}'$ that contains all $\kappa(A_i)$. Then $\overline{D} \otimes_{\mathbb{F}} \mathbb{F}'' \cong \bigoplus_i \overline{D}_i \otimes_{\kappa(A_i)} \mathbb{F}''$. However, $\overline{D}$ is split over $\mathbb{F}$ and so all $\overline{D}_i$ are defined over $\mathbb{F}$, and this shows $\kappa(A_i) = \mathbb{F}$ for all i by Lemma 4.4.8. We deduce $A_i \in \mathcal{A}dm_{\mathbb{F}}$. □

Corollary 4.8.7. *Let κ be a finite or a local field, and let $\rho\colon G \to \mathrm{GL}_n(\kappa)$ be a continuous absolutely irreducible homomorphism with associated pseudocharacter D. Suppose that $\Phi_{\overline{D}}$ holds for $\overline{D}$ attached to D as in Lemma 4.8.1. Then the natural map $R^{\mathrm{univ}}_{\Lambda,\rho} \to R^{\mathrm{univ}}_{\Lambda,D}$ induced from $\rho_A \mapsto D_{\rho_A}$ for $A \in \mathcal{A}r_\kappa$ is an isomorphism.*

Proof. If κ is finite, the assertion is [Che14, Example 3.4]. For local κ, we need to show that the natural transformation of functors $\mathcal{A}r_\kappa \to \mathit{Sets}$ defined by

$$\begin{tikzcd} & \{\text{continuous deformations } \rho_A \text{ of } \rho \text{ to } A\} \arrow[d, "\rho_A \mapsto D_{\rho_A}"] \\ A \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & \{\text{continuous pseudodeformations } D_A \text{ of } D \text{ to } A\} \end{tikzcd}$$

is an isomorphism. Well definedness is clear. Injectivity follows from Theorem 4.3.10(a) since ρ_D is absolutely irreducible. To prove surjectivity, consider a pseudodeformation $D_A\colon A[G] \to A$ of D and note that by Theorem 4.3.10(a) there exists a deformation ρ_A of ρ_D to A with $D_A = D_{\rho_A}$. The continuity of ρ_A follows from Corollary 4.8.2(b). □

We now give an analog of Theorem 3.3.1 for pseudocharacters.

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Corollary 4.8.8. *Let $\mathbb{F}$ be a finite field and let $\overline{D} \in \mathcal{P}s\mathcal{R}^n_G(\mathbb{F})$ be continuous. For $x \in X^{\mathrm{univ}}_{\Lambda,\overline{D}}$ such that $\kappa(x)$ is a local field and with residue map $\pi_x\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to \kappa(x)$ and associated pseudocharacter $D_x\colon \kappa(x)[G] \to \kappa(x)$, define the morphism $f_x = \pi_x \otimes \mathrm{id}\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa(x) \to \kappa(x)$, and the completion $\widehat{R}$ of $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa(x)$ at $\mathfrak{p} = \mathrm{Ker}(f_x)$. Denote by*

(i) *$\widehat{D}\colon G \to R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to \widehat{R}$ the completion at $\mathfrak{p}$ of the pseudocharacter $D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \kappa(x)$, and by*

(ii) *$D^{\mathrm{univ}}_{D_x}\colon G \to R^{\mathrm{univ}}_{\kappa(x),D_x}$ the universal pseudodeformation from Proposition 4.7.4 attached to D_x.*

Then the map $\varphi\colon R^{\mathrm{univ}}_{D_x} \to \widehat{R}$ induced from the universal property of $R^{\mathrm{univ}}_{D_x}$ is an isomorphism.

Proof. We adapt the proof of Theorem 3.3.1. Thus, we need to show that φ is formally étale. We abbreviate $\kappa = \kappa(x)$ and let $\mathcal{O}$ be the ring of integers of κ. Consider the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "D^{\mathrm{univ}}_{D_x}"] \arrow[d, "D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \kappa"'] \arrow[rr, bend left, "D"] & R^{\mathrm{univ}}_{D_x,\kappa} \arrow[r, "\alpha_A"] \arrow[d, "\varphi"] & A \arrow[d] & \\
R^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \kappa \arrow[r, "\widehat{(\cdot)}"] & \widehat{R} \arrow[r, "\hat{\alpha}_{A/I}"] \arrow[ur, dashed, "\hat{\alpha}_A"] & A/I \arrow[r] & \kappa,
\end{tikzcd}$$

where $A \in \widehat{\mathcal{A}r}_\kappa$, the ideal $I \subset A$ satisfies $I^2 = 0$, and $D\colon A[G] \to A$ is continuous pseudodeformation of D_x. The maps $\hat{\alpha}_{A/I}$ and α_A are homomorphisms in $\widehat{\mathcal{A}r}_\kappa$. We will construct the dashed arrow $\hat{\alpha}_A$ so that the extended diagram commutes and show its uniqueness. Our first claim is that there is an $\mathcal{O}$-algebra $A_0 \subset A$ that as an $\mathcal{O}$-module is an $\mathcal{O}$-lattice in A and such that D is valued in A_0.

The proof is an induction over j for the composition $D_j = D (\mathrm{mod}\ \mathfrak{m}_A^j)$ of D with the quotient map $A \to A_j = A/\mathfrak{m}_A^j$. For $j = 1$, the claim holds because $D_x = D_1$ is valued in $\mathcal{O} \subset \kappa$. Suppose in the induction step that we have defined $A_{j,0} \subset A_j$ as in the claim and we wish to construct $A_{j+1,0} \subset A_{j+1}$. Because the characteristic polynomial coefficients $\Lambda_{i,D_{j+1}}$, $i = 1, \ldots, n$, are continuous and G is profinite and hence compact, the joint image of the $\Lambda_{i,D_{j+1}}(G)$ is bounded in A_{j+1} and hence there exists an $\mathcal{O}$-sublattice $L \subset A_{j+1}$ that contains this joint image as well as $\mathcal{O} \cdot 1$, and such that $L (\mathrm{mod}\ \mathfrak{m}_A^j) = A_{j,0}$. Now, using that $A_{j,0}$ is a ring and that $\mathfrak{m}_A^j$ is annihilated by $\mathfrak{m}_A$, it follows, for instance by considering a suitable $\mathcal{O}$-basis of L and the quotient $L + \mathfrak{m}_A^j/L$ inside $\mathfrak{m}_A^j/(\mathfrak{m}_A^{j+1} + L \cap \mathfrak{m}_A^j)$, that $L \cdot L$ is an $\mathcal{O}$-algebra that can be taken as $A_{j+1,0}$.

Let C_x be the ring of definition of D_x. It is a quotient of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ and a subring of $\mathcal{O} \subset \kappa$. We let B be the subalgebra of A_0 that is the inverse image of C_x under the reduction map $A_0 \to \mathcal{O}$ modulo $\mathfrak{m}_A$. Then $B \in \widehat{\mathcal{A}r}_\Lambda$ and B is a coefficient ring for D. Denote by $D_B\colon B[G] \to B$ the pseudocharacter such that $D_B \otimes_B A = D$. If κ has positive characteristic, it follows from Lemma 4.8.5 that D_B is continuous with B carrying the $\mathfrak{m}_B$-adic topology. The same holds if κ is a p-dic field: Then B is a finite free $\mathbb{Z}_p$-module, and so the p-adic topology on B agrees with the $\mathfrak{m}_B$-adic topology on B. Therefore, the continuity of D with respect to the topology on A as a continuous κ-module implies the continuity of D_B for the $\mathfrak{m}_B$-adic topology on B.

Now, the universal property of $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ yields a unique homomorphism $\beta_B\colon R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to B$ such that $D = \beta_B \circ D_B$. Then $\beta_B \otimes_\Lambda \mathrm{id}_\kappa$ is a homomorphism $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa \to A$. Its composition with $A \to \kappa$ has kernel $\mathfrak{p}$, and thus it induces a map $(R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa)_{\mathfrak{p}} \to A$. If m denotes the length of A, then $\mathfrak{p}^m$ maps to zero under that map so that it factors via $\widehat{R} \to A$. This is the wanted map $\hat{\alpha}_A$: The top triangle in the diagram commutes by construction, the bottom triangle because $\beta_A (\mathrm{mod}\ I)$ and the map $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \to B/(I \cap B) \subset A/I$ must agree since they both give rise to $D (\mathrm{mod}\ I)$.

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Let us show the uniqueness of $\hat{\alpha}_A$, and so let α'_A be a second map $\widehat{R} \to A$ that makes as a dashed arrow the diagram commute. Observe that by construction we have the equality of pseudocharacters $\varphi \circ D^{\mathrm{univ}}_{D_x} = \widehat{(\cdot)} \circ (D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \kappa)$. Now composing the equality with either $\hat{\alpha}_A$ or α'_A gives the same pseudocharacter. We claim that $\hat{\alpha}_A \circ \widehat{(\cdot)} = \alpha'_A \circ \widehat{(\cdot)}$ as maps $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa \to A$. If the claim is shown, then uniqueness follows, because the induced diagonal map is reconstructed by localization and completion – the ideal $\mathfrak{p}^m$ is mapped to zero in A.

To prove the latter claim, by the universal property of the tensor product of rings, it suffices to understand the ring maps on both factors of $R^{\mathrm{univ}}_{\Lambda,\overline{D}} \otimes_\Lambda \kappa \to A$ separately. On the second factor, both maps are the scalar multiplication isomorphism $\kappa \to \kappa \cdot 1, \alpha \mapsto \alpha \cdot 1$, by the definition of $\widehat{(\cdot)}$ and the condition that the diagonal map be in $\widehat{\mathcal{A}r}_\kappa$. The restriction of either map $\hat{\alpha}_A \circ \widehat{(\cdot)}$ or $\alpha'_A \circ \widehat{(\cdot)}$ to the first factor $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ gives when composed with $D^{\mathrm{univ}}_{\overline{D}} \otimes_\Lambda \kappa$ the pseudocharacter D. Both restrictions to $R^{\mathrm{univ}}_{\Lambda,\overline{D}}$ are subject to the universal property of this ring, and hence these restrictions agree, and the claim is shown. □

Remark 4.8.9. We think that [Che14, Corollary 2.23(ii)] has to be formulated in a way similar to Corollary 4.8.8; only if $\kappa(x)$ is a p-adic field, one can simply complete $(R^{\mathrm{univ}}_{\Lambda,\overline{D}})_{\mathfrak{p}_x}$ to obtain a universal pseudodeformation ring for D_x. In Corollary 4.8.8 we have only verified this for dimension 1 points.

Corollary 4.8.10. *Let κ be a local field and let $D \in \mathcal{P}s\mathcal{R}^n_G(\kappa)$ be continuous. Suppose that condition $\Phi_{\overline{D}}$, for $\overline{D}$ from Lemma 4.8.1, is satisfied. Then the following hold:*

(a) *The ring $R^{\mathrm{univ}}_{\kappa,D}$ is Noetherian.*
(b) *Suppose that D is irreducible and that $H^2(G, \mathrm{ad}_\rho) = 0$ for $\rho := \rho_{D\otimes_\kappa \kappa^{\mathrm{alg}}}$. Then $R^{\mathrm{univ}}_{\kappa,D}$ is formally smooth over κ of relative dimension $\dim_{\kappa^{\mathrm{alg}}} H^1(G, \mathrm{ad}_\rho)$.*

Proof. Let C be the ring of definition of D, let $D_C : C[G] \to C$ the continuous pseudocharacter from Lemma 4.8.1 such that $D_C \otimes_C \kappa = D$ and let $\overline{D}' := D_C \otimes_C \kappa(C)$; note that $\kappa(C)$ is finite; note also that continuity is clear because C carries the subspace topology of κ and we only require the continuity of the characteristic polynomial coefficients as functions $G \to C$. By Proposition 4.7.4 and our hypotheses, the ring $R^{\mathrm{univ}}_{W(\kappa(C)),\overline{D}'}$ is Noetherian. Part (a) follows by choosing $x \in X^{\mathrm{univ}}_{W(\kappa(C)),\overline{D}'}$ as the point corresponding to D, and by applying Corollary 4.8.8 with this x; note that $\widehat{R}$ (in Corollary 4.8.8) is Noetherian as the completion of a Noetherian local ring.

To see Part (b), let $\kappa' \supset \kappa$ be a finite extension over which D is split. Let $\rho := \rho_{D\otimes_\kappa \kappa'} : G \to \mathrm{GL}_n(\kappa')$ be the continuous and absolutely irreducible representation with $D_{\rho'}. = D \otimes_\kappa \kappa'$. Our hypotheses gives $H^2(G, \mathrm{ad}_{\rho'}) = 0$ and it will suffice to show that $R^{\mathrm{univ}}_{\kappa',D}$ is formally smooth over κ' of dimension $\dim_{\kappa'} H^1(G, \mathrm{ad}_{\rho'})$. This follows from Corollary 4.8.7 and Theorem 3.2.4(e). □

For later use, we also need variants of Corollary 4.8.7 and Corollary 4.8.8 for deformations of pairs of representations and pseudocharacters. Let $\overline{D}_1, \overline{D}_2 : \mathbb{F}[G] \to \mathbb{F}$ be continuous pseudocharacters of dimensions n_1 and n_2 such that $\Phi_{\overline{D}_i}$ holds for $i = 1, 2$. Consider the functor

$$\mathcal{P}sD_{(\overline{D}_1,\overline{D}_2)} : \widehat{\mathcal{A}r}_\Lambda \to Sets, \quad A \longmapsto \{(D_1, D_2) \mid D_i : G \longrightarrow A \text{ is a pseudodeformation of } \overline{D}_i\}.$$

It is straightforward to see that $\mathcal{P}sD_{(\overline{D}_1,\overline{D}_2)}$ is represented by $R^{\mathrm{univ}}_{(\Lambda,\overline{D}_1,\overline{D}_2)} := R^{\mathrm{univ}}_{\Lambda,\overline{D}_1} \widehat{\otimes} R^{\mathrm{univ}}_{\Lambda,\overline{D}_2}$ and that the ring $R^{\mathrm{univ}}_{(\Lambda,\overline{D}_1,\overline{D}_2)}$ is Noetherian, using Propositions 4.7.4 and 4.7.11.

Let $x \in X^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2)} := \operatorname{Spec} R^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2)}$ be a point of dimension 1 such that $D_{i,x}$ is irreducible for $i = 1, 2$ for the corresponding pair $(D_{1,x}, D_{2,x})$. As above, one can define a deformation functor for this pair an $\mathcal{A}r_{\kappa(x)}$. It is representable by $R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})} := R^{\mathrm{univ}}_{\kappa(x),D_{1,x}} \widehat{\otimes}_{\kappa(x)} R^{\mathrm{univ}}_{\kappa(x),D_{2,x}}$, which is again complete local Noetherian. Let $\pi : R_x := R^{\mathrm{univ}}_{\Lambda,(\overline{D}_1,\overline{D}_2)} \otimes_\Lambda \kappa(x) \to \kappa(x)$ be the homomorphism induced from x, and let

$$\varphi\colon R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})} \to \widehat{R}_x$$

be the natural homomorphism constructed as in Corollary 4.8.8, where $\widehat{R}_x$ denotes the completion of R_x at $\mathfrak{p}_x := \operatorname{Ker}\pi$.

Let finally L be a finite extension of $\kappa(x)$ over which there exist absolutely irreducible representations $\rho_i\colon G \to \mathrm{GL}_{n_i}(L)$ such that $D_{\rho_i} = D_{i,x} \otimes_{\kappa(x)} L$ for $i = 1, 2$. Define the functor

$$\mathcal{D}_{(\rho_1,\rho_2)}\colon \mathcal{A}r_\Lambda \longrightarrow \mathcal{S}ets,\quad A \longmapsto \{(\rho_{1,A},\rho_{2,A}) \mid \rho_{i,A}\colon G \longrightarrow \mathrm{GL}_n(A) : \rho \text{ is a deformation of } \rho_i\},$$

Since the ρ_i are absolutely irreducible, it is represented by $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)} := R^{\mathrm{univ}}_{L,\rho_1} \widehat{\otimes}_L R^{\mathrm{univ}}_{L,\rho_2}$, and the latter ring is Noetherian local by Proposition 3.2.3 and Theorem 3.2.4 since we assume $\Phi_{\overline{D}_i}$, $i = 1, 2$. As in Corollary 4.8.7 one has a natural homomorphism

$$\psi\colon R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)} \to R^{\mathrm{univ}}_{\kappa(x),(D_{1,x},D_{2,x})} \otimes_{\kappa(x)} L.$$

Proposition 4.8.11. *The following hold:*

(a) *The maps ψ and φ are isomorphisms.*
(b) *Suppose $G = G_K$ and $H^0(G, \overline{\mathrm{ad}}_{\rho_i}) = 0$ for $i = 1, 2$. Then $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2),\mathrm{red}}$ is formally smooth over L of dimension $d(n_1^2 + n_2^2) + 2$ and hence x is a smooth point on $X^{\mathrm{univ}}_{(\overline{D}_1,\overline{D}_2),\mathrm{red}}$ with tangent space dimension $d(n_1^2 + n_2^2) + 1$.*

Proof. The two assertions in (a) are proved exactly as Corollary 4.8.7 and Corollary 4.8.8, and we omit the details. The first assertion in (b) follows from our description of $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2),\mathrm{red}}$ as a completed tensor product and from Corollary 3.4.3. The second assertion now is a consequence of (a), of Proposition 4.7.6 and of Lemma 3.3.5. □

5. Equidimensionality and density of the regular locus

This section proves the main result of this work, the equidimensionality of the special fiber of universal pseudodeformation rings of expected dimension. The proof follows the steps of Chenevier's proof of the equidimensionality of the generic fiber of the universal pseudocharacter space from [Che11]. The main contribution is to overcome the complications that arise in the special fiber.

There are certain points in the special fiber that have no counterpart in the generic fiber. We call them *special points* and describe them in Subsection 5.1; see Definition 5.1.2. nonspecial (irreducible) points will take the role of irreducible points in Chenevier's work. Subsection 5.1 also contains some technical result, Lemma 5.1.6, on the comparison of universal pseudodeformation and universal deformation rings over local fields where the residual pseudocharacter is a sum of two irreducible ones.

In Subsection 5.2, we prove the inductive Theorem 5.2.1 to obtain our main result. If the nonspecial irreducible points are Zariski open dense in universal pseudodeformation spaces for $\overline{D}$ of dimension less than n, then irreducible points are Zariski dense for $\overline{D}$ of dimension n. Subsection 5.3 gives an alternative proof of Theorem 5.2.1 following a suggestion of the referee. The main point of Subsection 5.4 is to show in Theorem 5.4.1 that the nonspecial irreducible points are dense open in the irreducible points. This uses induction of pseudocharacters from Subsection 4.6 as a main tool, and the proof in dimension n relies on results for dimension $n' < n$.

Then in Subsection 5.5, we complete the proof of our main theorem Theorem 5.5.1 in a straightforward manner. In Theorem 5.5.5, we determine the singular locus of $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ when $\zeta_p \notin K$. This allows us in Theorem 5.5.7 to establish Serre's condition (R_2) for $\overline{R}^{\mathrm{univ}}_{\overline{D}}$ except for one single $\overline{D}$.

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In this section, we use the notation $K \supset \mathbb{Q}_p$, d, G_K, ζ_p, $\overline{D}\colon G_K \to \mathbb{F}$ (continuous) with $\mathbb{F}$ finite, as before. Often, we write n for $\dim \overline{D}$. To emphasize K in universal objects, we sometimes write $\overline{R}^{\mathrm{univ}}_{K,\overline{D}}$ for $R^{\mathrm{univ}}_{G_K,\mathbb{F},\overline{D}}$ and $\overline{X}^{\mathrm{univ}}_{K,\overline{D}}$ for Spec $\overline{R}^{\mathrm{univ}}_{K,\overline{D}}$. All results of this section only concern the special fiber of pseudodeformation spaces.

5.1. Special points

Let $\chi_{\mathrm{cyc}}\colon G_K \to \mathbb{Z}_p^\times$ denote the p-adic cyclotomic character. Let A be in $\widehat{\mathcal{A}r}_{W(\mathbb{F})}$ (or a localization of such a ring), let $\rho\colon G_K \to \mathrm{GL}_n(A)$ be a continuous representation and $D\colon G_K \to A$ be a continuous pseudocharacter. For $i \in \mathbb{Z}$, we shall denote by $\rho(i)$ and $D(i)$ the twist by χ^i_{cyc} of ρ and D, respectively. An elementary but crucial observation in [Che11] was that $H^2(G_K, \mathrm{ad}_\rho) = 0$ whenever a $\rho\colon G_K \to \mathrm{GL}_n(E)$ is a continuous absolutely irreducible representation into a p-adic field E; this follows from local Tate duality in the form given in Theorem 3.4.1, which gives

$$H^2(G_K, \mathrm{ad}_\rho)^\vee = \mathrm{Hom}_{G_K}(\rho, \rho(1)), \tag{28}$$

together with the fact that χ_{cyc} has infinite order. For representations into local (or finite) fields of characteristic p the order of $\chi_{\mathrm{cyc}} (\mathrm{mod}\ p)$ is finite, and so the situation has to be further analyzed.

Lemma 5.1.1. *Let E be a finite or local field of characteristic p, and let $\rho\colon G_K \to \mathrm{GL}_n(E)$ be a continuous absolutely irreducible representation. Then the following hold:*

Suppose that *$\zeta_p \notin K$* ***(Case I).*** *Then the following assertions are equivalent:*

(i) *$H^2(G_K, \mathrm{ad}_\rho)$ is nonzero.*
(ii) *The G_K-representations ρ and $\rho(1)$ are isomorphic.*
(iii) *There exists a finite separable extension E' of E such that $\rho \otimes_E E'$ is induced from a continuous representation τ of $G_{K'}$ over E' for $K' = K(\zeta_p)$.*

Suppose that *$\zeta_p \in K$* ***(Case II).*** *Then the map $H^2(\mathrm{Tr})\colon H^2(G_K, \mathrm{ad}_\rho) \to H^2(G_K, E) \cong E$ is surjective, and the following assertions are equivalent:*

(i') *$H^2(G_K, \mathrm{ad}^0_\rho)$ is nonzero.*
(ii') *$H^0(G_K, \overline{\mathrm{ad}}_\rho)$ is nonzero.*
(iii') *There exists a finite extension E' of E and a Galois extension K' of K of degree p such that $\rho \otimes_E E'$ is induced from a continuous representation τ over E' of $G_{K'}$.*
(iv') *The restriction $\rho \otimes_E E^{\mathrm{alg}}|_{G_{K'}}$ is reducible for some Galois extension K' of K of degree p.*

In both cases, if τ exists, then it is absolutely irreducible, and in particular $\mathrm{End}_{G_{K'}}(\tau) = E$.

Proof. The equivalence of (i) and (ii) follows from the isomorphism (28) and the absolute irreducibility of ρ. The duality in Theorem 3.4.1 also yields the equivalence of (i') and (ii'), in a similar way. In all cases, the continuity and absolute irreducibility of τ, if it exists, is implied by Lemma 2.1.4(b) and (f).

The equivalence of (ii) and (iii) now follows from Theorem 2.2.1. The equivalence of (iii') and (iv') is a consequence of Lemma 2.3.1. The implication (ii')⇒(iii') follows from Lemma 2.3.2(g) and (j), and the implication (iii')⇒(ii') is shown in Example 4.6.4. □

Definition 5.1.2. We call $x \in (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$ *special* if one of the following two conditions holds

(i) $\zeta_p \notin K$ and $D_x = D_x(1)$,
(ii) $\zeta_p \in K$ and $D_x|_{G_{K'}}$ is reducible for some degree p Galois extension K' of K;

otherwise x is called *nonspecial*. We write $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ for $\{x \in (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}} \mid \text{ is special}\}$ and $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\text{n-spcl}}$ for $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}} \backslash (\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$.

Lemma 5.1.3. *The set $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{spcl}}$ is closed in $(\overline{X}^{\mathrm{univ}}_{K,\overline{D}})^{\mathrm{irr}}$.*

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Proof. If $\zeta_p \notin K$, then the condition $D = D(1)$ is a closed condition in $\overline{X}^{\text{univ}}_{K,\overline{D}}$ by Corollary 4.5.11, and this concludes the argument.

If $\zeta_p \in K$, then note first that the set of Galois extensions K' of K of degree p is finite. Since by class field theory $(G_{K^{\text{ab}}})/(G_K^{\text{ab}})^{\times p}$ is finite if K is a p-adic field. By [Che14, 2.20], the reducibility of a pseudocharacter over a field can be detected by the vanishing of certain determinants whose entries are traces of the pseudocharacter, evaluated at certain elements of the group in question. If $n = \dim D$, then $x \in (\overline{X}^{\text{univ}}_{K,\overline{D}})^{\text{spcl}}$ if and only if for some degree p Galois extensions K' over K and all n^2 tuples $(g_i) \in G_{K'}^{n^2}$, one has

$$\det\Big(\Lambda_{D,1}(g_i g_j)_{i,j=1,\dots,n^2}\Big) = 0.$$

Hence, $(\overline{X}^{\text{univ}}_{K,\overline{D}})^{\text{spcl}}$ is Zariski closed in $(\overline{X}^{\text{univ}}_{K,\overline{D}})^{\text{irr}}$ as a finite union (over K') of closed subsets. □

For $X \subset \overline{X}^{\text{univ}}_{\overline{D}}$ locally closed, we set $\dot{X} := X \backslash \{\mathfrak{m}_{\overline{R}^{\text{univ}}_{\overline{D}}}\}$. The following holds:

Facts 5.1.4.

(a) $\mathfrak{m}_{\overline{R}^{\text{univ}}_{\overline{D}}}$ is the unique closed point of $\overline{X}^{\text{univ}}_{\overline{D}}$.

(b) For $i \geq 1$, the dimension i points of $\overline{X}^{\text{univ}}_{\overline{D}}$ are the dimension $i-1$ points of $\dot{\overline{X}}^{\text{univ}}_{\overline{D}}$.

(c) If $\dot{X}$ is nonempty, then the dimension 0 points of $\dot{X}$ are very dense in $\dot{X}$; see Lemma A.1.8.

We call $x \in \overline{X}^{\text{univ}}_{\overline{D}}$ regular, if $\overline{R}^{\text{univ}}_{\overline{D}}$ is regular at x, and singular otherwise.

Notation 5.1.5. Let X be a locally closed subset of $\overline{X}^{\text{univ}}_{\overline{D}}$.

(a) We use the superscripts irr, red, reg, sing on X to denote the subset of irreducible, reducible, regular and singular points, respectively; cf. Definition 4.7.16.

(b) We write X_{red} (subscript!) for X with its induced reduced subscheme structure.

(c) For attributes a, b, c of X, if they apply, we write $X^{a,b}$ for $X^a \cap X^b$, X^a_b for $X^a \cap X_b$, $X_{a,b}$ for $X_a \cap X_b$, and so on.

The remaining results in this section concern dimension 1 points on $\overline{X}^{\text{univ}}_{\overline{D}}$.

Lemma 5.1.6. *Let x be a closed point of $U := (\dot{\overline{X}}^{\text{univ}}_{\overline{D}})^{\text{irr}}$, let D'_x be the pseudocharacter $\kappa(x)[G] \to \kappa(x), g \mapsto 1 \otimes_{W(\mathbb{F})} D_x(g)$ and let $\widehat{R}$ be the universal pseudodeformation ring for D'_x from Corollary 4.8.8. Then the following hold:*

(a) *Suppose that $\zeta_p \notin K$ and that x is nonspecial. Then $\widehat{R}$ is regular of dimension dn^2+1. If in addition $U^{\text{n-spcl}}$ is nonempty, it is regular and equidimensional of dimension dn^2.*

(b) *Suppose that $\zeta_p \notin K$ and that x is special. Then $\widehat{R}$ is a complete intersection ring with* $\dim \widehat{R} \in$ *$\{dn^2+1, dn^2+2\}$. Moreover, U is of dimension at most dn^2+1.*

(c) *Suppose that $\zeta_p \in K$ and that x is nonspecial. Then $\widehat{R}_{\text{red}}$ is regular of dimension dn^2+1. If in addition $U^{\text{n-spcl}}_{\text{red}}$ is nonempty, it is regular and equidimensional of dimension dn^2.*

Proof. Consider the Galois representation $\rho_x \colon G_K \to \mathrm{GL}_n(L)$ with $D_{\rho_x} = D'_x$ from Theorem 4.3.10 that is defined over a finite extension L of $\kappa(x)$. For (a) note that we have $H^2(G_K, \mathrm{ad}_{\rho_x}) = 0$ by Lemma 5.1.1, Case I and the definition of special. The Euler characteristic formula of Theorem 3.4.1 now yields

$$\dim \widehat{R} = \dim_L H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + \dim_L H^0(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + 1.$$

It follows from Lemma 3.3.5 and Remark 3.3.2 that x is a regular point of $\overline{X}^{\text{univ}}_{\overline{D}}$ of dimension $dn^2+1-1 = dn^2$. Since x lies on U, it is also a regular point of U. To see that U is regular, let $Y \subset U$ be the closed subscheme of singular points. We know that points of dimension at most 1 will be dense in the

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constructible set Y. Since the unique closed point of $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ is not in U, points of dimension 1 are dense in $Y \subset U$. However, as we just saw, such points are regular and cannot lie in Y. Therefore, Y must be empty. And again by the density of dimension 1 points in U, it follows that U is regular and equidimensional of dimension dn^2, proving (a).

For (b), we observe $H^2(G_K, \mathrm{ad}_{\rho_x})^\vee = H^0(G_K, \mathrm{ad}_{\rho_x}(1)) = \mathrm{Hom}_{G_K}(\rho_x, \rho_x(1)) \cong L$ using Theorem 3.4.1, and in the last step that $\rho_x \cong \rho_x(1)$ and that ρ_x is absolutely irreducible. This time, the Euler characteristic formula provides a presentation

$$0 \to I \to \kappa(x)[[X_1, \dots, X_{dn^2+2}]] \to \widehat{R}_{\mathfrak{p}} \to 0,$$

where the ideal I is generated by at most one element over $\kappa(x)[[X_1, \dots, X_{dn^2+2}]]$. This proves the claims on $\widehat{R}_{\mathfrak{p}}$. The remaining assertion follows from the density of dimension 1 points in U and Lemma 3.3.5.

For (c), it follows from the nonspecialness of D_x and from Corollary 3.4.3 that $(\overline{R}^{\mathrm{univ}}_{\rho_x})_{\mathrm{red}}$ is regular local of dimension dn^2+1. From Proposition 4.7.6 and Corollary 4.8.7 we deduce $\overline{R}^{\mathrm{univ}}_{D_x} \otimes_{\kappa(x)} L \cong \overline{R}^{\mathrm{univ}}_{\rho_x}$, and the assertion on $\widehat{R}^{\mathfrak{p}}_{\mathrm{red}}$ follows. The remaining assertion follows from the density of dimension 1 points in U and Lemma 3.3.5. □

We also need a similar result in certain reducible cases. It is adapted from [Che11, Lemma 2.2].

Lemma 5.1.7. *For $i = 1, 2$, let $\overline{D}_i \colon G_K \to \mathbb{F}$ be pseudocharacters over a finite field $\mathbb{F}$, let $x_i \in \overline{X}^{\mathrm{univ}}_{\overline{D}_i}$ be irreducible nonspecial dimension 1 points and let L be a finite extension of both $\kappa(x_i)$ over which there exist absolutely irreducible representations $\rho_i \colon G_K \to \mathrm{GL}_{n_i}(L)$ with $D_{\rho_i} = D_{x_i} \otimes_{\kappa(x)} L$. Let $\rho \colon G_K \to \mathrm{GL}_n(L)$ be a nontrivial extension of ρ_2 by ρ_1. Assume that $D_{x_1} \neq D_{x_2}(m)$ for any $m \in \{1, \dots, p-1\}$. Then the following hold:*

(a) *The representation ρ exists; it satisfies $L = \mathrm{End}_{G_K}(\rho)$; one has $D_\rho = D_{\rho_1} \oplus D_{\rho_2}$ as pseudocharacters into L; the functor $\mathcal{D}_\rho \colon \mathcal{A}r_L \to \mathcal{S}ets$ is pro-representable.*

We write R^{univ}_ρ for the representing universal ring of $\mathcal{D}_\rho$ and $\rho^{\mathrm{univ}}_\rho \colon G_K \to \mathrm{GL}_n(R^{\mathrm{univ}}_\rho)$ for a universal deformation and X^{univ}_ρ for $\mathrm{Spec}\, R^{\mathrm{univ}}_\rho$. Denote by $\widehat{R}$ the universal pseudodeformation ring for D_ρ to $\mathcal{A}r_L$, by $\varphi \colon X^{\mathrm{univ}}_\rho \to \widehat{X} := \mathrm{Spec}\widehat{R}$ the map of L-schemes induced by sending $\rho^{\mathrm{univ}}_\rho$ to its associated pseudocharacter $D_{\rho^{\mathrm{univ}}_\rho}$, and by $\mathrm{d}\,\varphi \colon \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} \to \mathbf{t}_{\widehat{X}, x}$ the induced L-linear map on tangent spaces.

(b) *Suppose that $\rho' \in \ker \mathrm{d}\,\varphi \subset \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} \cong \mathcal{D}_\rho(L[\varepsilon])$, that is, that $D_{\rho'} = D_\rho$. Then with respect to a suitable basis ρ' is upper triangular and is the trivial deformations on the diagonal blocks.*

(c) *If $\zeta_p \notin K$, then R^{univ}_ρ is formally smooth over L of dimension $\dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} = dn^2 + 1$,*

$$\dim_L \ker \mathrm{d}\varphi = dn_1 n_2 - 1 \quad \textit{and} \quad \dim_L \mathrm{im}\, \mathrm{d}\varphi = dn^2 - dn_1 n_2 + 2.$$

(d) *If $\zeta_p \in K$, then $R^{\mathrm{univ}}_{\rho,\mathrm{red}}$ is formally smooth over L of relative dimension $h - 1$ for $h := \dim_L \mathbf{t}_{X^{\mathrm{univ}}_\rho, \rho} = dn^2 + 2$. Denoting by $\varphi_{\mathrm{red}} \colon (X^{\mathrm{univ}}_\rho)_{\mathrm{red}} \to (\widehat{X})_{\mathrm{red}}$ the morphism on reduced L-schemes associated to φ and by $\mathrm{d}\,\varphi_{\mathrm{red}} \colon \mathbf{t}_{(X^{\mathrm{univ}}_\rho)_{\mathrm{red}}, \rho} \to \mathbf{t}_{(\widehat{X})_{\mathrm{red}}, x}$ the induced map on tangent spaces, there furthermore exists $\delta \in \{0, 1\}$ such that*

$$\dim_L \ker \mathrm{d}\varphi_{\mathrm{red}} = dn_1 n_2 - 1 - \delta \quad \textit{and} \quad \dim_L \mathrm{im}\, \mathrm{d}\varphi_{\mathrm{red}} = dn^2 - dn_1 n_2 + 2 + \delta.$$

Proof. We begin with (a). The Euler characteristic formula in Theorem 3.4.1 now gives

$$\begin{aligned} \dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) &= \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) \\ &= dn_1 n_2 + \dim_L H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) + \dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee), \end{aligned}$$

which is strictly positive. Thus, there exists an nonzero element $c \in \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1})$. Setting $\rho = \begin{pmatrix} \rho_{x_1} & c \\ 0 & \rho_{x_2} \end{pmatrix}$ and applying Lemma 3.4.4 and Theorem 3.2.4 completes the proof of (a). We observe for later, that in fact our assumptions imply that

$$\dim_L H^2(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) = \dim_L H^0(G_K, \rho_{x_1}^\vee \otimes \rho_{x_2}(1)) = \dim_L \mathrm{Hom}_{G_K}(\rho_{x_1}, \rho_{x_2}(1)) = 0,$$

and $H^0(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) = \mathrm{Hom}_{G_K}(\rho_{x_2}, \rho_{x_1}) = 0$, so that $\dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) = dn_1n_2$.

For the proof of (b), we use the canonical identifications (see [Maz97, Proposition, p. 271])

$$\mathcal{D}_\rho(L[\varepsilon]) \cong \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} \quad \text{and} \quad \mathcal{P}sD_{D_\rho}(L[\varepsilon]) \cong \mathbf{t}_{\widehat{X},D_\rho} \tag{29}$$

to identify $\ker \mathrm{d}\varphi$ with the L-subspace of $\mathcal{D}_\rho(L[\varepsilon])$, which consists of the deformations of ρ to $L[\varepsilon]$ that map under $\mathrm{d}\varphi$ to the trivial pseudodeformation to $L[\varepsilon]$ of the residual pseudocharacter D_ρ associated with ρ. Let ρ' be a deformation of ρ whose associated pseudocharacter satisfies $D_{\rho'} = D_\rho$. The linearization of ρ' gives a continuous homomorphism

$$L[\varepsilon][G_K] \longrightarrow \begin{pmatrix} \mathrm{Mat}_{n_1\times n_1}(L[\varepsilon]) & \mathrm{Mat}_{n_1\times n_2}(\mathcal{A}_{12}) \\ \mathrm{Mat}_{n_2\times n_1}(\mathcal{A}_{21}) & \mathrm{Mat}_{n_2\times n_2}(L[\varepsilon]) \end{pmatrix},$$

which when composed with the determinant gives $D_{\rho'}$ so that by Theorem 4.3.10(b) ρ' factors via a GMA. By hypothesis, we must have $\mathcal{A}_{12} = L[\varepsilon]$ and $\mathcal{A}_{21} \subset \varepsilon L$. Also by hypothesis, the residual pseudocharacter D_ρ is multiplicity free and split so that by Proposition 4.3.9(b) the ideal of total reducibility $\mathcal{A}_{12}\mathcal{A}_{21}$ vanishes, and hence $\mathcal{A}_{21} = 0$, and ρ' is upper triangular. Let D_1 and D_2 be the pseudocharacters described by the upper left and lower right diagonal blocks of ρ'. then again by Proposition 4.3.9(b) (and by the nonsplitness of ρ) we have $D_i = D_{\rho_i}$, $i = 1, 2$, and hence by Theorem 4.3.10(a), ρ' is the trivial deformations on the diagonal blocks.

For (c) and (d), we first compute $\mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} = \dim_L H^1(G_K, \mathrm{ad}_\rho)$. It follows from Lemma 3.4.4 that $H^0(G_K, \mathrm{ad}_\rho) \cong L$, and now formula Theorem 3.4.1(c) yields

$$\dim_L H^1(G_K, \mathrm{ad}_\rho) = dn^2 + 1 + \dim_L H^2(G_K, \mathrm{ad}_\rho).$$

By Theorem 3.4.1(b), we have $\dim_L H^2(G_K, \mathrm{ad}_\rho) = \dim_L \mathrm{Hom}_{G_K}(\rho, \rho(1))$. The claimed expressions for $\dim_L \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$ now follow from Lemma 3.4.4 with $\chi = \mathbb{F}(1)$ and our hypotheses. The claim on R_ρ^{univ} in (c) now follows from Theorem 3.2.4. The claim on R_ρ^{univ} in (d) follows from Corollary 3.4.3 provided that we show that $H^0(G_K, \overline{\mathrm{ad}}_\rho) = 0$. But under our hypotheses this follows from Corollary 2.3.3.

For the assertions on $\mathrm{d}\varphi$ and $\mathrm{d}\varphi_{\mathrm{red}}$ in (c) and (d), we first give a formula for $\dim_L \ker \mathrm{d}\varphi$ in either case. We consider lifts ρ_1, ρ_2 of ρ to $L[\varepsilon]$ whose associated deformation classes satisfy $[\rho_1] = [\rho_2] \in \ker \mathrm{d}\varphi \subset \mathbf{t}_{X_\rho^{\mathrm{univ}}} \cong \mathcal{D}_\rho(L[\varepsilon])$. By assertion, (b) we have $\rho_i = \rho + \varepsilon \begin{pmatrix} 0 & c_i \\ 0 & 0 \end{pmatrix}$ for some cocycle $c_i \in Z^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$. In order to obtain $\dim_L \ker \mathrm{d}\varphi$, we determine when ρ_1 is equivalent to ρ_2. In this case, there exists a matrix $U \in \mathrm{Mat}_{n\times n}(L)$ such that

$$\begin{aligned} \rho + \varepsilon \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} &= \rho_2 \\ &= (1 + \varepsilon U)\rho_1(1 - \varepsilon U) \\ &= (1 + \varepsilon U)(\rho + \varepsilon \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix})(1 - \varepsilon U) \\ &= \rho + \varepsilon(U\rho - \rho U + \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix}). \end{aligned}$$

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If we write $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ with matrices $U_{ij} \in \mathrm{Mat}_{n_i \times n_j}(L)$ for $1 \le i, j \le 2$, then the above equality is equivalent to

$$\begin{pmatrix} 0 & c_2 - c_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U_{11}\rho_{x_1} & U_{11}c + U_{12}\rho_{x_2} \\ U_{21}\rho_{x_1} & U_{21}c + U_{22}\rho_{x_2} \end{pmatrix} - \begin{pmatrix} \rho_{x_1}U_{11} + cU_{21} & \rho_{x_1}U_{12} + cU_{22} \\ \rho_{x_2}U_{21} & \rho_{x_2}U_{22} \end{pmatrix}.$$

Because $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_j}^\vee) = 0$ and $\dim_L H^0(G_K, \rho_{x_i} \otimes \rho_{x_i}^\vee) = 1$ for $1 \le i, j \le 2$ and $i \neq j$, we deduce that $U_{21} = 0$ and that U_{11} and U_{22} are scalar matrices. Finally, the map

$$-\rho_{x_1}U_{12} + U_{12}\rho_{x_2} \in B^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$$

is a coboundary. Therefore, $c_2 = (U_{11} + U_{22})c + c_1 \in H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee)$ and

$$\dim_L \ker \mathrm{d}\varphi = \dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_2}, \rho_{x_1}) - 1 = \dim_L H^1(G_K, \rho_{x_1} \otimes \rho_{x_2}^\vee) - 1 = dn_1n_2 - 1, \tag{30}$$

by the computation for (a). Using $\dim V = \dim \ker \psi + \dim \operatorname{im} \psi$ for a vector space V and a linear map ψ with domain V, and the already computed dimension of $\mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$, the proof of (c) is complete.

For (d), consider the following diagram with left exact rows and where the middle and right vertical arrows are injective (by definition of $\mathbf{t}$):

$$\begin{tikzcd}
0 \arrow[r] & \ker\varphi \arrow[r] & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} \arrow[r, "\mathrm{d}\varphi"] & \mathbf{t}_{\widehat{X},D_\rho} \\
0 \arrow[r] & \ker\varphi_{\mathrm{red}} \arrow[r] \arrow[u] & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} \arrow[r, "\mathrm{d}\varphi_{\mathrm{red}}"] \arrow[u, hook] & \mathbf{t}_{(\widehat{X})_{\mathrm{red}},D_\rho}. \arrow[u, hook]
\end{tikzcd}$$

By a simple diagram, chase one deduces $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} \subset \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho}$. Next, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\mathrm{d}\varphi \arrow[r] & \mathbf{t}_{X_\rho^{\mathrm{univ}},\rho} \arrow[r] & \operatorname{im}\mathrm{d}\varphi \arrow[r] & 0 \\
0 \arrow[r] & \ker\mathrm{d}\varphi_{\mathrm{red}} \arrow[r] \arrow[u, hook, "\alpha"] & \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho} \arrow[r] \arrow[u, hook, "\beta"] & \operatorname{im}\mathrm{d}\varphi_{\mathrm{red}} \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

with exact rows and where the left and middle vertical arrows are injective. Because of $\ker\varphi_{\mathrm{red}} = \ker\varphi \cap \mathbf{t}_{(X_\rho^{\mathrm{univ}})_{\mathrm{red}},\rho}$ the map γ is injective, and we deduce from the 9-Lemma that $\dim\operatorname{coker}\alpha + \dim\operatorname{coker}\gamma = \dim\operatorname{coker}\beta$. From the tangent space computations for (d) made so far, we deduce $\dim\operatorname{coker}\beta = 1$. Letting $\delta := \dim\operatorname{coker}\alpha$, we must have $0 \le \delta \le 1$ and $\dim\operatorname{coker}\gamma = 1 - \delta$. Arguing as for (c) and using $\dim_L \mathbf{t}_{X_\rho^{\mathrm{univ}},\mathrm{red},\rho} = dn^2 + 1$, the proof of (d) is complete, as well. $\square$

5.2. *Zariski density of the irreducible locus*

The aim of this subsection is to formulate an inductive procedure to prove Zariski density of the irreducible locus the special fibers of universal pseudodeformation spaces, and to establish some key steps. Our procedure is an adaption of an analogous result of Chenevier for the generic fiber; see [Che11, Théorème 2.1]. We shall prove the following main result.

Theorem 5.2.1. *Let $n \ge 2$ be an integer. Suppose that for all pseudocharacters $\overline{D}' : G_K \to \mathbb{F}$ on G_K of dimension $n' < n$ the following hold:*

(a) $\overline{X}^{\mathrm{univ}}_{\overline{D}'}$ *is equidimensional of dimension $[K : \mathbb{Q}_p](n')^2 + 1$,*

(b) $(\overline{X}^{\mathrm{univ}}_{\overline{D}'})^{\text{n-spcl}}$ *is Zariski dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}'}$.*

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Then for all n-dimensional pseudocharacters $\overline{D}\colon G_K \to \mathbb{F}$ *on* G_K *the subspace* $(X_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}} \subset X_{\overline{D}}^{\mathrm{univ}}$ *is Zariski dense.*

Let us begin with some preparations. Let $n_1, n_2 \geq 1$ be integers such that $n = n_1+n_2$. Let $\overline{D}_i\colon G_K \to \mathbb{F}$ be residual pseudocharacters on G_K of dimension n_i. Addition $(D_1, D_2) \mapsto D_1 \oplus D_2$ of pseudocharacters yields a morphism

$$\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2} \longrightarrow \overline{X}^{\mathrm{univ}}_{\overline{D}} \tag{31}$$

for $\overline{D} := \overline{D}_1 \oplus \overline{D}_2$. If $\overline{D}_1 \neq \overline{D}_2$, we define $\overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} := \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ and write $\iota_{\overline{D}_1,\overline{D}_2}$ for the above morphism. In the other case we let $\mathbb{Z}/2$ act on $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1}$ by exchanging the factors; it preserves the diagonal, which we denote by $\Delta^{\mathrm{univ}}_{\overline{D}_1}$, and one has an induced morphism

$$\iota_{\overline{D}_1,\overline{D}_1}\colon \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_1} := (\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1})/(\mathbb{Z}/2) \longrightarrow \overline{X}^{\mathrm{univ}}_{\overline{D}} \tag{32}$$

Note that away from the $\Delta^{\mathrm{univ}}_{\overline{D}_1}$, the morphism $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_1}$ is an étale Galois cover with monodromy group $\mathbb{Z}/2$.

Lemma 5.2.2 [Che11, Lemme 1.1.]. *For* $i = 1, 2$, *let* $x_i \in \overline{X}^{\mathrm{univ}}_{\overline{D}_i}$ *be irreducible points of dimension* 1, *and let L be a finite common extension of the residue fields* $\kappa(x_i)$. *If* $\overline{D}_1 = \overline{D}_2$, *assume also that* $x_1 \neq x_2$. *Let* $x \in \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *be the point of dimension* 1 *with* $D_{x_2} \otimes_{\kappa(x_1)} L$. *Let* $\overline{x}\colon \mathrm{Spec}\,\kappa(x) \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$ *be a geometric point over x.*

Then there is an étale neighborhood $(U, \overline{u}, \varphi_U : U \to \overline{X}^{\mathrm{univ}}_{\overline{D}})$ *of* $\overline{x}$ *such that the base change*

$$U' := U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}, \iota_{\overline{D}_1,\overline{D}_2}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} \xrightarrow{\iota_U} U$$

of $\iota_{\overline{D}_1,\overline{D}_2}$ *along* φ_U *is a closed immersion with image* $U^{\mathrm{red}} = \{u \in U \mid u \text{ is reducible}\}$. *Moreover, if* $\overline{D}_1 = \overline{D}_2$, *then we may choose U such that* $\varphi_U(U)$ *is disjoint from* $\iota_{\overline{D}_1,\overline{D}_1}(\Delta^{\mathrm{univ}}_{\overline{D}_1})$.

Proof. As recalled above Definition 4.7.8, the universal pseudodeformation $D^{\mathrm{univ}}_{\overline{D}}$ factors via the universal Cayley–Hamilton pseudodeformation and CH-representation

$$\overline{R}^{\mathrm{univ}}_{\overline{D}}[[G_K]] \xrightarrow{\rho^{\mathrm{CH}}_{\overline{D}}} \overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}} \xrightarrow{D^{\mathrm{CH\text{-}univ}}_{\overline{D}}} \overline{R}^{\mathrm{univ}}_{\overline{D}}.$$

Consider the strictly local ring $\mathcal{O}^{\mathrm{sh}}_{\overline{x}} := \mathrm{colim}_{(V,\overline{v})}\mathcal{O}(V)$ for $\mathcal{O}(V) := \mathcal{O}_{\overline{X}^{\mathrm{univ}}_{\overline{D}}}(V)$, where $(V, \overline{v})$ runs over all connected étale neighborhoods of $\overline{x}$ in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ [Sta18, Lemma 04HX]. Since by Proposition 4.1.22 the formation of the Cayley–Hamilton quotient $\overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}}$ commutes with arbitrary base change, for any étale neighborhood $(V, \overline{v})$ of $\overline{x}$ there is an isomorphism

$$\mathcal{O}(V)[[G_K]]/\mathrm{CH}(D^{\mathrm{univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V)) \xrightarrow{\sim} \overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}} \otimes_{\overline{R}^{\mathrm{univ}}_{\overline{D}}} \mathcal{O}(V) =: \overline{S}_V.$$

From Theorem 4.3.10, it follows that $\overline{S}_{\overline{x}} := \mathrm{colim}_{(V,\overline{v})}\overline{S}_V$ is a GMA of type (n_1, n_2). In particular, there exists idempotents $e_1, e_2 \in \overline{S}_{\overline{x}}$ with $e_1 + e_2 = 1$ and for $i = 1, 2$ an isomorphism $\psi_{\overline{x},i}\colon e_i\overline{S}_{\overline{x}}e_i \to \mathrm{Mat}_{n_i\times n_i}(\mathcal{O}^{\mathrm{sh}}_{\overline{x}})$. Denote by $\mathcal{E}_{\overline{x}} := (e_i, \psi_{\overline{x},i})_{i=1,2}$, then also the induced pseudocharacter to $\mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ factors via the natural Cayley–Hamilton pseudocharacter $D_{\overline{S}_{\overline{x}},\mathcal{E}_{\overline{x}}}$ from Proposition 4.3.5.

By Proposition 4.7.11, the ring $\overline{S}^{\mathrm{CH\text{-}univ}}_{\overline{D}}$ is module-finite as an $\overline{R}^{\mathrm{univ}}_{\overline{D}}$-algebra and Noetherian. Note also that we constructed $\overline{S}_{\overline{s}}$ and $\mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ as direct limits over étale neighborhoods. Using spreading out principles

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from [Gro66, §8.5], we can thus find a connected affine étale neighborhood $(\overline{u}, U, \varphi\colon U \to \overline{X}^{\mathrm{univ}}_{\overline{D}})$ of $\overline{x}$ such that the e_i can be defined $e_1 + e_2 = 1$, and such that one has isomorphism

$$\psi_{U,i}\colon e_i \overline{S}_U e_i \to \mathrm{Mat}_{n_i \times n_i}(\mathcal{O}(U)),$$

whose base change under $\mathcal{O}(U) \to \mathcal{O}^{\mathrm{sh}}_{\overline{x}}$ is $\psi_{\overline{x},i}$. Hence, $\overline{S}_U$ together with $\mathcal{E}_U := (e_i, \psi_{U,i})_{i=1,2}$ is a GMA. By choosing U sufficiently 'small', we may also assume that the pseudocharacter $D_U\colon \mathcal{O}(U)[G] \to \mathcal{O}(U)$ induced from $D^{\mathrm{univ}}_{\overline{D}}$ factors via the induced CH-representation $G \to (\overline{S}_U)^\times$ composed with the natural Cayley–Hamilton pseudocharacter $D_{\overline{S}_U, \mathcal{E}_U}$.

Let us write

$$\overline{S}_U \cong \begin{pmatrix} \mathrm{Mat}_{n_1 \times n_1}(\mathcal{O}(U)) & \mathrm{Mat}_{n_1 \times n_2}(\mathcal{A}_{12}) \\ \mathrm{Mat}_{n_2 \times n_1}(\mathcal{A}_{21}) & \mathrm{Mat}_{n_2 \times n_2}(\mathcal{O}(U)) \end{pmatrix}.$$

with finitely generated $\mathcal{O}(U)$-modules $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ together with the structure of a GMA described in Lemma 4.3.3. Let $I = \mathcal{A}_{12}\mathcal{A}_{21} + \mathcal{A}_{21}\mathcal{A}_{12} = \mathcal{A}_{12}\mathcal{A}_{21}$ be the ideal of total reducibility. From Proposition 4.3.9(b), we deduce that there exist unique pseudocharacters $D_i\colon e_i \overline{S}_U e_i \to \mathcal{O}(U)/I$ for $i = 1, 2$ such that

$$(D_U \bmod I) = D_1 \oplus D_2.$$

Denote by $Z := \mathrm{Spec}(\mathcal{O}(U)/I)$ the locus of total reducibility, by $f\colon Z \to U$ the induced closed immersion and by $g\colon Z \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1, \overline{D}_2}$ the morphism corresponding to the $\mathcal{O}(Z)$-valued pseudocharacters (D_1, D_2). Then the morphism $\varphi_U \circ f$ corresponds to the $\mathcal{O}(U)/I$-valued pseudocharacter $D_U \bmod I$ and there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & \overline{X}^{\mathrm{univ}}_{\overline{D}_1, \overline{D}_2} \arrow[d, "\iota_{\overline{D}_1, \overline{D}_2}"] \\ U \arrow[r, "\varphi_U"] & \overline{X}^{\mathrm{univ}}_{\overline{D}} \end{tikzcd} \tag{33}$$

since $\varphi_U \circ f$ and $\iota_{\overline{D}_1, \overline{D}_2} \circ g$ both correspond to $D_U \bmod I = D_1 \oplus D_2$. We need to show that this diagram is Cartesian; then $\iota_U = f$ is a closed immersion, by construction, that is, by [Sta18, Definition 01JP] given any connected affine scheme W together with morphisms $f'\colon W \to U$ and $g'\colon W \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1, \overline{D}_2}$ such that in the following diagram the solid square commutes

$$\begin{tikzcd} W \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\ & Z \arrow[r, "g"'] \arrow[d, "f"] & \overline{X}^{\mathrm{univ}}_{\overline{D}_1, \overline{D}_2} \arrow[d, "\iota_{\overline{D}_1, \overline{D}_2}"] \\ & U \arrow[r, "\varphi_U"] & \overline{X}^{\mathrm{univ}}_{\overline{D}}, \end{tikzcd}$$

we need to check that there exists a unique dashed arrow h making the diagram commute.

The morphism $\varphi_U \circ f' = \iota_{\overline{D}_1, \overline{D}_2} \circ g'$ defines an $\mathcal{O}(W)$-valued pseudocharacter D'_W, and the morphism g' a pair (D'_1, D'_2) of $\mathcal{O}(W)$-valued pseudocharacters of dimension n_j for $j = 1, 2$. By Lemma 4.3.4, the base change $\overline{S}_W$ of $\overline{S}_U$ along f' is a generalized matrix algebra over $\mathcal{O}(W)$ of type (n_1, n_2). The

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definition of $\iota_{\overline{D}_1,\overline{D}_2}$ implies that $D'_W = D'_1 \oplus D'_2$, and from Proposition 4.3.9(b) we conclude that the ideal

$$I' := I \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(W) = \mathcal{A}_{12}\mathcal{A}_{21} \otimes_{\mathcal{O}(U),(f')^*} \mathcal{O}(W)$$

of total reducibility of $\overline{S}_W$ vanishes. Hence, there exists a unique morphism $h\colon W \to Z$ such that $(f')^*$ factors as $\mathcal{O}(U) \xrightarrow{f^*} \mathcal{O}(Z) \xrightarrow{h^*} \mathcal{O}(W)$. Note the $g^* \circ h^*$ determines a pair (D''_1, D''_2) of pseudocharacters $G \to \mathcal{O}(W)$ on G. From Proposition 4.3.9(b), we deduce $\{D'_1, D'_2\} = \{D''_1, D''_2\}$. The universal property of $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$, and our definition of $(\iota_{\overline{D}_1,\overline{D}_2}, \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2})$ implies that $(h')^* = g^* \circ h^*$.

Next, we prove $Z = U^{\mathrm{red}}$ under the closed immersion f. By the definition of $\iota_{\overline{D}_1,\overline{D}_2}$ the inclusion $\subseteq$ is obvious. Let therefore y be any point of U^{red}. To see that y lies on $f(Z)$, let D_y be the reducible pseudocharacter corresponding $\overline{R}^{\mathrm{univ}}_{\overline{D}} \to \mathcal{O}(U) \to \kappa(y)$. By Lemma 4.3.4, the base change $S_y := \overline{S}_U \otimes_{\mathcal{O}(U)} \kappa(y)^{\mathrm{alg}}$ of $\overline{S}_U$ is also a generalized matrix algebra of type (n_1, n_2). Since D_y is reducible, there exists pseudocharacters $D_1, D_2\colon G_K \to \kappa(y)^{\mathrm{alg}}$ on G_K such that $D_y = D_1 \oplus D_2$. By again applying Proposition 4.3.9, we find that the ideal of total reducibility of the generalized matrix algebra S_y vanishes. Hence, $\mathcal{O}(U) \to \kappa(y)$ factors via $\mathcal{O}(Z)$ as was to be shown.

For the final assertion, suppose from now on that $\overline{D}_1 = \overline{D}_2$ so that $m := n_1 = n_2$. Consider the maps

$$\Lambda^j_i\colon G \longrightarrow \overline{S}_U \xrightarrow{\psi_{U,i}} \mathrm{End}_{\mathcal{O}(U)}(\mathcal{O}(U)^m) \xrightarrow{\bigwedge^j} \mathrm{End}_{\mathcal{O}(U)}(\mathcal{O}(U)^{\binom{m}{j}}) \xrightarrow{\mathrm{tr}} \mathcal{O}(U)$$

for $i = 1, 2$ and $j = 1, \ldots, m$, where $\bigwedge^j$ denotes the exterior power map on endomorphisms. For every $g \in G$, the vanishing locus of $\Lambda^j_1(g) - \Lambda^j_2(g) \in \mathcal{O}(U)$ is a closed subscheme Y_g of U, and hence the intersection $Y := \bigcap_{g \in G} Y_g$ is closed in U. Since $x_1 \neq x_2$, we have $(x_1, x_2) \notin \varphi(Y)$, and thus $U' = U \backslash Y$ is an étale neighborhood of $\overline{x}$ as required for the last assertion. $\square$

Proof (First proof of Theorem 5.2.1). We suppose to the contrary that there exists a nonempty open affine $V \subset \overline{X}^{\mathrm{univ}}_{\overline{D}}$ such that $(\overline{X}^{\mathrm{univ}}_{\overline{D}})^{\mathrm{irr}} \cap V = \emptyset$. Since $V \neq \mathrm{Spec}\mathbb{F}$ and the points of dimension 1 are very dense in $\overline{X}^{\mathrm{univ}}_{\overline{D}}$ by Lemma A.1.8, there exists a point $x \in V$ of dimension 1 that defines a reducible pseudodeformation

$$D_x\colon G_K \longrightarrow \kappa(x)$$

of $\overline{D}$. By Corollary 4.8.4, there exist a finite extension $L'/\kappa(x)$ with finite residue field $\mathbb{F}' \supset \mathbb{F}$, residual pseudocharacter $\overline{D}_i\colon G_K \to \mathbb{F}'$ on G_K of dimension n_i for some $n_i \in \mathbb{N}_0$ with $n_1 + n_2 = n$, and pseudocharacters $D_1, D_2\colon G_K \to \mathcal{O}_{L'}$ of G_K corresponding to points $(x_1, x_2) \in \overline{X}^{\mathrm{univ}}_{\overline{D}_1} \hat{\times} \overline{X}^{\mathrm{univ}}_{\overline{D}_2}$ such that $D_x \otimes_{\kappa(x)} L' = (D_1 \oplus D_2) \otimes_{\mathcal{O}_{L'}} L'$. By Lemma 5.2.6, we may and will assume $\mathbb{F} = \mathbb{F}'$.

The inverse image of V under $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \widehat{\times} \overline{X}^{\mathrm{univ}}_{\overline{D}_2} \to \overline{X}^{\mathrm{univ}}_{\overline{D}}$, $(D_1, D_2) \mapsto D_1 + D_2$ is an open neighborhood of (x_1, x_2). By hypothesis (b) of Theorem 5.2.1, we may within this neighborhood replace the initially chosen pair by (x_1, x_2) such that both are irreducible and nonspecial, and by Lemma A.1.7 we may further assume that D_1 is not isomorphic to any of the finitely many $D_2(m)$, $m \in \{1, \ldots, p-1\}$, since $\dim \overline{X}^{\mathrm{univ}}_{\overline{D}_i} \geq 2$. Let $U_i := (\overline{X}^{\mathrm{univ}}_{\overline{D}_i})^{\text{n-spcl}}$. Then we observe that by Lemma 5.1.6 the schemes $(U_i)_{\mathrm{red}}$ are regular, and if $\zeta_p \notin K$, then $U_i = (U_i)_{\mathrm{red}}$.

Let $\overline{x}$ be a geometric point above x. By Lemma 5.2.2, there exists an étale neighborhood $(U, \bar{u}, \varphi_U : U \to \overline{X}^{\mathrm{univ}}_{\overline{D}})$ of $\overline{x}$ such that the pullback of $\iota_{\overline{D}_1,\overline{D}_2}$ along φ_U

$$W := U \times_{\varphi_U, \overline{X}^{\mathrm{univ}}_{\overline{D}}, \iota_{\overline{D}_1,\overline{D}_2}} \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} \longrightarrow U$$

is a closed immersion with image U^{red}. We may replace U by $\varphi_U^{-1}(V)$, which is nonempty since $x \in V$, and is étale over V, and we may shrink W accordingly. By further replacing U by an open subset (and accordingly W), we can assume that U is connected and affine. Since $W \to U$ is a closed immersion, the scheme W is affine. But we also have that $W \to U$ is surjective as a map of topological spaces since all points of V are reducible. Hence, the nil-reduction of $W \to U$ is an isomorphism of schemes $W_{\mathrm{red}} \to U_{\mathrm{red}}$, and as a map of topological spaces $W \to U$ is a homeomorphism. Since the base change of étale morphisms is étale, so is the map $W \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2}$ that is the base change of φ_U under $\iota_{\overline{D}_1,\overline{D}_2}$. Let $\widetilde{U}_i$ be the preimage of U_i under the i-th projection $\overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} \to \overline{X}^{\mathrm{univ}}_{\overline{D}_i}$. We shrink W (and hence U) to a connected affine open so that the image of W in $\overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2}$ lies in $\widetilde{U}_1 \cap \widetilde{U}_2$.[7] We display the situation in the following diagram:

$$\begin{tikzcd}
\widetilde{U}_1 \cap \widetilde{U}_2 \arrow[r, hook] & \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2} \arrow[r, "\iota_{\overline{D}_1,\overline{D}_2}"] & \overline{X}^{\mathrm{univ}}_{\overline{D}} & V \arrow[l, hook'] \\
& W \arrow[ul] \arrow[u] \arrow[r] & U \arrow[ur, "\varphi_U"] &
\end{tikzcd}$$

Note also that $\varphi_U(U)$ intersects trivially with $\iota_{\overline{D}_1,\overline{D}_1}(\Delta^{\mathrm{univ}}_{\overline{D}_1})$ if $\overline{D}_1 = \overline{D}_2$. Hence, in all cases, the morphism $\overline{X}^{\mathrm{univ}}_{\overline{D}_1} \widehat{\times}_{\mathbb{F}} \overline{X}^{\mathrm{univ}}_{\overline{D}_2} \to \overline{X}^{\mathrm{univ}}_{\overline{D}_1,\overline{D}_2}$ is an étale Galois cover above $\widetilde{U}_1 \cap \widetilde{U}_2$ with group $\mathbb{Z}/2$ or trivial group.

Let $w \in W$ be the point corresponding to $u \in U$ under the homeomorphism $W \to U$. We complete at w and its images and pass to nil reductions. This gives

$$\widehat{\mathcal{O}}_{\widetilde{U}_1\cap\widetilde{U}_2,(x_1,x_2),\mathrm{red}} \xrightarrow{\ \alpha\ } \widehat{\mathcal{O}}_{W,w,\mathrm{red}} \xrightarrow{\ \simeq\ } \widehat{\mathcal{O}}_{U,u,\mathrm{red}} \xleftarrow{\ \beta\ } \widehat{\mathcal{O}}_{V,x,\mathrm{red}}.$$

By Lemma A.1.14, the maps α and β are finite étale. The completion $\widehat{\mathcal{O}}_{\widetilde{U}_1\cap\widetilde{U}_2,(x_1,x_2),\mathrm{red}}$ can be compared with the deformation ring $R^{\mathrm{univ}}_{L,(\rho_1,\rho_2)}$; using Proposition 4.8.11, it follows that the ring $\widehat{\mathcal{O}}_{\widetilde{U}_1\cap\widetilde{U}_2,(x_1,x_2),\mathrm{red}}$ is formally smooth over L of dimension $d(n_1^2+n_2^2)+1$, because by Lemma 5.1.6 and by hypothesis Theorem 5.2.1(a) the rings $R^{\mathrm{univ}}_{\rho_1,\mathrm{red}}$ are formally smooth over L of dimension dn_i^2+1. Hence, by Lemma A.1.14 all local rings in the above diagram will be formally smooth over L of dimension $d(n_1^2+n_2^2)+1$.

Let now $\rho\colon G_K \to \mathrm{GL}_n(L)$ be a nontrivial extension of ρ_2 by ρ_1 for $n = n_1+n_2$ as constructed in Lemma 5.1.7 (a). It possesses a universal deformation ring R^{univ}_ρ for deformation to $\mathcal{A}r_L$, because $L = H^0(G_K, \mathrm{ad}_\rho)$. Let also $\widehat{R}$ be the universal pseudodeformation ring for D_ρ, and write φ for the natural morphism between associated space:

$$\varphi\colon X^{\mathrm{univ}}_\rho := \operatorname{Spec} R^{\mathrm{univ}}_\rho \longrightarrow \widehat{X} := \operatorname{Spec} \widehat{R}.$$

The relation to the above is given by the following isomorphism obtained by combining Corollary 4.8.8 and Lemma 3.3.5

$$\widehat{R} = \widehat{\mathcal{O}}_{V,x}[[T]]. \tag{34}$$

We now consider the map $\mathrm{d}\varphi\colon \mathbf{t}_{X^{\mathrm{univ}}_\rho,\rho} \to \mathbf{t}_{\widehat{X},x}$ induced from φ on tangent spaces at ρ and D_ρ, respectively, or rather the induced map on nil-reductions

$$\mathrm{d}\varphi_{\mathrm{red}}\colon \mathbf{t}_{X^{\mathrm{univ}}_\rho,\mathrm{red}}, \rho \to \mathbf{t}_{\widehat{X}_{\mathrm{red}},x}.$$

By Lemma 5.1.7 (c) and (d) we have $\delta \in \{0,1\}$ (and $\delta = 0$ if $\zeta_p \notin K$) such that

$$dn^2 - dn_1n_2 + 2 + \delta = \dim_L \mathrm{Im}(\mathrm{d}\varphi_{\mathrm{red}})$$

[7]The intersection $\widetilde{U}_1 \cap \widetilde{U}_2$ is strictly bigger than $U_1 \widehat{\times} U_2$. If for instance $X_i = \operatorname{Spec} L[[T_i]]$, $i = 1,2$ and $U_i = \operatorname{Spec} L((T_i))$. Then $\widetilde{U}_1 \cap \widetilde{U}_2 = \operatorname{Spec} L[[T_1,T_2]][\frac{1}{T_1T_2}]$ contains all but 3 points of $\operatorname{Spec} L[[T_1,T_2]]$.

From (34) and the dimension found for $\widehat{\mathcal{O}}_{V,x}$, we have $\dim \mathbf{t}_{\widehat{X}_{\mathrm{red}},x} = 1 + d(n_1^2 + n_2^2)$. This gives the inequality

$$dn^2 - dn_1n_2 + 2 + \delta \leq d(n_1^2 + n_2^2) + 2$$

Using $n = n_1 + n_2$, we deduce $dn_1n_2 + \delta \leq 0$, which is absurd since both $n_i > 0$. □

5.3. *Alternative proof of Theorem 5.2.1*

Following a suggestion of the referee, we now present a second proof of Theorem 5.2.1. It is technically easier than the proof given above, and might be of independent interest. The approach makes no assertion on the geometry of $X_{\overline{D}}^{\mathrm{univ}}$ near a reducible point x as in Lemma 5.2.2 but focuses directly on the completed deformation ring at such an x and a dimension estimate.

Let R be a Noetherian $\mathbb{F}$-algebra, and let A be an associative (possible noncommutative) unital R-algebra, which is finitely generated as an R-module. For $x \in \operatorname{Spec} R$, write $A_x := A \otimes_R \kappa(x)$, where $\kappa(x)$ is the residue field of x; that is, A_x is the fiber at x and not a localization. Let

$$U = \{x \in \operatorname{Spec} R : A_x \otimes_{\kappa(x)} A_x^{\mathrm{op}} \to \operatorname{End}_{\kappa(x)}(A_x) \text{ is an isomorphism}\}$$

be the *Azumaya locus of A in* Spec R.

Lemma 5.3.1. *The Azumaya locus U is a constructible subset of* Spec R.

Proof. By [Sta18, Lemma 051Z] and Noetherian induction, we can find a flattening stratification of A as an R-module, that is, a finite increasing chain of open subsets $U_0 \subset \ldots \subset U_n = \operatorname{Spec} R$ such that if R_i is the reduced quotient of R with $\operatorname{Spec} R_i = \operatorname{Spec} R \backslash U_i$, then $U_{i+1} \backslash U_i = \operatorname{Spec}\,(R_i)_{f_i}$ for some $f_i \in R_i$ and $A \otimes_R (R_i)_{f_i}$ is finite flat over $\operatorname{Spec}\,(R_i)_{f_i}$. Hence, to prove the assertion on U, we may assume that A is finite flat over R.

Now, let C and K be R-modules fitting in the exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes_R A^{\mathrm{op}} \longrightarrow \operatorname{End}_R(A) \longrightarrow C \to 0. \tag{35}$$

Because C and K are finitely generated modules over the Noetherian ring R, their support is closed, and we deduce that $U = \operatorname{Spec} R \backslash (SuppC \cup SuppK)$ is open in Spec R and hence constructible. □

Lemma 5.3.2. *Let x be in* Spec R, *let* $\widehat{R}$ *be the completion of* $\kappa(x) \otimes_{\mathbb{F}} R$ *at the kernel of the natural map* $\kappa(x) \otimes_{\mathbb{F}} R \to \kappa(x)$ *and let* $\widehat{A} = A \otimes_R \widehat{R}$. *Let* $y \in \operatorname{Spec} \widehat{R}$ *be such that* $\widehat{A}_y$ *is an Azumaya algebra. Then U is nonempty and x lies in the closure of U in* Spec R.

Proof. Let z be the image of y under $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$, and note that we have induced maps $\iota\colon R/\mathfrak{p}_z \to \widehat{R}/\mathfrak{p}_y$ and $\kappa(z) \to \kappa(y)$ for $\mathfrak{p}_z \subset R$ and $\mathfrak{p}_y \subset \widehat{R}$ the primes corresponding to z and y, respectively. Since $\widehat{A}_y = A_z \otimes_{\kappa(z)} \kappa(y)$, and $\kappa(z) \to \kappa(y)$ is faithfully flat, the diagram (35) with A_z and $\kappa(z)$ in place of A and R, respectively, implies $z \in U$ so that U is nonempty.

Moreover, by our definitions, the residue map $R \to \widehat{R} \to \kappa(x)$ factors via $R/\mathfrak{p}_z \xrightarrow{\iota} \widehat{R}/\mathfrak{p}_y \to \kappa(x)$ with ι injective. Hence, $x \in \operatorname{Spec} R/\mathfrak{p}_z = \overline{\{z\}} \subset \operatorname{Spec} R$ so that x lies in the closure of $\{z\}$ and hence in the closure of $U \supset \{z\}$. □

Let now $R := R_{\overline{D}}^{\mathrm{univ}}$ be the special fiber of the universal deformation ring for $\overline{D}$ so that R is complete Noetherian local with finite residue field. Let $A = S_{\overline{D}}^{\mathrm{CH\text{-}univ}}$ be the corresponding Cayley–Hamilton R-algebra so that by Proposition 4.7.11 the ring A is finitely generated as an R-module and hence Noetherian. In this setting, U is precisely the absolutely irreducible locus of $X_{\overline{D}}^{\mathrm{univ}}$ as explained in [Che14, Corollary 2.23]; see also Proposition 4.7.18.

Proposition 5.3.3. *In the setting just described, the closure of U in* Spec R *contains all points $x \in$* Spec R *of dimension* 1 *such that $D_x = D_1 + D_2$, where the D_i are nonspecial pseudocharacters of absolutely irreducible representations ρ_i, $i = 1, 2$ such that $\rho_1 \not\cong \rho_2(j)$ for $j \in \{0, \pm 1\}$.*

Proof. In view of Lemma 5.3.2, we only have to explain how to find y. By Corollary 4.8.8 the ring $\widehat{R}$ can be identified with the universal pseudodeformation ring of D_x, and we need to show that the reducible locus in Spec $\widehat{R}$ is not the whole of Spec $\widehat{R}$.

To show this, we compare $\widehat{R}$ with the universal deformation ring $\overline{R}_\rho^{\mathrm{univ}}$, where ρ is a nonsplit extension of ρ_2 by ρ_1. The existence of ρ and of $\overline{R}_\rho^{\mathrm{univ}}$ was established in Lemma 5.1.7(a). Moreover, by Lemma 5.1.7(c) and (d), respectively, the ring $R' := (\overline{R}_\rho^{\mathrm{univ}})_{\mathrm{red}}$ is formally smooth over $\kappa(x)$ of dimension $dn^2 + 1$. The map that sends a representation to its associated pseudocharacter induces a map Spec $\overline{R}_\rho^{\mathrm{univ}} \to \mathrm{Spec}\widehat{R}$, and it will suffice to show that the generic point of $R' = (\overline{R}_\rho^{\mathrm{univ}})_{\mathrm{red}}$ gives rise to an absolutely irreducible representation of G_K. Let $\mathfrak{m}' \subset R'$ be its maximal ideal.

Denote by $\rho' \colon G_K \to \mathrm{GL}_n(R')$ a representation corresponding to R'. Because D_ρ is multiplicity free, by Theorem 4.3.10(b) the linearization of ρ' factors via a GMA of type (n_1, n_2) inside $\mathrm{Mat}_{n\times n}(R')$ and thus gives a continuous surjective homomorphism

$$R'[G_K] \longrightarrow \begin{pmatrix} \mathrm{Mat}_{n_1\times n_1}(R') & \mathrm{Mat}_{n_1\times n_2}(J') \\ \mathrm{Mat}_{n_2\times n_1}(I') & \mathrm{Mat}_{n_2\times n_2}(R') \end{pmatrix}$$

for suitable ideals I', J' of R'. The reduction modulo $\mathfrak{m}'$ of the right-hand side arises from the nonsplit extension ρ of ρ_2 by ρ_1 so that we must have $J' = R'$, and $I' \subset \mathfrak{m}'$.

If I' is nonzero, then after passing from the regular local ring R' to its fraction field, say E', we obtain $\mathrm{Mat}_{n\times n}(E')$ as the image of the linearization map, and by the theorem of Burnside the corresponding representation is absolutely irreducible. If on the other hand $I' = 0$, then ρ' is reducible and we apply Proposition 3.4.6 and the discussion preceding it. It follows that $\dim(R^{\mathrm{univ}}_{\rho_1\subset\rho})_{\mathrm{red}} = d(n^2 - n_1n_2) + 1$ and that the induced map of reduced rings $R' \to (R^{\mathrm{univ}}_{\rho_1\subset\rho})_{\mathrm{red}}$ is an isomorphism. This contradicts $\dim R' = dn^2 + 1$ found above, and so the case $I' = 0$ cannot occur. □

Proof (Second proof of Theorem 5.2.1). Let $U = (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ be the open locus of irreducible points on $\overline{X}_{\overline{D}}^{\mathrm{univ}}$. We need to show that the closure of U is the whole space. By the reduction steps given in the first two paragraphs of the first proof of Theorem 5.2.1 on page 68, it suffices to show that all points $x \in \overline{X}_{\overline{D}}^{\mathrm{univ}}$ of dimension 1 such that $D_x = D_1 + D_2$ with D_i irreducible nonspecial and $D_2 \neq D_1(j)$ for $j \in \{0, \pm 1\}$ lie in the closure of U. This follows from Proposition 5.3.3. □

5.4. *A dimension bound for the special locus*

As before, we denote by $\overline{D} \colon G_K \to \mathbb{F}$ a residual pseudocharacter on G_K, and we let n be its dimension. Theorem 5.2.1 of the previous subsection provided part of an inductive procedure to prove the equidimensionality of $\overline{X}_{K,\overline{D}}^{\mathrm{univ}}$ for the dimension $[K : \mathbb{Q}_p] \cdot n^2 + 1$. It remains to be proved that $(\overline{X}_{K,\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}} \subset (\overline{X}_{K,\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ is Zariski dense. In this subsection, we shall prove the following result.

Theorem 5.4.1. *Let $n \geq 2$ be an integer. Suppose that for all pseudocharacters $\overline{D}' \colon G_{K'} \to \mathbb{F}$ on G_K of dimension $n' < n$ with K' a p-adic field the Krull dimension of the space $\overline{X}_{K',\overline{D}'}^{\mathrm{univ}}$ is bounded by $[K' : \mathbb{Q}_p](n')^2 + 1$, Then for all n-dimensional pseudocharacters $\overline{D} \colon G_K \to \mathbb{F}$ on G_K one has:*

(a) *The Zariski closure of $(\overline{X}_{K,\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ has dimension at most $\frac{1}{2}[K : \mathbb{Q}_p]n^2 + 1$.*
(b) *$(\overline{X}_{K,\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}} \subset (\overline{X}_{K,\overline{D}}^{\mathrm{univ}})^{\mathrm{irr}}$ is Zariski dense.*

Before giving the proof, we need the following auxiliary result.

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Lemma 5.4.2. *Let $\overline{R}_{G,\mathbb{F},\overline{D}} \to A$ be a surjective homomorphism such that A is a domain with field of fractions $\mathbb{K}$, and set $D_A := D^{\text{univ}}_{\overline{D}} \otimes_{\overline{R}_{G,\mathbb{F},\overline{D}}} A$. Let $H \subset G$ be an open normal subgroup and suppose the following hold:*

(i) *$\overline{D}|_H$ is split over $\mathbb{F}$ and condition $\Phi_{\overline{D}_H}$ is satisfied.*
(ii) *$D_{\mathbb{K}} := D_A \otimes_A \mathbb{K}$ is irreducible and $\rho := \rho_{D_{\mathbb{K}} \otimes \mathbb{K}^{\text{alg}}}$ is induced from H.*

Then there exist a domain $A' \in \widehat{\mathcal{A}r}_{\mathbb{F}}$ that contains A and is finite over A, and a continuous irreducible pseudocharacter $D' \colon A'[H] \to A'$ that is residually equal to a direct summand $\overline{D}'$ of $\overline{D}_H$ such that the following hold:

(a) $\operatorname{Ind}_H^G D' = D_A \otimes_A A'$
(b) *The homomorphism $\overline{R}_{H,\mathbb{F},\overline{D}'} \to A'$ that results from D' is surjective.*

In particular, $\dim A = \dim A' \le \dim \overline{R}_{H,\mathbb{F},\overline{D}'}$.

Proof. Note first by Lemma 2.1.4(b) and (f) that $\rho = \operatorname{Ind}_H^G \rho'$ for some irreducible representation $\rho' \colon H \to \mathrm{GL}_{n'}(\mathbb{K}^{\text{alg}})$ such that the representations $(\rho')^g$, $g \in G/H$, are pairwise nonisomorphic, and that $\operatorname{Res}_H^G \rho = \bigoplus_{g \in G/H} (\rho')^g$. Hence, $D_{\mathbb{K}}|_H$ is multiplicity free so that we can apply Proposition 4.8.6 to it.

By what we just observed, conjugation by G/H acts simply transitively on the continuous pseudocharacters D_i' from Proposition 4.8.6, and so the A_i from Proposition 4.8.6 are independent of i. Define A' as any of the A_i, and let $D' \colon A'[H] \to A'$ be that pseudocharacter D_i' for which $D_i' \otimes_{A'} \mathbb{K}^{\text{alg}}$ is the pseudocharacter attached to ρ'. Then $\operatorname{Ind}_H^G D' \otimes_{A'} \mathbb{K}^{\text{alg}} = D_A \otimes_A \mathbb{K}^{\text{alg}}$. Now, $\operatorname{Ind}_H^G D'$ is defined over A' and A is the minimal field of definition of D_A by Corollary 4.7.13. Hence, A is contained in A'. By Proposition 4.8.6 it is then clear that A' is finite integral over A and lies in $\widehat{\mathcal{A}r}_{\mathbb{F}}$, and moreover that $\oplus_{g \in G/H} (\overline{D}')^g = \overline{D}$ for $\overline{D}' := D' \otimes_{A'} \kappa(A')$. Part (a) is now clear.

It is also clear that D' is a deformation of $\overline{D}'$. Since $A' \in \widehat{\mathcal{A}r}_{\mathbb{F}}$, we have a corresponding homomorphism $\overline{R}_{H,\mathbb{F},\overline{D}'} \to A'$, and the latter must be surjective by Corollary 4.7.13 since A' is the ring of definition of D'. Now, by Lemma A.1.2 we have $\dim A' = \dim A$, and the inequality $\dim A' \le \dim \overline{R}_{H,\mathbb{F},\overline{D}'}$ is trivial. □

Proof of Theorem 5.4.1. By Lemma 3.2.6, by possibly enlarging $\mathbb{F}$, we may assume that $\overline{D}$ is split over $\mathbb{F}$. Since the number of Galois extensions K' of K of degree p is finite, we may, by the same reasoning, also assume that $D|_{G'_K}$ is split for any such K' and for $K' = K(\zeta_p)$. It is also clear that Mazur's condition Φ_p holds over any such K' and hence $\Phi_{\overline{D}|_{G_{K'}}}$ holds.

To prove (a), let η be any generic point of $(\overline{X}^{\text{univ}}_{\overline{D}})^{\text{spcl}}$. Let

$$\varphi \colon \overline{R}_{K,\mathbb{F},\overline{D}} \to A$$

be the corresponding surjective ring homomorphism so that $\eta = \operatorname{Ker}(\varphi)$. Because D_η is irreducible, $\rho := \rho_{D_\eta \otimes_{\kappa(\eta)} \kappa(\eta)^{\text{alg}}}$ is defined. Since η is special, there exists a Galois extension K' of K such that either $K' = K(\zeta_p)$ or K' has degree p over K and such that ρ is induced from $G_{K'}$. From Lemma 5.4.2, we deduce

$$\dim A \le \dim \overline{R}_{K',\mathbb{F},\overline{D}'} = [K' : \mathbb{Q}_p](n/[K' : K])^2 + 1 = \frac{1}{[K' : K]}\left([K : \mathbb{Q}_p]n^2\right) + 1.$$

As the schemes $\operatorname{Spec} A$ cover $(\overline{X}^{\text{univ}}_{K,\overline{D}})^{\text{spcl}}$ and as $[K' : K] \ge 2$, the proof of (a) is complete.

To prove (b), we argue by contradiction and assume that there exists an open subset $V \subset \overline{X}^{\text{univ}}_{K,\overline{D}}$ that is entirely contained in $(\overline{X}^{\text{univ}}_{K,\overline{D}})^{\text{spcl}}$. Then $\dim \overline{V} \le \frac{1}{2}[K : \mathbb{Q}_p]n^2 + 1$ by (b), for $\overline{V}$ the Zariski closure of V. Let x be any dimension 1 point of V, and let $\rho \colon G_K \to \mathrm{GL}_n(L)$ be an absolutely irreducible representation over a local field L containing $\kappa(x)$ such that $D_\rho = D_x \otimes_{\kappa(x)} L$. Let R^{univ}_ρ be the universal

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ring for deformations of ρ to $\mathcal{A}r_L$. Then $\widehat{\mathcal{O}}_{\overline{V},x}[[T]] \cong R_\rho^{\mathrm{univ}}$ by Corollary 4.8.8 and Lemma 3.3.5. On the other hand, $\dim R_\rho^{\mathrm{univ}} \geq [K:\mathbb{Q}_p]n^2+1$ by a standard argument using Theorem 3.4.1. It follows that

$$\frac{1}{2}[K:\mathbb{Q}_p]n^2+1+1 \geq [K:\mathbb{Q}_p]n^2+1,$$

and hence $2 \geq [K:\mathbb{Q}_p]n^2$, which implies $n=1$. But then x cannot be induced, and hence not special, and we reach a contradiction. □

5.5. *Main results*

Let K be a p-adic field, let $\overline{D}\colon G_K \to \mathbb{F}$ be a residual pseudocharacter on G_K, and set $n := \dim \overline{D}$.

Theorem 5.5.1 (Theorem 1]. *The following assertions hold:*

(a) $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is equidimensional of dimension* $[K:\mathbb{Q}_p]n^2+1$.
(b) $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}} \subset \overline{X}_{\overline{D}}^{\mathrm{univ}}$ *is open and Zariski dense.*
(c) *If* $\zeta_p \notin K$, *then* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}}$ *is regular.*
(d) *If* $\zeta_p \in K$, *then* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\text{n-spcl}}_{\mathrm{red}}$ *is regular, and* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{reg}}$ *is empty.*

Proof. Part (a) follows from Corollary 3.4.3, Theorem 5.2.1 and Theorem 5.4.1 by induction on $\dim \overline{D}$ and $[K:\mathbb{Q}_p]$. The same results also prove (b). Parts (c) and (d) follow from Lemma 5.1.6; the last part of (d) uses Corollary 3.4.3(a). □

Lemma 5.5.2. *One has the following estimates:*

(a) *If* $n>1$, *then*

$$\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} = \dim \overline{X}_{\overline{D}}^{\mathrm{univ}} - 2[K:\mathbb{Q}_p](n-1)+1,$$

and in particular $\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} \leq \dim \overline{X}_{\overline{D}}^{\mathrm{univ}} - 2$ *unless* $n=2$ *and* $K=\mathbb{Q}_p$. *In the latter case* $\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} = \dim \overline{X}_{\overline{D}}^{\mathrm{univ}} - 1$.
(b) $\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}} \leq \dim \overline{X}_{\overline{D}}^{\mathrm{univ}} - 2$.

Proof. Since $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ is empty for $n=1$, because nontrivially induced representations have dimension at least 2, Part (b) is immediate from Theorem 5.4.1. For Part (a), we may assume that $\overline{D}$ is split by Lemma 5.2.6. Then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} \subset \bigcup_{\overline{D}_1 \oplus \overline{D}_2 = \overline{D}} \iota_{\overline{D}_1,\overline{D}_2}(\overline{X}_{\overline{D}_1,\overline{D}_2}^{\mathrm{univ}})$, and now Theorem 5.5.1(a) yields

$$\dim(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} = \max_{\substack{n_1+n_2=n\\ n_1,n_2>0}} \dim \overline{X}_{n_1} + \dim \overline{X}_{n_2} = \max_{\substack{n_1+n_2=n\\ n_1,n_2>0}} d(n_1^2+n_2^2)+2 = d((n-1)^2+1)+2.$$

The wanted estimate in (a) is immediate. For the remaining assertion note that $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ is empty when $n=1$. □

Corollary 5.5.3. *Suppose that* $\zeta_p \notin K$ *and that* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ *is nonempty so that* $e := [K':K]$ *divides* n, *for* $K' = K(\zeta_p)$. *Then the Zariski closure of* $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ *has dimension* $\frac{1}{e}[K:\mathbb{Q}_p]n^2+1$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} \overline{R}_{\overline{D}}^{\mathrm{univ}}$ be a generic point of $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$, let $A = \operatorname{Spec} \overline{R}_{\overline{D}}^{\mathrm{univ}}/\mathfrak{p}$ with fraction field $\mathbb{K}$ and let D_A be the corresponding pseudocharacter. Then $\operatorname{Spec} A$ contains a dense subset of dimension 1 points at which D_A is irreducible and special. But then $D_A \otimes_A \mathbb{K}$ must be irreducible and it also must be special, that is, it is invariant under twisting by the mod p cyclotomic character $\chi\colon \mathrm{Gal}(K'/K) \to \mathbb{F}_p^\times$.

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Let $\mathbb{K}' \supset \mathbb{K}$ be a finite extension over which there is an absolutely irreducible representation $\rho\colon G_K \to \mathrm{GL}_n(\mathbb{K}')$ such that $D_A \otimes_A \mathbb{K}' = D_\rho$, and so that $\rho \cong \rho \otimes \chi$. Then by Theorem 2.2.1 we have $e|n$ and after possibly enlarging $\mathbb{K}'$ there exists an absolutely irreducible representation $\rho'\colon G_{K'} \to \mathrm{GL}_{n'}(\mathbb{K}')$ with $n' = n/e$ such that $\mathrm{Ind}_{G_{K'}}^{G_K} \rho' \cong \rho$. Moreover, letting $D' = D_{\rho'}$, the pseudocharacters $(D')^g$, $g \in G_K/G_{K'}$ are pairwise nonisomorphic and $\mathrm{Res}_{G_K}^{G_{K'}} D = \oplus_{g \in G_K/G_{K'}} (D')^g$. In particular, $\mathrm{Res}_{G_K}^{G_{K'}} D$ is multiplicity free. Moreover, $(D_A)_{|G_{K'}}$ is a continuous pseudodeformation of $\overline{D}_{|G_{K'}}$, and so it arises from a map $\overline{R}^{\mathrm{univ}}_{K',\overline{D}} \to A$ in $\widehat{\mathcal{A}r}_{\mathbb{F}}$.

We deduce from Proposition 4.8.6 (and its proof) that after possibly enlarging $\mathbb{K}'$ again there is a continuous pseudocharacter $D'_{A'}\colon G_{K'} \to A'$ on G_K for A' the integral closure of A in $\mathbb{K}'$ and with $D'_{A'} \otimes_{A'} \mathbb{K}' = D_{\rho'}$, and moreover A' lies in $\widehat{\mathcal{A}r}_{\mathbb{F}'}$ for a finite extension $\mathbb{F}'$ of $\mathbb{F}$. Letting $\overline{D}' = D'_{A'} \pmod{\mathfrak{m}_{A'}}$, there is a map $\alpha'\colon \overline{R}^{\mathrm{univ}}_{K',\overline{D}'} \to A'$ in $\widehat{\mathcal{A}r}_{\mathbb{F}'}$ inducing $D'_{A'}$. Moreover, the pseudocharacters $(D'_{A'})^g$, $g \in G_K/G_{K'}$ are pairwise distinct.

Let $\mathfrak{q}$ be a generic point of $\overline{R}^{\mathrm{univ}}_{K',\overline{D}'}$ that lies in the kernel of α', let $B = \overline{R}^{\mathrm{univ}}_{K',\overline{D}'}/\mathfrak{q}$ with quotient field L, and let D'_B be the associated pseudocharacter. Then $D_B \otimes_B L$ is irreducible and the pseudocharacters $(D'_B)^g$, $g \in G_K/G_{K'}$ are pairwise distinct. Then by Theorem 4.6.7, $D_B = \mathrm{Ind}_{G_{K'}}^{G_K} D'_B\colon B[G] \to B$ is a continuous pseudodeformation of $\mathrm{Ind}_{G_{K'}}^{G_K} \overline{D}' = \overline{D} \otimes_{\mathbb{F}} \mathbb{F}'$ such that $D_B \otimes_B \mathrm{Quot}(B)$ is irreducible and special (invariant by the twist with χ). In particular, D_B arises from a homomorphism $\alpha\colon \overline{R}^{\mathrm{univ}}_{K,\overline{D}\otimes_{\mathbb{F}}\mathbb{F}'} \to B$ and the point $\mathrm{Ker}\,\alpha$ of $\overline{X}^{\mathrm{univ}}_{\mathbb{F}',\overline{D}\otimes_{\mathbb{F}}\mathbb{F}'}$ must be special. By our construction we have a commutative diagram

$$\begin{tikzcd} & B \arrow[dr] & \\ \overline{R}^{\mathrm{univ}}_{K,\overline{D}\otimes_{\mathbb{F}}\mathbb{F}'} \arrow[ur] \arrow[dr] & & A', \\ & A\mathbb{F}' \arrow[uu, dashed] \arrow[ur] & \end{tikzcd}$$

where we write $A\mathbb{F}'$ for the $\mathbb{F}'$ subalgebra of A' generated by $\mathbb{F}'$ and A, and initially without the dashed arrow. We deduce that $\mathrm{Ker}\,\alpha$ is also the kernel of the map to $A\mathbb{F}'$, and so the dashed arrow exists and is injective (by the definition of A). But then restricting the corresponding pseudocharacters to $G_{K'}$, one deduces that the maps from $\overline{R}^{\mathrm{univ}}_{K,\overline{D}'}$ to B and to a finite extension of $A\mathbb{F}$ have the same kernel, and so B and A', and hence A must have the same dimension. But $\dim B = \frac{1}{e}[K:\mathbb{Q}_p]n^2 + 1$ by Theorem 5.5.1, and this concludes the proof. $\square$

Lemma 5.5.4. *Let κ be a local or a finite field. Suppose $p > 2$. Let $D_i\colon G_{\mathbb{Q}_p} \to \kappa$, $i = 1, 2$, be continuous pseudocharacters on $G_{\mathbb{Q}_p}$ of dimension 1, and let $D = D_1 \oplus D_2$. Then*

(a) *If $D_1 \neq D_2(m)$ for $m \in \{0, \pm 1\}$, then*
  (1) *there exists a unique nontrivial extension $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\kappa)$ of D_2 by D_1,*
  (2) *the natural map $\overline{R}^{\mathrm{univ}}_D \to \overline{R}^{\mathrm{univ}}_\rho$ is an isomorphism,*
  (3) *and both rings are formally smooth over κ of dimension 5.*

(b) *If $D_1 = D_2(m)$ for some $m \in \{\pm 1\}$, then $\overline{R}^{\mathrm{univ}}_D$ is not regular.*

(c) *If $D_1 = D_2$, then $\overline{R}^{\mathrm{univ}}_D$ is regular.*

Proof. The idea for (a) stems from the proof of [Che11, Corollary 4.4] and goes back to Kisin. We regard the D_i exchangeably as pseudocharacters or as representations because they are of dimension 1. Lemma 5.1.7(a) guarantees the existence of ρ as in (1). Since $D_1 \notin \{D_2, D_2(\pm 1)\}$, Theorem 3.4.1 yields $\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(D_i, D_j) = 1$ for $i \neq j$, and this implies the uniqueness of ρ up to isomorphism. Note that once (2) is proved, Part (3) follows from Lemma 5.1.7(c). To see (2), let $X_\rho := \mathrm{Spec}\, \overline{R}^{\mathrm{univ}}_\rho$, $X_D := \overline{R}^{\mathrm{univ}}_D$ for $D := D_1 \oplus D_2$, write φ for the map in Part (1), and denote by

$$\mathrm{d}\varphi\colon \mathbf{t}_{X_\rho,\rho} \to \mathbf{t}_{X_D,D}$$

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the induced map on tangent spaces. By the formula in Lemma 5.1.7(c), the kernel of $\mathrm{d}\varphi$ is zero. Because $p > 2$, we also have $\dim \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(D_i, D_i) = 2$ for $i = 1, 2$. Consider now the following exact sequence from [Bel12, Theorem 2] with $\rho_i = D_i$

$$0 \longrightarrow \bigoplus_{i=1,2} \mathrm{Ext}^1_{G_K}(\rho_i, \rho_i) \longrightarrow \dim \mathbf{t}_{X_D,D} \otimes_{\kappa(x)} L \tag{36}$$

$$\longrightarrow \mathrm{Ext}^1_{G_K}(\rho_1, \rho_2) \otimes \mathrm{Ext}^1_{G_K}(\rho_2, \rho_1) \xrightarrow{h} \bigoplus_{i=1,2} \mathrm{Ext}^2_{G_K}(\rho_i, \rho_i).$$

It implies $\dim \mathbf{t}_{X_D,D} \leq 5$. Hence, $\mathrm{d}\varphi$ must be an isomorphism and $\dim \mathbf{t}_{X_D,D} = 5$. This implies that φ must be surjective, and hence an isomorphism since the target is formally smooth over κ.

To prove (b), note that we have $\mathrm{Ext}^2(\rho_i, \rho_i) = 0$ and $\mathrm{Ext}^1(\rho_i, \rho_i)$ is of dimension 1, while $\mathrm{Ext}^1(\rho_i, \rho_i(m))$ is 2 for $m = 1$ and 1 for $m = -1$. Hence, diagram (36) yields $\dim_\kappa \mathbf{t}_{X_D,D} = 6$. However, $\dim X_D = 5$ by Theorem 5.5.1, and hence R_D^{univ} is not regular.

Finally, we show (c). Because $p > 2$, we may apply [Che11, Théorème 3.1] in exactly the same way, as done in [Che11, Lemme 2.5]: Using that the mod p reduction of $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ is isomorphic to $(\mathbb{Z}/p)^2$, one has $\dim_\kappa \mathrm{Hom}(G_{\mathbb{Q}_p}, \kappa) = 2$, $\dim_\kappa \mathrm{Sym}(G_{\mathbb{Q}_p}, \kappa) = 3$ and $\dim_\kappa \mathrm{Alt}(G_{\mathbb{Q}_p}, \kappa) = 0$, and hence $\dim_\kappa \mathbf{t}_{X_D,D} = 5$. We now conclude using $\dim X_D = 5$ by Theorem 5.5.1. □

We now characterize the singular locus when $\zeta_p \notin K$.

Theorem 5.5.5 (Theorem 2, [Che11, Théorème 2.3]). *If $\zeta_p \notin K$, then the following hold:*

(a) *The closure of $X_1 := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{spcl}}$ in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$ lies in $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$.*
(b) *If $n > 2$ or $[K : \mathbb{Q}_p] > 1$, then $X_2 := (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}} \subset (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$.*
(c) *If $n = 2$, $K = \mathbb{Q}_p$, and $x \in X_2$ corresponds to a pair (D_1, D_2) of one-dimensional pseudocharacters, then $x \in (\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$ if and only if $D_2 = D_1(m)$ for $m \in \{\pm 1\}$.*

Proof. We know from Proposition 4.7.11 that $\overline{R}_{\overline{D}}^{\mathrm{univ}}$ is a complete Noetherian local ring so that by Lemma A.1.1(a), $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{sing}}$ is closed in $\overline{X}_{\overline{D}}^{\mathrm{univ}}$. Observe that if $X_i \neq \varnothing$, then its Zariski closure $\overline{X}_i$ has dimension at least 2: for X_2, this is clear from Lemma 5.5.2(a), for X_1 from Corollary 5.5.3. Hence, Proposition A.1.11 shows that the points of X_i of dimension 1 are dense in X_i.

To prove (a), let $x \in X_1$ be of dimension 1. A standard computation of tangent spaces as in the proof of Lemma 5.1.7(c) shows $\dim H^1(G_K, \mathrm{ad}_{\rho_x}) = dn^2 + 2$, while $\dim R_{\rho_x}^{\mathrm{univ}} = dn^2 + 1$. It follows from Lemma 3.3.5 that x is not regular on $\overline{X}_{\overline{D}}^{\mathrm{univ}}$.

For the proof of (b), we assume without loss of generality that $\overline{D}$ is split. Then $(\overline{X}_{\overline{D}}^{\mathrm{univ}})^{\mathrm{red}}$ is the image of the maps $\iota_{\overline{D}_1,\overline{D}_2}$ from (32) for all $\overline{D}_1$, $\overline{D}_2$ such that $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$. Fix such a pair, and let n_i be the dimension of $\overline{D}_i$. Because of Theorem 5.5.1 it suffices to consider pairs $x = (x_1, x_2)$ with $x_i \in (\overline{X}_{\overline{D}_i}^{\mathrm{univ}})^{\mathrm{n\text{-}spcl}}$; and we may also assume that D_{x_1} is distinct from the finitely many $D_{x_2}(m)$, $m \in \{1, \dots, p-1\}$. We compute the tangent space dimension of $\overline{R}_{D_x}^{\mathrm{univ}}$ this time, using (36) from [Bel12, Theorem A] which also holds for ρ_{x_i} in place of ρ_i. We conclude as in the proof of [Che11, Lemme 2.4]: $\dim_L H^1(G_K, \mathrm{ad}_{\rho_{x_i}}) \geq 1 + dn_i^2$, $\dim_L \mathrm{Ext}^1_{G_K}(\rho_{x_i}, \rho_{x_{3-i}}) = dn_1 n_2$, and the second extension groups vanish since the D_{x_i} satisfy $D_{x_i} \neq D_{x_i}(1)$. Hence,

$$\mathbf{t}_{\mathrm{Spec}\, \overline{R}_{D_x}^{\mathrm{univ}}} = d(n_1^2 + n_2^2) + 2 + d^2 n_1^2 n_2^2 \geq dn^2 + 1 + (dn_1 n_2 - 1)^2.$$

This dimension is strictly larger than $dn^2 + 1$, unless $dn_1 n_2 = 1$, that is, $n_1 = n_2 = 1$ and $K = \mathbb{Q}_p$. However, $\dim \overline{R}_{D_x}^{\mathrm{univ}} = dn^2 + 1$ by Lemma 3.3.5 and Theorem 5.5.1, and it follows that x cannot be regular, proving (b).

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Concerning (c), note that if $x = (D_1, D_2)$ is any point of dimension at most 1, then the assertion follows from Lemma 5.5.4. Since such points are Zariski dense in the closure of any point of dimension at least 2, the assertion in (c) follows in general. $\square$

Remark 5.5.6. Note that Theorem 5.5.5 reproves a result of Paškūnas, namely [Paš13, Proposition B.17]: suppose that $n = 2$, $p > 2$, $K = \mathbb{Q}_p$, and $\overline{D}\colon G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}$ is a direct sum $\overline{D}_1 \oplus \overline{D}_2$ of one-dimensional characters $\overline{D}_i$ sucht that $\overline{D}_2 \neq \overline{D}_1(m)$ for $m = 0, \pm 1$. Then $R^{\mathrm{univ}}_{\overline{D}} \cong \mathbb{F}_q[[X_1, \ldots, X_5]]$.

Theorem 5.5.7 (Theorem 3). *The ring $R^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$ satisfies Serre's condition (R_2), unless $n = 2$, $K = \mathbb{Q}_2$ and $\overline{D}$ is trivial.*

Proof. By Theorem 5.5.1 and Lemma 5.1.6, the subset $(X^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\text{n-spcl}}$ is regular, open and Zariski dense in $X^{\mathrm{univ}}_{\overline{D},\mathrm{red}}$. Thus, Lemma 5.5.2 implies the theorem unless $n = 2$ and $K = \mathbb{Q}_p$. Also, if $\overline{D}$ is irreducible, then so is any lift, and so $(X^{\mathrm{univ}}_{\overline{D},\mathrm{red}})^{\mathrm{red}}$ is empty. Now, again we conclude by Lemma 5.5.2. Suppose from now on that $K = \mathbb{Q}_p$ and $\overline{D} = \overline{D}_1 \oplus \overline{D}_2$ for one-dimensional pseudocharacters $\overline{D}_i\colon G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}$ on $G_{\mathbb{Q}_p}$, and suppose now also $p > 2$ which was excluded in this case.

The locus of $x \in X_2 := (X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red}}$ corresponding to a pair (D_1, D_2) of one-dimensional pseudocharacters such that $D_2 = D_1(m)$ for $m \in \{\pm 1\}$, can be realized as the image of $X^{\mathrm{univ}}_{\overline{D}_1}$. Hence, it has dimension at most 2 because of Corollary 3.4.3. Outside this, locus points are smooth by Theorem 5.5.5 (and the density of points of dimension 1). It follows that $(X^{\mathrm{univ}}_{\overline{D}})^{\mathrm{red,sing}}$ has dimension at most 2 which is less than $5 - 2 = 3$ so that then $X^{\mathrm{univ}}_{\overline{D}}$ satisfies (R_2), also. $\square$

A. Appendix. Auxiliary results on rings, algebras and representations

In this appendix, we collect some results used in various parts of this work. We also prove some minor facts that could not be found directly in the literature.

A.1. *Commutative algebra*

Complete local rings, integral extensions and regularity

A domain B with quotient field $\mathbb{K}$ is said to satisfy N-2 if for any finite field extension L of $\mathbb{K}$, the integral closure of B in L is a finite over B. A ring A is called a *Nagata ring* if A is Noetherian and for every prime ideal $\mathfrak{p}$ of A the ring $A/\mathfrak{p}$ satisfies N-2; see [Sta18, §032E].

Lemma A.1.1. *If A is complete Noetherian local ring, then the following hold:*

(a) *A is a Nagata ring, and hence the set of regular points of* Spec A *is open in* Spec A.
(b) *If A is a domain with fraction field $\mathbb{K}$ and perfect residue field, then $[\mathbb{K} : \mathbb{K}^p] < \infty$.*

Proof. Part (a) is [Sta18, §032W] combined with [Gro65, Théorème (6.12.7)]. Part (b) is proved in [Hoc07, Proposition (d), (g)]. $\square$

Lemma A.1.2 [Mat80, 13.C, Theorem 20]. *If B is a domain and if $B' \subset B$ is a subring such that B if finite over B', then* $\dim B = \dim B'$.

Recall that for a prime $\mathfrak{p}$ of A, the *height of* $\mathfrak{p}$ is defined as $\operatorname{ht}\mathfrak{p} = \dim R_{\mathfrak{p}}$.

Definition A.1.3. A commutative ring A is said to *satisfy (Serre's) condition* (R_i), if A is *regular in codimension at most i*, that is, if the local ring $A_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p}$ of height $\leq i$.

Density of points of dimension one

The next series of results stems from [Gro66, §10.1–10.5], except in one case where we give a direct reference. Let X be a topological space. It is called *Noetherian* if every descending chain of closed subsets becomes stationary. It is called *irreducible* if it is not the union of two proper closed subsets. If X is Noetherian, a subset is called *constructible* if it is a finite union of *locally closed* subsets of X, that is, of subsets that are the intersection of an open and a closed subset of X. The closure of a subset $Z \subset X$ is denoted by $\overline{Z}$. For a subset Z of X, its dimension $\dim Z \in \mathbb{N} \cup \{\infty\}$ is the maximal length n of a chain $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n \subset \overline{Z}$ of irreducible closed subsets Y_i in X.

Definition A.1.4. A subset X_0 of X is called *very dense in X* if every nonempty locally closed subset $Z \subset X$ satisfies $Z \cap X_0 \neq \emptyset$.

If X_0 is very dense in X, it is clearly dense in X.

Lemma A.1.5. *If X_0 is very dense in X, then $X_0 \cap Z$ is very dense in Z and dense in $\overline{Z}$ for any locally closed set Z in X.*

Proposition A.1.6. *For a subset X_0 of X, the following conditions are equivalent:*

(a) *X_0 is very dense in X;*
(b) *Under $X' \mapsto X_0 \cap X'$, the open subsets in X are in bijection to those in X_0.*
(c) *Under $X' \mapsto X_0 \cap X'$, the closed subsets in X are in bijection to those in X_0.*

In the following, we set $X_{\leq 1} := \{x \in X : \dim x \leq 1\}$. Since the union of finitely many irreducible subsets of dimension at most i has dimension at most i, we find:

Lemma A.1.7. *If $U \subset X$ satisfies $\dim U \geq 2$, then no finite subset of $U_{\leq 1}$ is dense in U.*

An important source for very dense subset of schemes comes from the following result:

Lemma A.1.8 [Mat80, (33.F) Lemma 5]. *Let $X = \operatorname{Spec} A$ for a Noetherian ring A. Then the set $X_{\leq 1}$ is very dense in X.*

From Lemma A.1.8 and Lemma A.1.5, one deduces:

Corollary A.1.9. *Let $X = \operatorname{Spec} A$ for a Noetherian ring A, and let $Z \subset X$ be constructible. Then $X_{\leq 1} \cap Z$ is very dense in Z and dense in $\overline{Z}$.*

Definition A.1.10. The space X is called *Jacobson* if $\{x \in X : \dim x = 0\}$ is very dense in X.

A scheme is called *Jacobson* if the underlying topological space is Jacobson; a ring A is called *Jacobson* if the scheme $\operatorname{Spec} A$ is Jacobson. For us, the following result is of importance:

Proposition A.1.11. *For a Noetherian local ring with A maximal ideal $\mathfrak{m}_A$, the scheme $\operatorname{Spec} A \backslash \{\mathfrak{m}_A\}$ is Jacobson.*

Besides our reference to [Gro66], Proposition A.1.11 can also be found in [Sta18, 02IM]

Étale morphisms and étale neighborhoods

We recall some terminology and a result on étale morphisms to be used in Section 5.

Definition A.1.12 [Sta18, §00U0 and Definition 02GI].

(a) A ring map $A \to B$ is called *étale* if it is a smooth ring map of relative dimension zero.
(b) A morphism $f : X \to Y$ of schemes is called *étale at $x \in X$* if there is an affine open neighborhood $\operatorname{Spec}(B) = U \subset X$ of x and an affine open $\operatorname{Spec}(A) = V \subset Y$ with $f(U) \subset V$ so that the corresponding ring map $A \to B$ is étale. We say that f is *étale* if it is étale at each point $x \in X$.

Definition A.1.13 [Sta18, Definition 03PO]. Let X be a scheme.

(a) A *geometric point of X* is a morphism $\overline{x}$: $\operatorname{Spec} k \to X$, where k is an algebraically closed field.
(b) One says that $\overline{x}$ *is lies over* $x \in X$ to indicate that x is the image of $\overline{x}$.

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(c) An *étale neighborhood* $(U, \overline{u}, \varphi)$ of a geometric point $\overline{x} \in X$ is a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "\varphi"] \\ \operatorname{Spec} k \arrow[ur, "\bar{u}"] \arrow[r, "\overline{x}"] & X, \end{tikzcd}$$

where φ is an étale morphism of schemes and $\overline{u}$ is a geometric point of U.

Lemma A.1.14. *Let $\varphi\colon U \to X$ be an étale morphism between schemes U and X. Let u be a point of U and denote by x its image $\varphi(u)$. Consider the local homomorphism $\varphi_u\colon \mathcal{O}_{X,x} \to \mathcal{O}_{U,u}$ induced from φ. Then*

(a) *The completion $\widehat{\varphi_u}\colon \widehat{\mathcal{O}}_{X,x} \to \widehat{\mathcal{O}}_{U,u}$ of φ_u is finite étale; its degree is equal to $[\kappa(u) : \kappa(x)]$.*
(b) *The ring $\widehat{\mathcal{O}}_{X,x}$ is regular if and only if $\widehat{\mathcal{O}}_{U,u}$ is regular, and in this case both have the same dimension.*

Proof. Part (a) is [Sta18, Lemma 039M] and the remark following it. For Part (b), note that by étaleness the tangent spaces at the closed point have the same dimension, and by finite étaleness the ring $\widehat{\mathcal{O}}_{U,u}$ is free of finite rank over $\widehat{\mathcal{O}}_{X,x}$ and hence they have the same dimension. From this, (b) follows easily. □

A.2. Finite-dimensional algebras and modules

Let $\mathbb{K}$ be a field. We gather some results, mostly from [CR62], on not necessarily commutative $\mathbb{K}$-algebras S and modules M over them, assuming that either the algebra or the module have finite $\mathbb{K}$-dimension. Our intended applications are to $S = \mathbb{K}[G]$ for a possibly infinite group G, or to G-modules of finite $\mathbb{K}$-dimension; note that if G is profinite, $\mathbb{K}$ is a topological field and M is a $\mathbb{K}[G]$-module of finite $\mathbb{K}$-dimension with a continuous G-action, then all G-subquotients of M carry a continuous action. So we need not worry about continuity in the following.

Let first S be a $\mathbb{K}$-algebra of finite $\mathbb{K}$-dimension. In this case, the sum of all nilpotent left ideals of S is a two-sided ideal of S, the *radical* of S and denoted $\mathrm{Rad}(S)$, see [CR62, §24]. It is the maximal nilpotent two-sided ideal of S. The radical is zero if and only if S is semisimple; in this case, S is the product of simple $\mathbb{K}$-algebras (of finite $\mathbb{K}$-dimension). If $\mathbb{K}'$ is any field extension of $\mathbb{K}$, then

$$\mathrm{Rad}(S) \otimes_{\mathbb{K}} \mathbb{K}' \subset \mathrm{Rad}(S \otimes_{\mathbb{K}} \mathbb{K}'). \tag{37}$$

Definition A.2.1. We call a $\mathbb{K}$-algebra S of finite $\mathbb{K}$–dimension *absolutely semisimple* if $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is semisimple.

Remark A.2.2. Suppose that S is absolutely semisimple. Then by the containment (37) it is semisimple. By the theorem of Artin–Wedderburn, the algebra $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is a product of matrix algebras over $\mathbb{K}^{\mathrm{alg}}$. From this one deduces, by repeated application of the inclusion (37), that $S \otimes_{\mathbb{K}} \mathbb{K}'$ is semisimple for any field extension $\mathbb{K}'$ of $\mathbb{K}$. Suppose now that S is only semisimple. By considering simple factors D_i of S, one shows that S is absolutely semisimple if and only if the center of each D_i is separable over $\mathbb{K}$.

Lemma A.2.3. *Let S a $\mathbb{K}$-algebra of finite $\mathbb{K}$-dimension, and write S' for $S \otimes_{\mathbb{K}} \mathbb{K}'$ and any field extension $\mathbb{K}'$ of $\mathbb{K}$.*

(a) *There exists a finite extension $\mathbb{K}'$ of $\mathbb{K}$ such that $S'/\mathrm{Rad}(S')$ is absolutely semisimple.*
(b) *If $S/\mathrm{Rad}(S)$ is absolutely semisimple over $\mathbb{K}$, then there exists an extension $\mathbb{K}'$ of $\mathbb{K}$ with $[\mathbb{K}' : \mathbb{K}] \le (\dim_{\mathbb{K}} S)!$ such that $S'/\mathrm{Rad}(S')$ is a product of matrix algebras over $\mathbb{K}'$.*
(c) *If $\mathbb{K}$ is finite, and if we write $S/\mathrm{Rad}(S) \cong \prod_i \mathrm{Mat}_{d_i \times d_i}(\mathbb{K}_i)$ for $d_i \ge 1$ and $\mathbb{K}_i$ finite over $\mathbb{K}$, then we may find $\mathbb{K}'$ as in (b) so that $[\mathbb{K}' : \mathbb{K}]$ divides $\prod_i [\mathbb{K}_i : \mathbb{K}]$.*

Proof. For (a) note first that for $S^{\mathrm{alg}} := S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ the ring $S^{\mathrm{alg}}/\mathrm{Rad}(S^{\mathrm{alg}})$ is semisimple and trivially absolutely semisimple. Let $\mathbb{K}'$ be a finite extension of $\mathbb{K}$ over which $S' := S \otimes_{\mathbb{K}} \mathbb{K}'$ contains a sub-$\mathbb{K}'$

vector space I with $I \otimes_{\mathbb{K}'} \mathbb{K}^{\mathrm{alg}} = \mathrm{Rad}(S^{\mathrm{alg}})$. Considering I inside $\mathrm{Rad}(S^{\mathrm{alg}})$, it follows that I is a nilpotent ideal of S' so that $I \subset \mathrm{Rad}(S')$. But then using the inclusion (37) and the faithful flatness of $\mathbb{K}' \to \mathbb{K}^{\mathrm{alg}}$, it is straightforward to see that $I = \mathrm{Rad}(S')$ and that S'/I is absolutely simple.

To prove (b), note first that we may replace S by $S/\mathrm{Rad}(S)$, again by the inclusion (37) so that we may assume that S is absolutely semisimple. Write S as a product of division algebras D_i, for i in a finite index set I, and write $\mathbb{K}_i$ for the center of D_i and let $d_i \in \mathbb{N}$ be such that $d_i^2 = \dim_{\mathbb{K}_i} D_i$. We consider all finite field extensions of $\mathbb{K}$ as subfields of a fixed algebraic closure $\mathbb{K}^{\mathrm{alg}}$ of $\mathbb{K}$. Let $\mathbb{K}' \subset \mathbb{K}^{\mathrm{alg}}$ be the join of the normal hull of all $\mathbb{K}_i$. By Remark A.2.2, $\mathbb{K}'$ is separable over $\mathbb{K}$ and for each i we have $\mathbb{K}_i \otimes_{\mathbb{K}} \mathbb{K}' \cong (\mathbb{K}')^{m_i}$ for $m_i = [\mathbb{K}_i : \mathbb{K}]$. Note also that $[\mathbb{K}' : \mathbb{K}] \le \prod_{i\in I} m_i!$. Let $\mathbb{E}_i \subset D_i$ be a maximal subfield over $\mathbb{K}_i$ so that $D_i \otimes_{\mathbb{K}_i} \mathbb{E}_i \cong \mathrm{Mat}_{d_i \times d_i}(\mathbb{E}_i)$. Let $\mathbb{E}' \supset \mathbb{K}^{\mathrm{alg}}$ be the join of $\mathbb{K}'$ and the fields $\mathbb{E}_i$, $i \in I$. Then

$$S \otimes_{\mathbb{K}} \mathbb{E}' \cong \prod_{i\in I} (D_i \otimes_{\mathbb{K}_i} (\mathbb{K}_i \otimes_{\mathbb{K}} \mathbb{K}') \otimes_{\mathbb{K}'} \mathbb{E}') \cong \prod_{i\in I} (D_i \otimes_{\mathbb{K}_i} \mathbb{E}')^{m_i} \overset{\mathbb{E}' \supset \mathbb{E}_i}{\cong} \prod_{i \in I} (\mathrm{Mat}_{d_i\times d_i}(\mathbb{E}'))^{m_i}. \tag{38}$$

Hence, $\mathbb{E}'$ is a field as in (a). Moreover, $[\mathbb{E}' : \mathbb{K}] \le \prod_{i\in I}(d_i \cdot m_i!) \le \prod_{i\in I}(d_i \cdot [\mathbb{K}_i : \mathbb{K}])!$. Since $\sum_{i\in I}(d_i[\mathbb{K}_i : \mathbb{K}]) \le \sum_{i\in I} d_i^2 \cdot m_i = n$, using that multinomials are integers, we deduce $[\mathbb{E}' : \mathbb{K}] \le n!$, and this proves (b).

To see (c) note that each $\mathbb{K}_i$ is normal over $\mathbb{K}$ and for each degree there is a unique extension of $\mathbb{K}$ of that degree in a fixed choice $\mathbb{K}^{\mathrm{alg}}$. Hence, in the proof of (b) we find $[\mathbb{K}' : \mathbb{K}] \le \mathrm{lcm}_{i\in I}[\mathbb{K}_i : \mathbb{K}]$. Moreover, over $\mathbb{K}_i$ the ring D_i is already split, and so we can take $\mathbb{E}' = \mathbb{K}'$. The assertion now is clear. □

Remark A.2.4.

- **(a)** Note that the hypothesis in Lemma A.2.3(b) holds whenever $\mathbb{K}$ is perfect.
- **(b)** A version of Lemma A.2.3(a) only under algebraicity hypotheses for S over $\mathbb{K}$ can be found in [Che14, Lemma 2.14].
- **(c)** It is possible to give effective bounds in Lemma A.2.3(b) also without any separability hypotheses. But the proof is longer and we do not need the result.

Let now S be any $\mathbb{K}$-algebra, not necessarily of finite $\mathbb{K}$-dimension. Let M be an S-module of finite $\mathbb{K}$-dimension. If M is semisimple, the representation $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ need in general not be semisimple over $S^{\mathrm{alg}} := S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$.[8]

Definition A.2.5. We call M *absolutely semisimple*, if $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is semisimple as an S^{alg}-module.

We call M *absolutely completely reducible* if it is semisimple and all its irreducible summands are absolutely irreducible.

Remark A.2.6. If M is absolutely completely reducible, it is clearly absolutely semisimple. If M is absolutely semisimple, it is absolutely completely reducible if and only if for each irreducible summand N of M the natural map $\mathbb{K} \to \mathrm{End}_S(N)$ is an isomorphism, see [CR62, 29.13]; the latter condition is equivalent to $\mathrm{End}_S(M)$ being a product of matrix algebras over $\mathbb{K}$.

For the following note, that if N is a second S-modulo of finite $\mathbb{K}$-dimension and $\mathbb{K}'$ is any field extension of $\mathbb{K}$, then by [CR62, 29.2] one has

$$\mathrm{Hom}_S(M, N) \otimes_{\mathbb{K}} \mathbb{K}' \cong \mathrm{Hom}_{S\otimes_{\mathbb{K}}\mathbb{K}'}(M \otimes_{\mathbb{K}} \mathbb{K}', N \otimes_{\mathbb{K}} \mathbb{K}'). \tag{39}$$

Lemma A.2.7. *Suppose M is absolutely semisimple. Then the following hold:*

- **(a)** *The $\mathbb{K}$-algebra $\mathrm{End}_S(M)$ is absolutely semisimple.*
- **(b)** *If $\mathbb{K}' \supset \mathbb{K}$ is an extension such that $\mathrm{End}_S(M) \otimes_{\mathbb{K}} \mathbb{K}'$ is a product of matrix algebras, then $M \otimes_{\mathbb{K}} \mathbb{K}'$ is absolutely completely reducible.*

[8]If S is a purely inseparable finite field extension of $\mathbb{K}$ and $M = S$, then $S \otimes_{\mathbb{K}} S$ is not semisimple.

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Proof. To prove (a) it suffices to assume that M is irreducible. Then $D := \mathrm{End}_S(M)$ is a skew field of finite dimension over $\mathbb{K}$. By the isomorphism (39), we have

$$D \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}} \cong \mathrm{End}_{S\otimes_{\mathbb{K}}\mathbb{K}^{\mathrm{alg}}}(M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}).$$

By hypothesis, $M \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is semisimple over $S \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$. By Remark A.2.6, $D \otimes_{\mathbb{K}} \mathbb{K}^{\mathrm{alg}}$ is then a product of matrix algebras over $\mathbb{K}^{\mathrm{alg}}$. This proves (a). Part (b) is immediate from the isomorphism (39) since it implies $\mathbb{K}' \cong \mathrm{End}_{S\otimes_{\mathbb{K}}\mathbb{K}'}(N)$ for every irreducible summand of $M \otimes_{\mathbb{K}} \mathbb{K}'$. □

Remark A.2.7. For $\mathbb{K}'$ as in Lemma A.2.7(b), one can bound $[\mathbb{K}' : \mathbb{K}]$ by $((\dim_{\mathbb{K}} M)^2)!$ using Lemma A.2.3(b).

A.3. Absolutely irreducible mod p representations of the absolute Galois group of a p-adic field

This subsection gives the proof of the classification of irreducible finite-dimensional representations of G_K for a p-adic field K over a finite field of characteristic p, and some complements.

We begin with some preparations and reminders: Recall the classification of tame characters of the inertia group I_K of G_K from [Ser72]: Let m denote some natural number. Let k^{alg} be the residue field of K^{alg} and set $q := |k|$. Let in the following $\sigma \in G_K$ be any element that maps to Frobenius in G_k. Let $K^{\mathrm{nr}} \subset K^{\mathrm{t}} \subset K^{\mathrm{alg}}$ denote the maximal unramified and maximal tamely ramified extensions of K, respectively. Denote by $K_m \subset K^{\mathrm{nr}}$ the unique extension of K of degree m and by $k_m \subset k^{\mathrm{alg}}$ its residue field. If ϖ is a fixed choice of uniformizer of K and $K_m^t = K^{\mathrm{nr}}(\sqrt[q^m-1]{\varpi})$, then $K^{\mathrm{t}} = \varinjlim_{m\in\mathbb{N}_{\geq 1}} K_m^t$. The characters

$$\omega_m \colon I_{\mathrm{t}} := \mathrm{Gal}(K^{\mathrm{t}}/K^{\mathrm{nr}}) \to \mathrm{Gal}(K_m^t/K^{\mathrm{nr}}) \xrightarrow{\sim} \mu_{q^m-1}(K^{\mathrm{nr}}) \cong \mu_{q^m-1}(k^{\mathrm{alg}}) = k_m^\times, \sigma \mapsto \frac{\sigma(\sqrt[q^m-1]{\varpi})}{\sqrt[q^m-1]{\varpi}},$$

form an inverse system, $I_{\mathrm{t}} \cong \varprojlim\{k_m^\times : m \in \mathbb{N}\}$ is pro-cyclic and $I_{\mathrm{t}}^p = I_{\mathrm{t}}$; see [Ser72, Propositions 1 and 2].

A continuous character $\omega \colon I_{\mathrm{t}} \to (k^{\mathrm{alg}})^\times$ is called *of level m (with respect to k)* if m is the smallest integer such that ω factors as $\omega = \varphi \circ \omega_m$ for some homomorphism $\varphi \colon k_m^\times \to (\mathbb{F}_p^{\mathrm{alg}})^\times$; since I_{t} is procyclic this is equivalent to ω having order a divisor of $q^m - 1$ with m minimal; in particular, the number of such characters is finite. For any $m \geq 1$, let $\mathcal{P}_m := \mathrm{Hom}_k(k_m, \mathbb{F}_p^{\mathrm{alg}})$, and set $\omega_{m,\tau} := \tau \circ \omega_m$ for $\tau \in \mathcal{P}_m$. For any $\tau \in \mathcal{P}_m$ we have $\mathcal{P}_m = \{\tau^{q^i} \mid i = 0, \ldots, m-1\}$. Moreover, $\sigma \in G_K$ as fixed above satisfies $\sigma\tau\sigma^{-1} = \tau^q$.

If ω is of level dividing m, it can be written as $\omega = \omega_{m,\tau}^r$ for any $\tau \in \mathcal{P}_m$ and some $r \in \{1, \ldots, q^m - 2\}$ (that depends on τ). Call $r \in \{1, 2, \ldots, q^m - 2\}$ *primitive for m (and q)* if there is no proper divisor d of m such that r is a multiple of $(q^m - 1)/(q^d - 1)$; equivalently, r is primitive, if its base q expansion $r = [e_{m-1}e_{n-2} \ldots e_1 e_0]_q$, with digits $e_j \in \{0, \ldots, q-1\}$, is preserved under no cyclic digit permutation but the identity. Then the level m is minimal for $\omega = \omega_{m,\tau}^r$ if and only if r is primitive for m. In the latter case, the orbit of ω under conjugation by σ has exact length m.

To extend $\omega_{m,\tau}$ to G_{K_m}, recall that the local Artin map is an isomorphism $\widehat{K}_m^\times \xrightarrow{\sim} G_{K_m}^{\mathrm{ab}}$ that maps $\mathcal{O}_{K_m}^\times$ to the inertia subgroup of $G_{K_m}^{\mathrm{ab}}$; the latter surjects onto $I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1}$. The choice of ϖ gives an isomorphism $\widehat{K}_m^\times \cong \widehat{\mathbb{Z}} \times \mathcal{O}_{K_m}^\times$; it induces a homomorphism $\mathrm{pr}_2 \colon G_{K_m} \to I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1}$. We define

$$\hat{\omega}_{m,\tau} \colon G_{K_m} \xrightarrow{\mathrm{pr}_2} I_{\mathrm{t}}/(I_{\mathrm{t}})^{q^m-1} \xrightarrow{\omega_m} k_m^\times \xrightarrow{\tau} (k^{\mathrm{alg}})^\times.$$

Finally, for $\lambda \in (k^{\mathrm{alg}})^\times$ and a finite extension field $K' \supset K$, we write $\overline{\mu}_{K',\lambda} \colon G_{K'} \to (k^{\mathrm{alg}})^\times$ for the unramified character of $G_{K'}$ that sends a Frobenius automorphism to $\lambda^{-1} \in k^{\mathrm{alg}}$.

The following is the main result of this subsection.

Lemma A.3.1 (Berger, Muller). *Let $\overline{\rho} \colon G_K \to \mathrm{GL}_n(k^{\mathrm{alg}})$ be an n-dimensional irreducible continuous representation. Let $\mathbb{F} \subset k^{\mathrm{alg}}$ be a finite field that contains k_n. Then the following hold:*

(a) *There exists* $\lambda \in (k^{\mathrm{alg}})^\times$, $\tau \in \mathcal{P}_n$ *and a primitive number* $r \in \{1, 2, \ldots, q^n - 2\}$ *such that*

$$\overline{\rho} \cong \overline{\mu}_{K,\lambda} \otimes \mathrm{Ind}_{G_{K_n}}^{G_K} \widehat{\omega_{n,\tau}}^r.$$

(b) $\bar{\rho}$ *can be defined over* $\mathbb{F}$ *if and only if* $\lambda^n \in \mathbb{F}$.

In particular, given n there are only finitely many isomorphism classes of absolutely irreducible representations $G_K \to \mathrm{GL}_n(\mathbb{F})$.

Proof. The proof of (a) is essentially that of [Ber10, Corollary 2.1.5] for $K = \mathbb{Q}_p$ as extended in [Mul13, Proposition 2.1.1] to any K. We give a complete proof of (a), since it also serves to prove (b). Note that the last assertion is immediate from (a) and (b).

To prove (a), let $\overline{\rho}\colon G_K \to \mathrm{GL}_n(\mathbb{F}^{\mathrm{alg}})$ be irreducible. Then the wild ramification subgroup P_K of G_K acts trivially via $\bar{\rho}$: the group P_K is normal in G_K and a pro-p subgroup. If its action on $(\mathbb{F}^{\mathrm{alg}})^n$ was not trivial, then the invariants $((\mathbb{F}^{\mathrm{alg}})^n)^{P_K}$ would be a nontrivial proper subrepresentation of G_K. But this is impossible, since $\bar{\rho}$ is irreducible.

We deduce that the restriction $\bar{\rho}|_{I_K}$ factors via I_{t}, and hence is a direct sum of one-dimensional continuous characters of I_{t}. Fix one such character ω and write $\omega = \omega^r_{m,\tau}$ for m the level of ω, some $\tau \in \mathcal{P}_m$ and $r \in \{1, \ldots, q^m - 2\}$ primitive, and let $\widehat{\omega} := \widehat{\omega}^r_{m,\tau}$. It follows that $0 \neq (\bar{\rho}|_{G_{K_m}} \otimes \widehat{\omega}^{-1})^{I_K}$, and hence we can find $\lambda' \in (k^{\mathrm{alg}})^\times$ such that $\bar{\mu}_{K_m,\lambda'} \otimes \widehat{\omega}$ is a subrepresentation of $\bar{\rho}|_{G_{K_m}}$. Let $\lambda \in k^{\mathrm{alg}}$ be such that $\lambda^m = \lambda'$ so that $\bar{\mu}_{K_m,\lambda'} = \bar{\mu}_{K,\lambda}|_{G_{K_m}}$. Then by Frobenius reciprocity

$$\mathrm{Ind}_{G_{K_m}}^{G_K}(\widehat{\omega} \otimes \bar{\mu}_{K_m,\lambda'}) \cong \big(\mathrm{Ind}_{G_{K_m}}^{G_K} \widehat{\omega}\big) \otimes \bar{\mu}_{K,\lambda}$$

admits a nonzero homomorphism to the irreducible representation $\bar{\rho}$. By the primitivity of r, the orbit of ω under conjugation by σ has length $m = [G_K : G_{K_m}]$, and it follows that $\mathrm{Ind}_{G_{K_m}}^{G_K} \widehat{\omega}$ is irreducible by the criterion of Mackey, see Lemma 2.1.4(e). This yields the isomorphism

$$\overline{\rho} \cong \big(\mathrm{Ind}_{G_{K_m}}^{G_K} \widehat{\omega}^r_{m,\tau}\big) \otimes \overline{\mu}_{K,\lambda},$$

and moreover that $m = n$, proving (a).

For (b), assume first that $\bar{\rho}$ is defined over $\mathbb{F}$. From our definitions and our hypothesis on $|\mathbb{F}|$ is it clear that $\bar{\rho}' := \mathrm{Ind}_{G_{K_m}}^{G_K} \widehat{\omega}^r_{m,\tau}$ is defined over $k_n \subset \mathbb{F}$. It follows that $\det\bar{\rho}'(\sigma), \det\bar{\rho}(\sigma) \in \mathbb{F}^\times$. Since $\det\bar{\rho}(\sigma) = \lambda^n \cdot \det\bar{\rho}'(\sigma)$, we deduce $\lambda^n \in \mathbb{F}$. For the converse, let $\lambda \in (\mathbb{F}_p^{\mathrm{alg}})^\times$ satisfy $\lambda^n \in \mathbb{F}$. From Lemma 4.6.6, one deduces that the characteristic polynomial of any $\sigma \in G_K$ acting via $\bar{\rho}$ lies in $\mathbb{F}[t]$. It follows from the triviality of the Brauer group of a finite field and [CR62, Section 70] that the representation $\bar{\rho}$ can be defined over $\mathbb{F}$. □

A.4. A variant of a result of Vaccarino

In Theorem A.4.4 of this subsection, we prove a variant of the main theorem of Vaccarino from [Vac09] for group rings of free groups instead of free associative algebras. We use this result in the construction of induction for general pseudocharacters in Theorem 4.6.7.

Let us first introduce some notation. For a set X, let $\mathrm{FM}(X)$ be the free monoid over X and let $\mathrm{FG}(X)$ be the free group over X; we regard $\mathrm{FM}(X)$ as a submonoid of $\mathrm{FG}(X)$. We define $\mathbb{Z}\{X\} := \mathbb{Z}[\mathrm{FM}(X)]$ as the monoid ring of $\mathrm{FM}(X)$ over $\mathbb{Z}$; in other words, $\mathbb{Z}\{X\}$ is the free associative $\mathbb{Z}$-algebra in the indeterminates $x \in X$. We also define $\mathbb{Z}\{X^\pm\} := \mathbb{Z}[\mathrm{FG}(X)]$ as the group ring of $\mathrm{FG}(X)$ over $\mathbb{Z}$ and note that $\mathbb{Z}\{X\}$ is a subring of $\mathbb{Z}\{X^\pm\}$ via the inclusion $\mathrm{FM}(X) \subset \mathrm{FG}(X)$. Let further $F_X(n)$ be the polynomial ring $\mathbb{Z}[\xi_{x,i,j} : x \in X, 1 \le i, j \le n]$ in indeterminates $\xi_{x,i,j}$, that is, the commutative ring of matrix coefficients of generic $n \times n$-matrices over X. Then one has the natural generic matrices representation

$$\rho_X : \mathbb{Z}\{X\} \longrightarrow \mathrm{Mat}_{n\times n}(F_X(n)), \quad x \mapsto \xi_x := (\xi_{x,i,j})_{1\le i,j\le n}.$$

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Let $E_X(n) \subset F_X(n)$ be the subring generated by the coefficients of the characteristic polynomials of the matrices $\rho_X(w)$, $w \in \mathrm{FM}(X)$. The associated degree n pseudocharacter D_{ρ_X}, cf. Definition 4.1.4, factors through a unique $E_X(n)$-valued pseudocharacter

$$D_X := D_{\rho_X} : \mathbb{Z}\{X\} \longrightarrow E_X(n),$$

as follows for instance from [Che14, Corollary 1.14] using Amitsur's formula.

Let $D_X^u : \mathbb{Z}\{X\} \to R^{\mathrm{univ}}_{\mathbb{Z}\{X\},n}$ be the universal n-dimensional pseudocharacter of $\mathbb{Z}\{X\}$ from Proposition 4.2.1 so that one has a unique homomorphism $\alpha_X : R^{\mathrm{univ}}_{\mathbb{Z}\{X\},n} \to E_X(n)$ in $\mathcal{CA}lg_{\mathbb{Z}}$ with $\alpha_X \circ D_X^u = D_X$. The following result is an important theorem of Vaccarino.

Theorem A.4.1 (Vaccarino; [Vac09, Theorem 28]). *The map $\alpha_X : R^{\mathrm{univ}}_{\mathbb{Z}\{X\},n} \to E_X(n)$ is an isomorphism, and in particular $R^{\mathrm{univ}}_{\mathbb{Z}\{X\},n}$ is a domain and a free $\mathbb{Z}$-module.*

In the remainder of this section, we shall extend the pseudocharacter $D_X : \mathbb{Z}[\mathrm{FM}(X)] \to E_X(n)$ to an explicit pseudocharacter $D_{X^\pm} : \mathrm{FG}(X) \to E_{X^\pm}(n)$ on $\mathrm{FG}(X)$ and prove that the extension has again a universal property. The following lemma provides some required auxiliary results.

Lemma A.4.2. *Let G be a group and let $M \subset G$ be a submonoid that is also a generating set of G (as a group). Let A be in $\mathcal{CA}lg_{\mathbb{Z}}$. Then the following hold:*

(a) *If $D : A[M] \to A$ is a pseudocharacter of degree n and if $m \in M$ is an element such that $\Lambda_{D,n}(m) \in A^\times$, then the class of m is a unit of $A[M]/\mathrm{CH}(D)$ and its inverse is the class of*

$$q_{D,m} := \Lambda_{D,n}(m)^{-1} \cdot \sum_{i=0}^{n-1} (-1)^{n-1-i} \Lambda_{D,i}(m)\, m^{n-i-1}.$$

(b) *Let $D, D' : A[G] \to A$ be pseudocharacters of degree n. Then we have:*
- **(1)** *The canonical map $A[M] \to A[G]/\mathrm{CH}(D)$ is surjective.*
- **(2)** *If $B \subset A$ is a subring such that $D(B[M]) \subset B$ and $\Lambda_{D,n}(M) \subset B^\times$, then $D(B[G]) \subset B$.*
- **(3)** *If $D|_{A[M]} = D'_{A[M]}$, then $D = D'$.*

Proof. For (a), simply note that the Cayley–Hamilton identity $\chi_D(m,t)|_{t=m} = 0$ holds in the ring $A[M]/\mathrm{CH}(D)$. Because $1 - m \cdot q_{D,m} = (-1)^n \Lambda_{D,n}(m)^{-1} \cdot \chi_D(m,t)|_{t=m}$, Part (a) follows.

To see (b)(1), it suffices to show that the class of any $g \in G$ in $A[G]/\mathrm{CH}(D)$ lies in the image of $A[M]$. Because M generates G, we can write $g = m_1^{\varepsilon_1} \cdot m_2^{\varepsilon_2} \cdot \ldots \cdot m_r^{\varepsilon_r}$ for suitable $m_1, \ldots, m_r \in M$ and $\varepsilon_1, \ldots, \varepsilon_r \in \{\pm 1\}$. Note that $\Lambda_{D,n}(m_i) \in A^\times$ as observed before Definition 4.1.5. So in the formula for g we can by (a), whenever $\varepsilon_i = -1$, replace the occurring m_i^{-1} by $q_{D,m_i} \in A[M]$, and this shows $g \in A[M] + \mathrm{CH}(D)$.

We turn to (b)(2). As we assume $\Lambda_{D,n}(M) \subset B^\times$, the argument in the previous paragraph now shows that $B[G] \subset B[M] + \mathrm{CH}(D)$. By the further hypothesis $D(B[M]) \subset B$ we deduce that all characteristic polynomial coefficients of any $g \in G$ lie in B, and Part (b)(2) now follows from Proposition 4.1.10 quoted from [Che14].

Finally, we prove (b)(3). Let us go back to the argument for (b)(1). It replaces an element x of $A[G]$ by using the Cayley–Hamilton identity by an element x' in $A[M]$ in such a way that in the replacement, which used $q_{D,m}$ from (a), only values involving $D|_{A[M]}$ were used. It follows that the construction of x' from x is the same whether we use D or D' since we assume $D|_{A[M]} = D'|_{A[M]}$. Therefore, we have $D(x) = D(x') = D'(x') = D(x)$ for all $x \in A[G]$, and we are done. □

Let us now turn to the construction of $\rho_{X^\pm}$ and its properties. To extend ρ_X to $\mathrm{FG}(X)$, we wish to invert $\xi_x \in \mathrm{Mat}_{d\times d}(F_X(n))$, and so we need $\det(\xi_x)$ to be a unit; observe that $\det(\xi_x)$ is a coefficient of the characteristic polynomial of $\rho_X(x)$ and hence lies in $E_X(n)$. We define the subrings $F_{X^\pm}(n) = F_X(n)[\det(\xi_x)^{-1} : x \in X]$ and $E_{X^\pm}(n) = E_X(n)[\det(\xi_x)^{-1} : x \in X]$ of the fraction field of the integral domain $F_X(n)$ by adjoining the inverses of $\det(\xi_x)$ for all $x \in X$ to $F_X(n)$ and to $E_X(n)$, respectively. It is now clear that the representation ρ_X has a canonical extension to a representation

$$\rho_{X^\pm}\colon \mathbb{Z}\{X^\pm\} \longrightarrow \mathrm{Mat}_{n\times n}\big(F_{X^\pm}(n)\big), \quad x \longmapsto \xi_x.$$

Proposition A.4.3. *The pseudocharacter*

$$D_{X^\pm} := D_{\rho_{X^\pm}}\colon \mathbb{Z}\{X^\pm\} \to F_{X^\pm}(n)$$

associated to $\rho_{X^\pm}$ takes values in $E_{X^\pm}(n) \subset F_{X^\pm}(n)$, and $E_{X^\pm}(n)$ is the minimal such ring.

Proof. Let $M = \mathrm{FM}(X) \subset G = \mathrm{FG}(X)$, $B = E_{X^\pm}(n) \subset A = F_{X^\pm}(n)$ and $D = D_{X^\pm}$ so that $D|_{B[M]} = D_X \otimes_{E_X(n)} E_{X^\pm}(n)$. Since D_X is defined over $E_X(n)$ we have $D(B[M]) \subset B$, and by definition of $E_{X^\pm}(n)$ we have $\Lambda_{D_X,n}(X) \subset B^\times$ and hence by multiplicativity of D also $\Lambda_{D_X,n}(M) \subset B^\times$. The first assertion on $D_{X^\pm}$ now follows by applying Lemma A.4.2(b)(2).

The minimality of $E_{X^\pm}(n)$ is straightforward: By Theorem A.4.1, the target ring has to contain $E_X(n)$, but it also has to contain the elements $D_{X^\pm}(x^{-1}) = \det(\xi_x)^{-1}$ for all $x \in X$. ◻

From now on, we regard $D_{X^\pm}$ as a pseudocharacter

$$D_{X^\pm}\colon \mathbb{Z}\{X^\pm\} \to E_{X^\pm}(n).$$

Let $D^u_{X^\pm}\colon \mathbb{Z}\{X^\pm\} \longrightarrow R^{\mathrm{univ}}_{\mathbb{Z}\{X^\pm\},n}$ be the universal n-dimensional pseudocharacter from Proposition 4.2.1. By the universal property of $R^{\mathrm{univ}}_{\mathbb{Z}\{X^\pm\},n}$, there is a unique homomorphism

$$\alpha_{X^\pm}\colon R^{\mathrm{univ}}_{\mathbb{Z}\{X^\pm\},n} \to E_{X^\pm}(n),$$

such that $D_{X^\pm} = \alpha_{X^\pm} \circ D^u_{X^\pm}$. The following variant of Theorem A.4.1 is the main result in this subsection.

Theorem A.4.4. *The map $\alpha_{X^\pm}$ is an isomorphism, and in particular:*

(a) $R^{\mathrm{univ}}_{\mathbb{Z}\{X^\pm\},n}$ *is a domain and a free $\mathbb{Z}$-module.*
(b) *The pseudocharacter $D^u_{X^\pm}$ is associated to the genuine representation $\rho_{X^\pm}$ of* $\mathrm{FG}(X)$.

Proof. We directly prove that the pair $(E_{X^\pm}(n), D_{X^\pm})$ has the universal property of the pair $(R^{\mathrm{univ}}_{\mathbb{Z}\{X^\pm\},n}, D^u_{X^\pm})$. So let $D^\pm\colon A[\mathrm{FG}(X)] \longrightarrow A$ be a pseudocharacter of degree n. Its restriction $D := D^\pm|_{A[\mathrm{FM}(X)]}$ is an A-valued pseudocharacter on $\mathrm{FM}(X)$. Hence, by the universal property of $E_X(n)$ from Theorem A.4.1 there is a unique homomorphism $\alpha\colon E_X(n) \longrightarrow A$ such that $D = \alpha \circ D_X$.

Now, $\Lambda_{D^\pm,n}(g) \in A^\times$ for all $g \in \mathrm{FG}(X)$ as noted above Definition 4.1.5. So for $x \in X$ the image of $\det(\xi_x) = \Lambda_{D_X,n}(x)$ under α is the unit $\Lambda_{D^\pm,n}(x) \in A^\times$. Therefore, α has a unique extension $\alpha^\pm\colon E_{X^\pm}(n) \longrightarrow A$. Let D' be the A-valued degree n pseudocharacter $\alpha^\pm \circ D_{X^\pm}$ on $\mathrm{FG}(X)$.

By construction, D' and $D^\pm$ agree when restricted to $A[\mathrm{FM}(X)]$. From Lemma A.4.2(b)(3) we conclude that $D' = D^\pm$, that is, that $D^\pm = \alpha^\pm \circ D_{X^\pm}$, and this shows the existence of an $\alpha^\pm$ as required for the universal property of $(E_{X^\pm}(n), D_{X^\pm})$. The uniqueness of $\alpha^\pm$ is clear, because its restriction to $E_X(n)$, that is, the map α, is unique, and the extension from α to $\alpha^\pm$ is also unique. ◻

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Competing interest. The authors have no competing interest to declare.

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