Canad. J. Math. Vol. **64** (4), 2012 pp. 721–754
http://dx.doi.org/10.4153/CJM-2012-011-9

Analysis of the Brylinski–Kostant Model for Spherical Minimal Representations

Dehbia Achab and Jacques Faraut

Abstract. We revisit with another view point the construction by R. Brylinski and B. Kostant of minimal representations of simple Lie groups. We start from a pair (V, Q), where V is a complex vector space and Q is a homogeneous polynomial of degree 4 on V. The manifold Ξ is an orbit of a covering of $\mathrm{Conf}(V, Q)$, the conformal group of the pair (V, Q), in a finite dimensional representation space. By a generalized Kantor–Koecher–Tits construction we obtain a complex simple Lie algebra $\mathfrak{g}$, and furthermore a real form $\mathfrak{g}_{\mathbb{R}}$. The connected and simply connected Lie group $G_{\mathbb{R}}$ with $\mathrm{Lie}(G_{\mathbb{R}}) = \mathfrak{g}_{\mathbb{R}}$ acts unitarily on a Hilbert space of holomorphic functions defined on the manifold Ξ.

Introduction

The construction of a realization for the minimal unitary representation of a simple Lie group by using geometric quantization has been the topic of many papers during the last thirty years (see [20, 23], and more recently [1, 16]). In a series of papers [6–9], R. Brylinski and B. Kostant introduced and studied a geometric quantization of minimal nilpotent orbits for simple real Lie groups that are not of Hermitian type. They have constructed the associated irreducible unitary representation on a Hilbert space of half forms on the minimal nilpotent orbit. This can be considered as a Fock model for the minimal representation. In this paper we revisit this construction with another point of view. We start from a pair (V, Q), where V is a complex vector space and Q is a homogeneous polynomial on V of degree 4. The structure group $\mathrm{Str}(V, Q)$, for which Q is a semi-invariant, is assumed to have a symmetric open orbit. The conformal group $\mathrm{Conf}(V, Q)$ consists of rational transformations of V whose differential belongs to $\mathrm{Str}(V, Q)$. The main geometric object is the orbit Ξ of Q under K, a covering of $\mathrm{Conf}(V, Q)$, on a space $\mathcal{W}$ of polynomials on V. Then, by a generalized Kantor–Koecher–Tits construction, starting from the Lie algebra $\mathfrak{k}$ of K, we obtain a simple Lie algebra $\mathfrak{g}$ such that the pair $(\mathfrak{g}, \mathfrak{k})$ is non-Hermitian. As a vector space $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{p} = \mathcal{W}$. The main point is to define a bracket

$$\mathfrak{p} \oplus \mathfrak{p} \to \mathfrak{k}, \quad (X, Y) \mapsto [X, Y]$$

such that $\mathfrak{g}$ becomes a Lie algebra. The Lie algebra $\mathfrak{g}$ is 5-graded:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Received by the editors January 13, 2011.
Published electronically June 9, 2012.
AMS subject classification: **17C36**, 22E46, 32M15, 33C80.
Keywords: minimal representation, Kantor–Koecher–Tits construction, Jordan algebra, Bernstein identity, Meijer G-function.

In the fourth part one defines a representation ρ of $\mathfrak{g}$ on the space $\mathcal{O}(\Xi)_{\text{fin}}$ of polynomial functions on Ξ. In a first step one defines a representation of an $\mathfrak{sl}_2$-triple (E, F, H). It turns out that this is only possible under a condition (T). In such a case one obtains an irreducible unitary representation of the connected and simply connected group $\widetilde{G}_{\mathbb{R}}$ whose Lie algebra is a real form of $\mathfrak{g}$. The representation is spherical. It is realized on a Hilbert space of holomorphic functions on Ξ. There is an explicit formula for the reproducing kernel of $\mathcal{H}$ involving a hypergeometric function ${}_1F_2$. Further the space $\mathcal{H}$ is a weighted Bergman space with a weight taking in general both positive and negative values.

The pairs satisfying (T) are the following:

Classical pairs	$\big((\mathfrak{sl}(n,\mathbb{R}), \mathfrak{so}(n)\big)\,, \big(\mathfrak{so}(p,p), \mathfrak{so}(p)\oplus\mathfrak{so}(p)\big)\,,$
Exceptional pairs	$\big(\mathfrak{e}_{6(6)}, \mathfrak{sp}(8)\big)\,, \big(\mathfrak{e}_{7(7)}, \mathfrak{su}(8)\big)\,, \big(\mathfrak{e}_{8(8)}, \mathfrak{so}(16)\big)\,.$

If $Q = R^2$ or $Q = R^4$ where R is a semi-invariant, then by considering a covering of order 2 or 4 of the orbit Ξ, one can obtain 1 or 3 other unitary representations of $\widetilde{G}_{\mathbb{R}}$. They are not spherical. If the condition T is not satisfied, by a modified construction, one still obtains an irreducible representation of $\widetilde{G}_{\mathbb{R}}$ that is not spherical. This last point is the subject of a paper in preparation by the first author.

The construction of a Schrödinger model for the minimal representation of the group $O(p,q)$ is the subject of a recent book by T. Kobayashi and G. Mano [15]. We should not wonder that there is a link between both the Fock and the Schrödinger models, and that there is an analogue of the Bargmann transform in this setting.

1 The Conformal Group and the Representation κ

Let V be a finite dimensional complex vector space and Q a homogeneous polynomial on V. Define

$$L = \mathrm{Str}(V, Q) = \big\{ g \in GL(V) \mid \exists \gamma = \gamma(g), Q(g\cdot x) = \gamma(g)Q(x) \big\}\,.$$

Assume that there exists $e \in V$ such that

(i) the symmetric bilinear form $\langle x, y\rangle = -D_xD_y \log Q(e)$ is non-degenerate;
(ii) the orbit $\Omega = L \cdot e$ is open;
(iii) the orbit $\Omega = L \cdot e$ is symmetric, *i.e.,* the pair (L, L_0), with $L_0 = \{g \in L \mid g \cdot e = e\}$, is symmetric, which means that there is an involutive automorphism ν of L such that L_0 is open in $\{g \in L \mid \nu(g) = g\}$.

We will equip the vector space V with a Jordan algebra structure. The Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $L = \mathrm{Str}(V, Q)$ decomposes into the $+1$ and -1 eigenspaces of the differential of ν : $\mathfrak{l} = \mathfrak{l}_0 + \mathfrak{q}$, where $\mathfrak{l}_0 = \{X \in \mathfrak{l} \mid X \cdot e = e\} = \mathrm{Lie}(L_0)$. Since the orbit Ω is open, the map $\mathfrak{q} \to V$, $X \mapsto X \cdot e$, is a linear isomorphism. If $X \cdot e = x \quad (X \in \mathfrak{q}, x \in V)$, one writes $X = T_x$. The product on V is defined by $xy = T_x\cdot y = T_x\circ T_y\cdot e$.

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

Theorem 1.1 *This product makes V into a semi-simple complex Jordan algebra:*

(J1) *for $x, y \in V, xy = yx$;*
(J2) *for $x, y \in V, x^2(xy) = x(x^2 y)$;*
(J3) *the symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ is associative: $\langle xy, z\rangle = \langle x, yz\rangle$.*

Proof The product is commutative. In fact

$$xy - yx = [T_x, T_y] \cdot e = 0,$$

since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{l}_0$.

Let τ be the differential of γ at the identity element of L: for $X \in \mathfrak{l}$,

$$\tau(X) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp tX).$$

Claim 1.2

(i) $(D_x \log Q)(e) = \tau(T_x)$,
(ii) $(D_x D_y \log Q)(e) = -\tau(T_{xy})$,
(iii) $(D_x D_y D_z \log Q)(e) = \frac{1}{2}\tau(T_{(xy)z})$.

The proof amounts to differentiating at e the relation $\log Q(\exp T_x \cdot e) = \tau(T_x) + \log Q(e)$ up to third order. (See [21, Exercise 5, p. 38].) Hence by (ii), $\langle x, y\rangle = \tau(T_{xy})$, and, by (iii), the symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ is associative.

Define the associator of three elements x, y, z in V by

$$[x, y, z] = x(zy) - (xz)y = [L(x), L(y)]z.$$

Then identity (J2) can be written as $[x^2, y, x] = 0$ for all $x, y \in V$. It can be shown by following the proof of [21, Theorem 8.5, p. 34], which is also the proof of [13, Theorem III.3.1, p. 50]. ■

The Jordan algebra V is a direct sum of simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i, \quad \text{and} \quad Q(x) = \prod_{i=1}^{s} \Delta_i(x_i)^{k_i} \quad (x = (x_1, \dots, x_s)),$$

where Δ_i is the determinant polynomial of the simple Jordan algebra V_i and the k_i are positive integers. The degree of Q is equal to $\sum_{i=1}^{s} k_i r_i$, where r_i is the rank of V_i.

The conformal group $\mathrm{Conf}(V, Q)$ is the group of rational transformations g of V generated by the translations $\tau_a\colon z \mapsto z + a$ ($a \in V$), the dilations $z \mapsto \ell \cdot z$ ($\ell \in L$), and the inversion $j\colon z \mapsto -z^{-1}$. A transformation $g \in \mathrm{Conf}(V, Q)$ is conformal in the sense that the differential $Dg(z)$ belongs to $L \in \mathrm{Str}(V, Q)$ at any point z where g is defined.

Let $\mathcal{W}$ be the space of polynomials on V generated by the translated $Q(z - a)$ of Q. We will define a representation κ on $\mathcal{W}$ of $\mathrm{Conf}(V, Q)$ or of a covering of order two of it.

Case 1: In case there exists a character χ of $\mathrm{Str}(V, Q)$ such that $\chi^2 = \gamma$, then let $K = \mathrm{Conf}(V, Q)$. Define the cocycle

$$\mu(g, z) = \chi(Dg(z)^{-1}) \quad (g \in K,\ z \in V),$$

and the representation κ of K on $\mathcal{W}$,

$$(\kappa(g)p)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

The function $\kappa(g)p$ belongs actually to $\mathcal{W}$. In fact the cocycle $\mu(g, z)$ is a polynomial in z of degree $\leq \deg Q$ and

$$(\kappa(\tau_a)p)(z) = p(z - a) \quad (a \in V),$$
$$(\kappa(\ell)p)(z) = \chi(\ell)p(\ell^{-1} \cdot z) \quad (\ell \in L),$$
$$(\kappa(j)p)(z) = Q(z)p(-z^{-1}).$$

Case 2: Otherwise, the group K is defined as the set of pairs (g, μ) with $g \in \mathrm{Conf}(V, Q)$, and μ is a rational function on V such that

$$\mu(z)^2 = \gamma(Dg(z))^{-1}.$$

We consider on K the product $(g_1, \mu_1)(g_2, \mu_2) = (g_1 g_2, \mu_3)$ with $\mu_3(z) = \mu_1(g_2 \cdot z)\mu_2(z)$. For $\widetilde{g} = (g, \mu) \in K$, define $\mu(\widetilde{g}, z) := \mu(z)$. Then $\mu(\widetilde{g}, z)$ is a cocycle:

$$\mu(\widetilde{g}_1\widetilde{g}_2, z) = \mu(\widetilde{g}_1, \widetilde{g}_2 \cdot z)\mu(\widetilde{g}_2, z),$$

where $\widetilde{g} \cdot z = g \cdot z$ by definition.

Proposition 1.3

(i) *The map* $K \to \mathrm{Conf}(V, Q)$, $\quad \widetilde{g} = (g, \mu) \mapsto g$ *is a surjective group morphism.*
(ii) *For* $g \in K$, $\mu(g, z)$ *is a polynomial in* z *of degree* $\leq \deg Q$.

Proof It is clearly a group morphism. We will show that the image contains a set of generators of $\mathrm{Conf}(V, Q)$. If g is a translation, then $(g, 1)$ and $(g, -1)$ are elements in K. If $g = \ell \in L$, then $Dg(z) = \ell$, and $(\ell, \alpha), (\ell, -\alpha)$, with $\alpha^2 = \gamma(\ell)^{-1}$, are elements in K. If $g \cdot z = j(z) := -z^{-1}$, then $Dg(z)^{-1} = P(z)$, where $P(z)$ denotes the quadratic representation of the Jordan algebra V: $P(z) = 2T_z^2 - T_{z^2}$, and $\gamma(P(z)) = Q(z)^2$. Then $(j, Q(z)), (j, Q(-z))$ are elements in K. ■

Let $P_{\max}$ denote the preimage in K of the maximal parabolic subgroup $L \ltimes N \subset \mathrm{Conf}(V, Q)$, where N is the subgroup of translations. For $g \in P_{\max}$, $\mu(g, z)$ does not depend on z, and $\chi(g) = \mu(g^{-1}, z)$ is a character of $P_{\max}$. If $g = (\ell, \alpha)$ with $\ell \in L$, then $\chi(g)^2 = \gamma(\ell)$.

Observe that the inverse in K of $\sigma = (j, Q(z))$ is $\sigma^{-1} = (j, Q(-z))$. If K is connected, then K is a covering of order 2 of $\mathrm{Conf}(V, Q)$. If not, the identity component K_0 of K is homeomorphic to $\mathrm{Conf}(V, Q)$.

The representation κ of K on $\mathcal{W}$ is then given by

$$\big(\kappa(g)p\big)(z) = \mu(g^{-1}, z)p(g^{-1}\cdot z).$$

In particular

$$\big(\kappa(g)p\big)(z) = \chi(g)p(g^{-1}\cdot z) \quad (g \in P_{\max}),$$
$$\big(\kappa(\sigma)p\big)(z) = Q(-z)p(-z^{-1}).$$

Hence $p_0 \equiv 1$ is a highest weight vector with respect to the parabolic subgroup $P_{\max}$, and $Q = \kappa(\sigma)p_0$ is a lowest weight vector. The representation κ is irreducible, since every highest weight vector in $\mathcal{W}$ is proportional to p_0.

Example 1 If $V = \mathbb{C}$, $Q(z) = z^n$, then $\mathrm{Str}(V, Q) = \mathbb{C}^*$, $\gamma(\ell) = \ell^n$, and $\mathrm{Conf}(V, Q) \simeq PSL(2, \mathbb{C})$ is the group of fractional linear transformations

$$z \mapsto g\cdot z = \frac{az+b}{cz+d}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Furthermore,

$$Dg(z) = \frac{1}{(cz+d)^2}, \quad \gamma\big(Dg(z)^{-1}\big) = (cz+d)^{2n}, \quad \mu(g, z) = (cz+d)^n.$$

Hence, if n is even, then $K = PSL(2, \mathbb{C})$, and, if n is odd, then $K = SL(2, \mathbb{C})$.

The space $\mathcal{W}$ is the space of polynomials of degree $\leq n$ in one variable. The representation κ of K on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = (cz+d)^n p\Big(\frac{az+b}{cz+d}\Big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Example 2 If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then $\mathrm{Str}(V, Q) = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ acting on V by

$$\ell \cdot z = \ell_1 z \ell_2^{-1} \quad \ell = (\ell_1, \ell_2).$$

Then $\gamma(\ell) = \det \ell_1 \, \det \ell_2^{-1}$, and γ is not the square of a character of $\mathrm{Str}(V, Q)$. Furthermore, $\mathrm{Conf}(V, Q) = PSL(2n, \mathbb{C})$ is the group of the rational transformations

$$z \mapsto g\cdot z = (az+b)(cz+d)^{-1}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2n, \mathbb{C}),$$

decomposed in $n \times n$-blocks. To determine the differential of such a transformation, let us write (assuming c to be invertible)

$$g\cdot z = (az+c)(cz+d)^{-1} = ac^{-1} - (ac^{-1}d - b)(cz+d)^{-1},$$

and we get

$$Dg(z)w = (ac^{-1}d - b)(cz+d)^{-1}cw(cz+d)^{-1}.$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

Notice that $Dg(z) \in \mathrm{Str}(V, Q)$:

$$Dg(z)w = \ell_1 w \ell_2^{-1}, \text{ with } \ell_1 = (ac^{-1}d - b)(cz+d)^{-1}c, \ \ell_2 = (cz+d).$$

Since $\det(ac^{-1}d - b) \det c = \det g = 1$,

$$\gamma\big(Dg(z)^{-1}\big) = \det(cz+d)^2.$$

It follows that $K = SL(2n, \mathbb{C})$ and $\mu(g, z) = \det(cz+d)$.

The space $\mathcal{W}$ is a space of polynomials of an $n \times n$ matrix variable, with degree $\leq n$. The representation κ of K on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = \det(cz+d)p\big((az+b)(cz+d)^{-1}\big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2 The Orbit Ξ and the Irreducible K-invariant Hilbert Subspaces of $\mathcal{O}(\Xi)$

Let Ξ be the K-orbit of Q in $\mathcal{W}$: $\Xi = \{\kappa(g)Q \mid g \in K\}$. Then Ξ is a conical variety. In fact, if $\xi = \kappa(g)Q$, then, for $\lambda \in \mathbb{C}^*$, $\lambda\xi = \kappa(g \circ h_t)Q$, where $h_t \cdot z = e^{-t}z$ ($t \in \mathbb{C}$) with $\lambda = e^{2t}$.

A polynomial $\xi \in \mathcal{W}$ can be written

$$\xi(v) = wQ(v) + \text{ terms of degree } < N = \deg Q \quad (w \in \mathbb{C}),$$

and $w = w(\xi)$ is a linear form on $\mathcal{W}$ that is invariant under the parabolic subgroup $P_{\max}$. The set $\Xi_0 = \{\xi \in \Xi \mid w(\xi) \neq 0\}$ is open and dense in Ξ. A polynomial $\xi \in \Xi_0$ can be written

$$\xi(v) = wQ(v - z) \quad (w \in \mathbb{C}^*, z \in V).$$

Hence we get a coordinate system $(w, z) \in \mathbb{C}^* \times V$ for Ξ_0.

Proposition 2.1 *In this system, the action of K is given by*

$$\kappa(g)\colon (w, z) \mapsto \big(\mu(g, z)w, g \cdot z\big).$$

Observe that the orbit Ξ can be seen as a line bundle over the conformal compactification of V.

Proof Recall that, for $\xi \in \Xi$,

$$\big(\kappa(g)\xi\big)(v) = \mu(g^{-1}, v)\xi(g^{-1} \cdot v),$$

and if $\xi(v) = wQ(v - z)$, then

$$= \mu(g^{-1}, v)wQ(g^{-1} \cdot v - z) = \mu(g^{-1}, v)wQ(g^{-1} \cdot v - g^{-1}g \cdot z).$$

By [12, Lemma 6.6],

$$\mu(g,z)\mu(g,z')Q(g\cdot z - g'\cdot z') = Q(z-z').$$

Therefore

$$(\kappa(g)\xi)(v) = \mu(g^{-1}, g\cdot z)^{-1} wQ(v - g.z) = \mu(g,z)wQ(v - g\cdot z)$$

by the cocycle property. ■

The group K acts on the space $\mathcal{O}(\Xi)$ of holomorphic functions on Ξ by

$$\big(\pi(g)f\big)(\xi) = f\big(\kappa(g)^{-1}\xi\big).$$

If $\xi \in \Xi_0$, *i.e.*, $\xi(v) = wQ(v-z)$, and $f \in \mathcal{O}(\Xi)$, we will write $f(\xi) = \phi(w,z)$ for the restriction of f to Ξ_0. In the coordinates (w,z), the representation π is given by

$$\big(\pi(g)\phi\big)(w,z) = \phi\big(\mu(g^{-1},z)w, g^{-1}\cdot z\big).$$

Let $\mathcal{O}_m(\Xi)$ denote the space of holomorphic functions f on Ξ, homogeneous of degree $m \in \mathbb{Z}$:

$$f(\lambda\xi) = \lambda^m f(\xi) \quad (\lambda \in \mathbb{C}^*).$$

The space $\mathcal{O}_m(\Xi)$ is invariant under the representation π. If $f \in \mathcal{O}_m(\Xi)$, then its restriction ϕ to Ξ_0 can be written $\phi(w,z) = w^m\psi(z)$, where ψ is a holomorphic function on V. We will write $\widetilde{\mathcal{O}}_m(V)$ for the space of the functions ψ corresponding to the functions $f \in \mathcal{O}_m(\Xi)$, and denote by $\widetilde{\pi}_m$ the representation of K on $\widetilde{\mathcal{O}}_m(V)$ corresponding to the restriction π_m of π to $\mathcal{O}_m(\Xi)$. The representation $\widetilde{\pi}_m$ is given by

$$\big(\widetilde{\pi}_m(g)\psi\big)(z) = \mu(g^{-1},z)^m\psi(g^{-1}\cdot z).$$

Observe that $(\widetilde{\pi}_m(\sigma)1)(z) = Q(-z)^m$.

Theorem 2.2

(i) $\mathcal{O}_m(\Xi) = \{0\}$ *for* $m < 0$.
(ii) *The space* $\mathcal{O}_m(\Xi)$ *is finite dimensional, and the representation* π_m *is irreducible.*
(iii) *The functions* ψ *in* $\widetilde{\mathcal{O}}_m(V)$ *are polynomials.*

Proof (i) Assume that $\mathcal{O}_m(\Xi) \neq \{0\}$. Let $f \in \mathcal{O}_m(\Xi)$, $f \not\equiv 0$, and $\phi(w,z) = \psi(z)w^m$ its restriction to Ξ_0. Then ψ is holomorphic on V, and

$$\big(\widetilde{\pi}_m(\sigma)\psi\big)(z) = Q(-z)^m\psi(-z^{-1})$$

is holomorphic as well. We may assume that $\psi(e) \neq 0$. The function $h(\zeta) = \psi(\zeta e)(\zeta \in \mathbb{C})$ is holomorphic on $\mathbb{C}$, $h(\zeta) = \sum_{k=0}^{\infty} a_k\zeta^k$, together with the function

$$Q(\zeta e)^m\psi\Big(-\frac{1}{\zeta}e\Big) = \zeta^{mN}h\Big(-\frac{1}{\zeta}\Big) = \zeta^{mN}\sum_{k=0}^{\infty} a_k\Big(-\frac{1}{\zeta}\Big)^k \quad (N = \deg Q).$$

It follows that $m \geq 0$ and that $a_k = 0$ for $k > mN$.

(ii) The subspace

$$\{f \in \mathcal{O}_m(\Xi) \mid \forall a \in V, \pi(\tau_a)f = f\}$$

reduces to the functions Cw^m, hence is one dimensional. By the theorem of the highest weight [14], it follows that $\mathcal{O}_m(\Xi)$ is finite dimensional and irreducible.

(iii) Furthermore it follows that the functions in $\mathcal{O}_m(\Xi)$ are of the form $w^m\psi(z)$, where ψ is a polynomial on V of degree $\leq m \cdot \deg Q$. ■

We fix a Euclidean real form $V_{\mathbb{R}}$ of the complex Jordan algebra V, denote by $z \mapsto \bar{z}$ the conjugation of V with respect to $V_{\mathbb{R}}$, and then consider the involution $g \mapsto \bar{g}$ of $\mathrm{Conf}(V, Q)$ given by: $\bar{g} \cdot z = \overline{g \cdot \bar{z}}$. For $(g, \mu) \in K$ define

$$\overline{(g, \mu)} = (\bar{g}, \bar{\mu}), \text{ where } \bar{\mu}(z) = \overline{\mu(\bar{z})}.$$

The involution α defined by $\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1}$ is a Cartan involution of K (see [19, Proposition 1.1.]), and

$$K_{\mathbb{R}} := \{g \in K \mid \alpha(g) = g\}$$

is a compact real form of K.

Example 1 If $V = \mathbb{C}$, $Q(z) = z^n$, then $V_{\mathbb{R}} = \mathbb{R}$, and $z \mapsto \bar{z}$ is the usual conjugation. We saw that $K = PSL(2, \mathbb{C})$ if n is even, and $SL(2, \mathbb{C})$ if n is odd. For $g \in SL(2, \mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = PSU(2)$ if n is even, and $K_{\mathbb{R}} = SU(2)$ if n is odd.

Example 2 If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then $V_{\mathbb{R}} = \mathrm{Herm}(n, \mathbb{C})$ and the conjugation is $z \mapsto z^*$. We saw that $K = SL(2n, \mathbb{C})$. For $g \in SL(2n, \mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = SU(2n)$.

We will define on $\mathcal{O}_m(\Xi)$ a $K_{\mathbb{R}}$-invariant inner product. Define the subgroup K_0 of K as $K_0 = L$ in Case 1, and the preimage of L in Case 2, relatively to the covering map $K \to \mathrm{Conf}(V, Q)$, and also $(K_0)_{\mathbb{R}} = K_0 \cap K_{\mathbb{R}}$. The coset space $M = K_{\mathbb{R}}/(K_0)_{\mathbb{R}}$ is a compact Hermitian space and is the conformal compactification of V. There is on M a $K_{\mathbb{R}}$-invariant probability measure for which $M \setminus V$ has measure 0. Its restriction m_0 to V is a probability measure with a density that can be computed by using the decomposition of V into simple Jordan algebras.

Let $H(z, z')$ be the polynomial on $V \times V$, holomorphic in z, anti-holomorphic in z' such that

$$H(x, x) = Q(e + x^2) \quad (x \in V_{\mathbb{R}}).$$

Put $H(z) = H(z, z)$. If z is invertible, then $H(z) = Q(\bar{z})Q(\bar{z}^{-1} + z)$.

Proposition 2.3 *For $g \in K_{\mathbb{R}}$,*

$$H(g \cdot z_1, g \cdot z_2)\mu(g, z_1)\overline{\mu(g, z_2)} = H(z_1, z_2),$$

and

$$H(g \cdot z)|\mu(g, z)|^2 = H(z).$$

Proof Recall that an element $g \in K_{\mathbb{R}}$ satisfies $\sigma \circ \bar{g} \circ \sigma^{-1} = g$, or $\sigma \circ \bar{g} = g \circ \sigma$. Recall also the cocycle property: for $g_1, g_2 \in K$, $\mu(g_1 g_2, z) = \mu(g_1, g_2 \cdot z)\mu(g_2, z)$. Since $\mu(\sigma, z) = Q(z)$, it follows that, for $g \in K_{\mathbb{R}}$,

$$\mu(g, \sigma \cdot z)Q(z) = Q(\bar{g} \cdot z)\mu(\bar{g}, z). \tag{2.1}$$

By [12, Lemma 6.6], for $g \in K$,

$$Q(g \cdot z_1 - g \cdot z_2)\mu(g, z_1)\mu(g, z_2) = Q(z_1 - z_2). \tag{2.2}$$

For $g \in K_{\mathbb{R}}$,

$$H(g \cdot z_1, g \cdot z_2) = Q(\bar{g} \cdot z_2)Q(g \cdot z_1 - \sigma\bar{g} \cdot \bar{z}_2) = Q(\bar{g} \cdot \bar{z}_2)Q(g \cdot z_1 - g\sigma\bar{z}_2),$$

and, by (2.2),

$$= Q(\bar{g} \cdot \bar{z}_2)\mu(g, z_1)^{-1}\mu(g, \sigma \cdot \bar{z}_2)^{-1}Q(z_1 - \sigma \cdot \bar{z}_2).$$

Finally, by (2.1),

$$= \mu(g, z_1)^{-1}\mu(\bar{g}, \bar{z}_2)^{-1}H(z_1, z_2). \qquad \blacksquare$$

We define the norm of a function $\psi \in \widetilde{\mathcal{O}}_m(V)$ by

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m} m_0(dz),$$

with

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

Proposition 2.4

(i) *This norm is $K_{\mathbb{R}}$-invariant. Hence, $\widetilde{\mathcal{O}}_m(V)$ is a Hilbert subspace of $\mathcal{O}(V)$.*

(ii) *The reproducing kernel of $\widetilde{\mathcal{O}}_m(V)$ is given by $\widetilde{\mathcal{K}}_m(z,z') = H(z,z')^m$.*

Proof (i) From Proposition 2.3 it follows that for $g \in K_{\mathbb{R}}$,

$$\begin{aligned}\|\widetilde{\pi}_m(g^{-1})\psi\|_m^2 &= \frac{1}{a_m}\int_V |\mu(g,z)|^{2m}|\psi(g^{-1}\cdot z)|^2 H(z)^{-m} m_0(dz)\\ &= \frac{1}{a_m}\int_V |\psi(g^{-1}\cdot z)|^2 H(g^{-1}\cdot z)^{-m} m_0(dz)\\ &= \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m} m_0(dz) = \|\psi\|_m^2.\end{aligned}$$

(ii) There is a unique function $\psi_0 \in \widetilde{\mathcal{O}}_m(V)$ such that, for $\psi \in \widetilde{\mathcal{O}}_m(V)$,

$$(\psi \mid \psi_0) = \psi(0).$$

The function ψ_0 is K_0-invariant, therefore constant $\psi_0(z) = C$. Taking $\psi = \psi_0$, one gets $C^2 = C$, hence $C = 1$. It means that, if $\widetilde{\mathcal{K}}_m(z,z')$ denotes the reproducing kernel of $\widetilde{\mathcal{O}}_m(V)$,

$$\widetilde{\mathcal{K}}_m(z,0) = \widetilde{\mathcal{K}}_m(0,z') = 1.$$

Since $\widetilde{\mathcal{K}}_m(z,z')$ and $H(z,z')$ satisfy the following invariance properties: for $g \in K_{\mathbb{R}}$,

$$\begin{aligned}\widetilde{\mathcal{K}}_m(g\cdot z, g\cdot z')\mu(g,z)^m\overline{\mu(g,z')}^m &= \widetilde{\mathcal{K}}_m(z,z'),\\ H(g\cdot z, g\cdot z')\mu(g,z)\overline{\mu(g,z')} &= (z,z'),\end{aligned}$$

it follows that $\widetilde{\mathcal{K}}_m(z,z') = H(z,z')^m$. ∎

Since $\mathcal{O}_m(\Xi)$ is isomorphic to $\widetilde{\mathcal{O}}_m(V)$, the space $\mathcal{O}_m(\Xi)$ becomes an invariant Hilbert subspace of $\mathcal{O}(\Xi)$, with reproducing kernel

$$\mathcal{K}_m(\xi,\xi') = \Phi(\xi,\xi')^m,$$

where

$$\Phi(\xi,\xi') = H(z,z')w\overline{w'} \qquad \big(\xi = (w,z), \xi' = (w',z')\big).$$

Theorem 2.5 *The group $K_{\mathbb{R}}$ acts multiplicity free on $\mathcal{O}(\Xi)$. The irreducible $K_{\mathbb{R}}$-invariant subspaces of $\mathcal{O}(\Xi)$ are the spaces $\mathcal{O}_m(\Xi)$ $(m \in \mathbb{N})$. If $\mathcal{H} \subset \mathcal{O}(\Xi)$ is a $K_{\mathbb{R}}$-invariant Hilbert subspace, the reproducing kernel of $\mathcal{H}$ can be written*

$$\mathcal{K}(\xi,\xi') = \sum_{m=0}^{\infty} c_m \Phi(\xi,\xi')^m,$$

with $c_m \geq 0$, such that the series $\sum_{m=0}^{\infty} c_m\Phi(\xi,\xi')^m$ converges uniformly on compact subsets in Ξ.

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

This multiplicity free property means that $K_{\mathbb{R}}$ acts multiplicity free on every $K_{\mathbb{R}}$-invariant Hilbert space $\mathcal{H} \subset \mathcal{O}(\Xi)$.

Proof The representation π of $K_{\mathbb{R}}$ on $\mathcal{O}(\Xi)$ commutes with the $\mathbb{C}^*$-action by dilations and the spaces $\mathcal{O}_m(\Xi)$ are irreducible and mutually inequivalent. It follows that $K_{\mathbb{R}}$ acts multiplicity free. ∎

In case of a weighted Bergman space there is an integral formula for the numbers c_m. For a positive function $p(\xi)$ on Ξ, consider the subspace $\mathcal{H} \subset \mathcal{O}(\Xi)$ of functions ϕ such that

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(w,z) m(dw) m_0(dz) < \infty,$$

where $m(dw)$ denotes the Lebesgue measure on $\mathbb{C}$.

Theorem 2.6 *Let F be a positive function on $[0,\infty[$, and define*

$$p(w,z) = F(H(z)|w|^2)H(z).$$

(i) *Then $\mathcal{H}$ is $K_{\mathbb{R}}$-invariant.*

(ii) *If*

$$\phi(w,z) = \sum_{m=0}^{\infty} w^m \psi_m(z),$$

then

$$\|\phi\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|\psi_m\|_m^2,$$

with

$$\frac{1}{c_m} = \pi a_m \int_0^{\infty} F(u) u^m du.$$

(iii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi,\xi') = \sum_{m=0}^{\infty} c_m \Phi(\xi,\xi')^m.$$

Proof (i) Observe first that the function defined on Ξ by

$$(w,z) \mapsto |w|^2 H(z),$$

is $K_{\mathbb{R}}$-invariant. In fact, for $g \in K$,

$$\kappa(g)\colon (w,g) \mapsto \big(\mu(g,z)w, g\cdot z\big)$$

and, by Propositiion 2.3, for $g \in K_{\mathbb{R}}$,

$$|\mu(g,z)|^2 H(g\cdot z) = H(z).$$

Furthermore, the measure $h(z)m(dw)m_0(dz)$ is also invariant under $K_{\mathbb{R}}$. In fact under the transformation $z = g \cdot z'$, $w = \mu(g, z')w'$ ($g \in K_{\mathbb{R}}$), we get

$$\begin{aligned} H(z)m(dw)m_0(dz) &= H(g \cdot z')|\mu(g, z')|^2 m(dw')m_0(dz') \\ &= H(z')m(dw')m_0(dz'). \end{aligned}$$

(ii) Assume that $p(w, z) = F\big(H(z)|w|^2\big)\,H(z)$. Then

$$\|\pi(g)\phi\|^2 = \int_{\mathbb{C}\times V} \big|\,\phi\big(\mu(g^{-1}, z)w, g^{-1} \cdot z\big)\,\big|^2 F\big(H(z)|w|^2\big)\,H(z)m(dw)m_0(dz).$$

We put

$$g^{-1} \cdot z = z', \quad \mu(g^{-1}, z)w = w'.$$

By the invariance of the measure $H(z)m(dw)m_0(dz)$, we obtain

$$\|\pi(g)\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w', z')|^2 F\big(H(g \cdot z')|\mu(g^{-1}, g \cdot z')|^{-2}|w'|^2\big)\,H(z')m(dw')m_0(dz').$$

Furthermore,

$$H(g \cdot z')|\mu(g^{-1}, g \cdot z')|^{-2} = H(g \cdot z')|\mu(g, z')|^2 = H(z'),$$

and, finally, $\|\pi(g)\phi\| = \|\phi\|$.

(iii) If $\phi(w, z) = w^m\psi(z)$, then

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |w|^{2m}|\psi(z)|^2 F\big(H(z)|w|^2\big)\,H(z)m(dw)m_0(dz).$$

We put $w' = \sqrt{H(z)}w$, then

$$\begin{aligned} \|\phi\|^2 &= \int_{\mathbb{C}\times V} H(z)^{-m}|w'|^{2m}|\psi(z)|^2 F(|w'|^2)m(dw')m_0(dz) \\ &= a_m\|\psi\|_m^2 \int_{\mathbb{C}} F(|w'|^2)|w'|^{2m} m(dw') \\ &= a_m\|\psi\|_m^2 \pi \int_0^\infty F(u)u^m du. \end{aligned}$$

■

3 Decomposition into Simple Jordan Algebras

Let us decompose the semi-simple Jordan algebra V into simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i.$$

Denote by n_i and r_i the dimension and the rank of the simple Jordan algebra V_i, and by Δ_i the determinant polynomial. Then $Q(z) = \prod_{i=1}^{s} \Delta_i(z_i)^{k_i}$. Let $H_i(z, z')$ be the polynomial on $V_i \times V_i$, holomorphic in z, antiholomorphic in z', such that

$$H_i(x, x) = \Delta_i(e_i + x^2) \quad \big(x \in (V_i)_{\mathbb{R}}\big),$$

and put $H_i(z) = H_i(z, z)$. The measure m_0 has a density with respect to the Lebesgue measure m on V

$$m_0(dz) = \frac{1}{C_0} H_0(z) m(dz),$$

with

$$H_0(z) = \prod_{i=1}^{s} H_i(z_i)^{-2\frac{n_i}{r_i}}, \qquad C_0 = \int_V H_0(z) m(dz).$$

The Lebesgue measure m will be chosen such that $C_0 = 1$.

Proposition 3.1 (i) *The polynomial Q satisfies the following Bernstein identity*

$$Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^{\alpha} = B(\alpha) Q(z)^{\alpha-1} \quad (z \in \mathbb{C}),$$

where the Bernstein polynomial B is given by

$$B(\alpha) = \prod_{i=1}^{s} b_i(k_i\alpha) b_i(k_i\alpha - 1) \cdots b_i(k_i\alpha - k_i + 1),$$

and b_i is the Bernstein polynomial relative to the determinant polynomial Δ_i.

(ii) *Furthermore,*

$$Q\Big(\frac{\partial}{\partial z}\Big) H(z)^{\alpha} = B(\alpha) \overline{Q(z)} H(z)^{\alpha-1}.$$

Proof (i) The Bernstein identity for Q follows from [13, Proposition VII.1.4].

(ii) For z invertible $H(z) = Q(\bar{z}) Q(\bar{z}^{-1} + z)$, and then, by (i),

$$\begin{aligned} Q\Big(\frac{\partial}{\partial z}\Big) H(z)^{\alpha} &= Q(\bar{z})^{\alpha} B(\alpha) Q(\bar{z}^{-1} + z)^{\alpha-1} \\ &= Q(\bar{z}) B(\alpha) H(z)^{\alpha-1}. \end{aligned}$$

∎

Example 1 If $V = \mathbb{C}$, $Q(z) = z^n$, then

$$\Big(\frac{d}{dz}\Big)^n z^{n\alpha} = B(\alpha) z^{n(\alpha-1)},$$

with $B(\alpha) = n\alpha(n\alpha - 1) \cdots (n\alpha - n + 1)$.

Example 2 If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then

$$\det\Big(\frac{\partial}{\partial z}\Big) (\det z)^{\alpha} = B(\alpha) (\det z)^{\alpha-1},$$

with $B(\alpha) = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$.

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

Recall that we have introduced the numbers

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

Proposition 3.2 *The numbers a_m are given by*

$$a_m = \prod_{i=1}^{s} \frac{\Gamma_{\Omega_i}(2\frac{n_i}{r_i})}{\Gamma_{\Omega_i}(\frac{n_i}{r_i})} \prod_{i=1}^{s} \frac{\Gamma_{\Omega_i}(mk_i + \frac{n_i}{r_i})}{\Gamma_{\Omega_i}(mk_i + 2\frac{n_i}{r_i})},$$

where Γ_{Ω_i} is the Gindikin gamma function of the symmetric cone Ω_i in the Euclidean Jordan algebra $(V_i)_{\mathbb{R}}$.

Proof If the Jordan algebra V is simple and $Q = \Delta$, the determinant polynomial, by [13, Proposition X.3.4],

$$\begin{aligned} a_m &= \int_V H(z)^{-m} m_0(dz) = \frac{1}{C_0} \int_V H(z)^{-m-2\frac{n}{r}} m(dz) \\ &= C \int_\Omega \Delta(e+x)^{-m-2\frac{n}{r}} m(dx). \end{aligned}$$

By [13, Exercice 4, Chapter VII] we obtain

$$a_m = C' \frac{\Gamma_\Omega(m + \frac{n}{r})}{\Gamma_\Omega(m + 2\frac{n}{r})}.$$

In the general case

$$a_m = \frac{1}{C_0} \prod_{i=1}^{s} \int_{V_i} H_i(z_i)^{-mk_i - 2\frac{n_i}{r_i}} m_i(dz_i),$$

and the formula of the proposition follows. ∎

4 Generalized Kantor–Koecher–Tits Construction

From now on, Q is assumed to be of degree 4. The group of dilations of $V : h_t \cdot z = e^{-t} z$ $(t \in \mathbb{C})$ is a one parameter subgroup of L, and $\chi(h_t) = e^{-2t}$. Put $h_t = \exp(tH)$. Then $\mathrm{ad}(H)$ defines a grading of the Lie algebra $\mathfrak{k}$ of K: $\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1$, with $\mathfrak{k}_j = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)X = jX\}$, $(j = -1, 0, 1)$. Notice that

$$\mathfrak{k}_{-1} = \mathrm{Lie}(N) \simeq V, \quad \mathfrak{k}_0 = \mathrm{Lie}(L), \quad \mathrm{Ad}(\sigma)\colon \mathfrak{k}_j \to \mathfrak{k}_{-j},$$

and also that H belongs to the centre $\mathfrak{z}(\mathfrak{k}_0)$ of $\mathfrak{k}_0$. The element H also defines a grading of $\mathfrak{p} := \mathcal{W}$:

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2,$$

where $\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(H)p = jp\}$ is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j+2$. The subspaces $\mathfrak{p}_j$ are invariant under K_0. Furthermore, $\kappa(\sigma)\colon \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-2} = \mathbb{C}, \quad \mathfrak{p}_2 = \mathbb{C}\,Q, \quad \mathfrak{p}_{-1} \simeq V, \quad \mathfrak{p}_1 \simeq V.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Put $E = Q$, $F = 1$.

Theorem 4.1 *There exists on* $\mathfrak{g}$ *a unique Lie algebra structure such that:*
(i) $[X, X'] = [X, X']_{\mathfrak{k}}$ $(X, X' \in \mathfrak{k})$,
(ii) $[X, p] = d\kappa(X)p$ $(X \in \mathfrak{k}, p \in \mathfrak{p})$,
(iii) $[E, F] = H$.

Proof Observe that (E, F, H) is an $\mathfrak{sl}_2$-triple, and that H defines a grading of

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

with

$$\mathfrak{g}_{-2} = \mathfrak{p}_{-2}, \quad \mathfrak{g}_{-1} = \mathfrak{k}_{-1} + \mathfrak{p}_{-1}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \quad \mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1, \quad \mathfrak{g}_2 = \mathfrak{p}_2.$$

It is possible to give a direct proof of Theorem 4.1 (see [2, Theorem 3.1.]). It is also possible to see this statement as a special case of constructions of Lie algebras by [5]. We will now describe this construction in our case.

(a) *Cayley–Dickson process*

Let $x \mapsto x^*$ denote the symmetry with respect to the one dimensional subspace $\mathbb{C}e$:

$$x^* = \frac{1}{2}\langle x, e\rangle\, e - x.$$

Observe that

$$\langle x, e\rangle = \tau(T_x) = D_x \log Q(e), \quad \langle e, e\rangle = 4.$$

On the vector space $W = V \oplus V$, one defines an algebra structure. If

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2),$$

then $z_1 z_2 = z = (x, y)$ with

$$x = x_1 x_2 - (y_1 y_2^*)^*, \quad y = x_1^* y_2 + (y_1^* x_2^*)^*,$$

and an involution

$$\bar{z} = \overline{(x, y)} = (x, -y^*).$$

This involution is an antiautomorphism: $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$. For $a, b \in W$, one introduces the endomorphisms $V_{a,b}$ and T_a given by

$$V_{a,b} z = \{a, b, z\} := (a\bar{b})z + (z\bar{b})a - (z\bar{a})b,$$
$$T_a z = V_{a,e} z = az + z(a - \bar{a}).$$

By [5, Theorem 6.6] the algebra W is structurable. This means that, for $a, b, c, d \in W$,

$$[V_{a,b}, V_{c,d}] = V_{V_{a,b}c,d} - V_{c,V_{b,a}d}. \tag{4.1}$$

Moreover the structurable algebra W is simple. By (4.1), the vector space spanned by the endomorphisms $V_{a,b}$ $(a, b \in W)$ is a Lie algebra denoted by $\mathrm{Instrl}(W)$. This algebra is the Lie algebra $\mathfrak{g}_0$ in the grading, and its subalgebra $\mathfrak{k}_0$ is the structure algebra of the Jordan algebra V. The space S of skew-Hermitian elements in W, $S = \{z \in W \mid \bar{z} = -z\}$, has dimension one. Its elements are proportionnal to $s_0 = (0, e)$. The subspace $\{(x, 0) \mid x \in V\}$ of W is identified to V, and any element $z = (x, y) \in W$ can be written $z = x + s_0 y$.

(b) *Generalized Kantor–Koecher–Tits construction*
One defines a bracket on the vector space

$$\mathcal{K}(W) = \widetilde{S} \oplus \widetilde{W} \oplus \mathrm{Instrl}(W) \oplus W \oplus S,$$

where $\widetilde{S}$ is a second copy of S, and $\widetilde{W}$ of W. This construction is described in [3], and, by Corollary 6 in that paper, $\mathcal{K}(W)$ is a simple Lie algebra. On the subspace $\mathcal{K}(V) = \widetilde{V} \oplus \mathfrak{str}(V) \oplus V$, this construction agrees with the classical Kantor–Koecher–Tits construction, which produces the Lie algebra $\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$. This algebra $\mathcal{K}(W)$ satisfies property (i). The restriction of the bracket of $\mathcal{K}(W)$ to $\mathcal{K}(V)$ coincides to the one of $\mathcal{K}(V)$. It satisfies (iii) as well: $[s_0, \widetilde{s_0}] = I$, the identity of $\mathrm{End}(W)$. It remains to check property (ii). This can be seen as a consequence of the theorem of the highest weight for irreducible finite dimensional representations of reductive Lie algebras. In fact, the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible with highest weight vector Q, with respect to any Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}_0 + \mathfrak{k}_1$:

- If $X \in \mathfrak{k}_1$, then $\mathrm{d}\kappa(X)Q = 0$.
- If $X \in \mathfrak{k}_0$, such that $d\gamma(X) = 0$, then $d\kappa(X)Q = 0$ and $d\kappa(H)Q = 2Q$.

On the other hand, for the bracket of $\mathcal{K}(W)$:

- If $u \in V$, $[u, s_0] = 0$.
- If $X \in \mathfrak{str}(V)$, such that $\mathrm{tr}(X) = 0$, then $[X, s_0] = 0$ and $[H, s_0] = 2s_0$.

It follows that the adjoint representation of $\mathcal{K}(V) = \widetilde{V} \oplus \mathfrak{str}(V) \oplus V$ on

$$\widetilde{S} \oplus \widetilde{s_0}\widetilde{V} \oplus T_W \oplus s_0 V \oplus S,$$

where $T_W = \{T_w = V_{w,e} \mid w \in W\}$ agrees with the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$. In the present case, $T_w = L(w) + \frac{1}{2}\langle v, e\rangle\,\mathrm{Id}$, if $w = u + s_0 v$ $(u, v \in V)$.

On the vector space $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with

$$\mathfrak{g}_1 = W, \quad \mathfrak{g}_{-1} = W, \quad \mathfrak{g}_2 = \mathbb{C}E, \quad \mathfrak{g}_{-2} = \mathbb{C}F, \quad \mathfrak{g}_0 = \mathrm{Instrl}(W),$$

one defines a bracket satisfying the following properties:

(1) $\mathfrak{g}_1 + \mathfrak{g}_2$ is a Heisenberg Lie algebra:

$$\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2, \quad (w_1, w_2) \mapsto w_1\bar{w}_2 - w_2\bar{w}_1 = \psi(w_1, w_2)s_0.$$

The bilinear form ψ is skew symmetric, and $[w_1, w_2] = \psi(w_1, w_2)E$.

(2) $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0, \quad (w, \widetilde{w}) \mapsto V_{w,\widetilde{w}}$.
(3) $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1, \quad (\lambda E, \widetilde{w}) \mapsto \lambda\widetilde{w}$. ■

With a different point of view the above construction is closely related to [11].

We now introduce a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ that will be considered in the sequel. In Section 2 we considered the involution α of K given by

$$\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1} \quad (g \in K),$$

and the compact real form $K_{\mathbb{R}}$ of K:

$$K_{\mathbb{R}} = \{g \in K \mid \alpha(g) = g\}.$$

Recall that $\mathfrak{p}$ has been defined as a space of polynomial functions on V. For $p \in \mathfrak{p}$, define $\bar{p} = \overline{p(\bar{z})}$, and consider the antilinear involution β of $\mathfrak{p}$ given by $\beta(p) = \kappa(\sigma)\bar{p}$. Observe that $\beta(E) = F$. The involution β is related to the involution α of K by the relation

$$\kappa\big(\alpha(g)\big) \circ \beta = \beta \circ \kappa(g) \quad (g \in K).$$

Hence, for $g \in K_{\mathbb{R}}$, $\kappa(g) \circ \beta = \beta \circ \kappa(g)$. Define

$$\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\}.$$

The real subspace $\mathfrak{p}_{\mathbb{R}}$ is invariant under $K_{\mathbb{R}}$, and irreducible for that action. The space $\mathfrak{p}$, as a real vector space, decomposes under $K_{\mathbb{R}}$ into two irreducible subspaces $\mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. One checks that $E + F \in \mathfrak{p}_{\mathbb{R}}$ (and hence $i(E - F)$ as well).

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{u} = \mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$. Then $\mathfrak{p}$ decomposes as

$$\mathfrak{p} = \mathfrak{p} \cap (i\mathfrak{u}) \oplus \mathfrak{p} \cap \mathfrak{u}$$

into two irreducible $K_{\mathbb{R}}$-invariant real subspaces. Looking at the subalgebra $\mathfrak{g}^0$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ generated by the triple (E, F, H), one sees that $E + F \in \mathfrak{p} \cap (i\mathfrak{u})$. Therefore $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p} \cap (i\mathfrak{u})$, and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ is a Lie algebra, real form of $\mathfrak{g}$, and the above decomposition is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. This real form $\mathfrak{g}_{\mathbb{R}}$ is not Hermitian, since the adjoint action of K on $\mathfrak{p}$ is irreducible.

For Table 1 we have used the notation

$$\varphi_n(z) = z_1^2 + \cdots + z_n^2, \quad (z \in \mathbb{C}^n).$$

In case of an exceptional Lie algebra $\mathfrak{g}$, the real form $\mathfrak{g}_{\mathbb{R}}$ has been identified by computing the Cartan signature.

5 Representation of the Generalized Kantor–Koecher–Tits Lie Algebra

Following the method of Brylinski and Kostant, we will construct a representation ρ of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ on the space of finite sums

$$\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi)$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

V	Q	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$
$\mathbb{C}^n$	$\varphi_n(z)^2$	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{sl}(n+2,\mathbb{C})$	$\mathfrak{sl}(n+2,\mathbb{R})$
$\mathbb{C}^p \oplus \mathbb{C}^q$	$\varphi_p(z)\varphi_q(z')$	$\mathfrak{so}(p+2,\mathbb{C}) \oplus \mathfrak{so}(q+2,\mathbb{C})$	$\mathfrak{so}(p+q+4,\mathbb{C})$	$\mathfrak{so}(p+2,q+2)$
$\mathrm{Sym}(4,\mathbb{C})$	$\det z$	$\mathfrak{sp}(8,\mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(6)}$
$M(4,\mathbb{C})$	$\det z$	$\mathfrak{sl}(8,\mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(7)}$
$\mathrm{Skew}(8,\mathbb{C})$	$\mathrm{Pfaff}(z)$	$\mathfrak{so}(16,\mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(8)}$
$\mathrm{Sym}(3,\mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sp}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{f}_4$	$\mathfrak{f}_{4(4)}$
$M(3,\mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(2)}$
$\mathrm{Skew}(6,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Pfaff}(z) \cdot z'$	$\mathfrak{so}(12,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-5)}$
$\mathrm{Herm}(3,\mathbb{O})_{\mathbb{C}} \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{e}_7 \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(-24)}$
$\mathbb{C} \oplus \mathbb{C}$	$z^3 \cdot z'$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{g}_2$	$\mathfrak{g}_{2(2)}$

Table 1

such that for all $X \in \mathfrak{k}$, $\rho(X) = d\pi(X)$. We define first a representation ρ of the subalgebra generated by E, F, H, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. In particular,

$$\rho(H) = d\pi(H) = \frac{d}{dt}\Big|_{t=0} \pi(\exp tH).$$

Hence, for $\phi \in \mathcal{O}_m(\Xi)$, $\rho(H)\phi = (\mathcal{E} - 2m)\phi$, where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w,z) = \frac{d}{dt}\Big|_{t=0} \phi(w, e^t z).$$

We introduce two operators, $\mathcal{M}$ and $\mathcal{D}$. The operator $\mathcal{M}$ is a multiplication operator $(\mathcal{M}\phi)(w,z) = w\phi(w,z)$, which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m+1}(\Xi)$, and $\mathcal{D}$ is a differential operator:

$$(\mathcal{D}\phi)(w,z) = \frac{1}{w}\Big(Q\Big(\frac{\partial}{\partial z}\Big)\phi\Big)(w,z),$$

which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m-1}(\Xi)$. (Recall that $\mathcal{O}_{-1}(\Xi) = \{0\}$.) We denote by $\mathcal{M}^\sigma$ and $\mathcal{D}^\sigma$ the conjugate operators:

$$\mathcal{M}^\sigma = \pi(\sigma)\mathcal{M}\pi(\sigma)^{-1}, \quad \mathcal{D}^\sigma = \pi(\sigma)\mathcal{D}\pi(\sigma)^{-1}.$$

Given a sequence $(\delta_m)_{m\in\mathbb{N}}$ one defines the diagonal operator δ on $\mathcal{O}(\Xi)_{fin}$ by

$$\delta\Big(\sum_m \phi_m\Big) = \sum_m \delta_m \phi_m,$$

and put

$$\rho(F) = \mathcal{M} - \delta \circ \mathcal{D}, \quad \rho(E) = \pi(\sigma)\rho(F)\pi(\sigma)^{-1} = \mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma.$$

(Observe that, since $\deg Q = 4$, then Q is even and $\sigma = \sigma^{-1}$.)

Lemma 5.1 *We have that* $[\rho(H), \rho(E)] = 2\rho(E)$, $[\rho(H), \rho(F)] = -2\rho(F)$.

Proof Since

$$\rho(H)\mathcal{M}\colon \psi(z)w^m \mapsto (\mathcal{E} - 2(m+1))\psi(z)w^{m+1},$$
$$\mathcal{M}\rho(H)\colon \psi(z)w^m \mapsto (\mathcal{E} - 2m)\psi(z)w^{m+1},$$

one obtains $[\rho(H), \mathcal{M}] = -2\mathcal{M}$. Since

$$\rho(H)\delta\mathcal{D}\colon \psi(z)w^m \mapsto \delta_{m-1}(\mathcal{E} - 2(m-1))Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1},$$
$$\delta\mathcal{D}\rho(H)\colon \psi(z)w^m \mapsto \delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)(\mathcal{E} - 2m)\psi(z)w^{m-1},$$

and, by using the identity

$$\Big[Q\Big(\frac{\partial}{\partial z}\Big), \mathcal{E}\Big] = 4Q\Big(\frac{\partial}{\partial z}\Big),$$

one gets

$$[\rho(H), \delta\mathcal{D}] \colon \psi(z)w^m \mapsto 2\delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1}.$$

Finally $[\rho(H), \rho(F)] = -2\rho(F)$. Since the operator δ commutes with $\pi(\sigma)$, and $\pi(\sigma)\rho(H)\pi(\sigma)^{-1} = -\rho(H)$, we get also $[\rho(H), \rho(E)] = 2\rho(E)$. ■

Let $\mathbb{D}(V)^L$ denote the algebra of L-invariant differential operators on V. This algebra is commutative. In fact it is isomorphic to the algebra of invariant differential operators on the symmetric cone in the Euclidean real form $V_{\mathbb{R}}$. If V is simple and $Q = \Delta$, the determinant polynomial, then $\mathbb{D}(V)^L$ is isomorphic to the algebra $\mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r}$ of symmetric polynomials in r variables. The map

$$D \mapsto \gamma(D), \quad \mathbb{D}(V)^L \to \mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r}$$

is the Harish-Chandra isomorphism (see [13, Theorem XIV.1.7]). In general V decomposes into simple ideals $V = \bigoplus_{i=1}^s V_i$, and $\mathbb{D}(V)^L$ is isomorphic to the algebra $\prod_{i=1}^s \mathcal{P}(\mathbb{C}^{r_i})^{\mathfrak{S}_{r_i}}$. The isomorphism is given by

$$D \mapsto \gamma(D) = \big(\gamma_1(D), \dots, \gamma_s(D)\big),$$

where γ_i is the isomorphism relative to the algebra V_i. For $D \in \mathbb{D}(V)^L$, we define the adjoint D^* by $D^* = J \circ D \circ J$, where $Jf(z) = f \circ j(z) = f(-z^{-1})$. Then $\gamma(D^*)(\lambda) = \gamma(D)(-\lambda)$. (See [13, Proposition XIV.1.8].)

In our setting we define the Maass operator $\mathbf{D}_\alpha$ as

$$\mathbf{D}_\alpha = Q(z)^{1+\alpha}Q\Big(\frac{\partial}{\partial z}\Big)Q(z)^{-\alpha}.$$

It is L-invariant. We write $\gamma_\alpha(\lambda) = \gamma(\mathbf{D}_\alpha)(\lambda)$. If V is simple and $Q = \Delta$, then

$$\gamma_\alpha(\lambda) = \prod_{i=1}^r \Big(\lambda_j - \alpha + \frac{1}{2}\Big(\frac{n}{r} - 1\Big)\Big)$$

([13, p. 296]). If V is simple and $Q = \Delta^k$, then

$$\mathbf{D}_\alpha = \Delta^{k+k\alpha}\Delta\Big(\frac{\partial}{\partial z}\Big)^k\Delta(z)^{-k\alpha} = \prod_{j=1}^k \Delta^{k\alpha+k-j+1}\Delta\Big(\frac{\partial}{\partial z}\Big)\Delta^{-(k\alpha+k-j)},$$

and

$$\gamma_\alpha(\lambda) = \prod_{j=1}^r \Big[\lambda_j - k\alpha + \frac{1}{2}\Big(\frac{n}{r} - 1\Big)\Big]_k.$$

(We have used the Pochhammer symbol $[a]_k = a(a-1)\cdots(a-k+1)$.)

Proposition 5.2 *In general*

$$\gamma_\alpha(\lambda) = \prod_{i=1}^{s}\prod_{j=1}^{r_i}\Big[\lambda_j^{(i)} - k_i\alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big)\Big]_{k_i}$$

for $\lambda = (\lambda^{(1)}, \dots, \lambda^{(s)})$, $\lambda^{(i)} \in \mathbb{C}^{r_i}$.

We say that the pair (V, Q) has property (T) if there is a constant η such that, for $X \in \mathfrak{l} = \mathrm{Lie}(L)$,

$$\mathrm{Tr}(X) = \eta\tau(X).$$

In such a case, for $g \in L$, $\mathrm{Det}(g) = \gamma(g)^\eta$, and, for $x \in V$, $\mathrm{Det}(P(x)) = Q(x)^{2\eta}$. Furthermore $Q(x)^{-\eta}m(dx)$ is an L-invariant measure on the symmetric cone $\Omega \subset V_\mathbb{R}$, and $H_0(z) = H(z)^{-2\eta}$.

Let $V = \oplus_{i=1}^{s} V_i$ be the decomposition of V into simple ideals. Property (T) is equivalent to the following: there is a constant η such that

$$\frac{n_i}{r_i} = \eta k_i \quad (i = 1, \dots, s).$$

In fact, for $x \in V$,

$$\mathrm{Tr}(T_x) = \sum_{i=1}^{s}\frac{n_i}{r_i}\,\mathrm{tr}_i(x_i), \quad \tau(T_x) = \sum_{i=1}^{s} k_i\,\mathrm{tr}_i(x_i),$$

with $x = (x_1, \dots, x_s)$, $x_i \in V_i$.

Property (T) is satisfied either if V is simple or if $V = \mathbb{C}^p \oplus \mathbb{C}^p$ and

$$Q(z) = (z_1^2 + \cdots + z_p^2)(z_{p+1}^2 + \cdots + z_{2p}^2).$$

Hence we get the following cases with property (T):

(1) $V = \mathbb{C}^n$, $Q(z) = (z_1^2 + \cdots + z_n^2)^2$, and then

$$\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C}), \quad \mathfrak{k} = \mathfrak{s}o(n+2, \mathbb{C}).$$

(2) $V = \mathbb{C}^p \oplus \mathbb{C}^p$, and then

$$\mathfrak{g} = \mathfrak{so}(2p+4, \mathbb{C}), \quad \mathfrak{k} = \mathfrak{so}(p+2, \mathbb{C}) \oplus \mathfrak{so}(p+2, \mathbb{C}).$$

(3) V is simple of rank 4, and $Q = \Delta$, the determinant polynomial. Then

$$(\mathfrak{g}, \mathfrak{k}) = \big(\mathfrak{e}_6, \mathfrak{sp}(8, \mathbb{C})\big), \quad \big(\mathfrak{e}_7, \mathfrak{sl}(8, \mathbb{C})\big), \quad \big(\mathfrak{e}_8, \mathfrak{so}(16, \mathbb{C})\big).$$

Observe that the case $V = \mathbb{C}^2$, $Q(z_1, z_2) = (z_1z_2)^2 = z_1^2z_2^2$ belongs both to (1) and (2). This corresponds to the isomorphisms:

$$\mathfrak{sl}(4, \mathbb{C}) \simeq \mathfrak{so}(6, \mathbb{C}), \quad \mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C}).$$

Proposition 5.3 *The subspaces $\mathcal{O}_m(\Xi)$ are invariant under $[\rho(E), \rho(F)]$, and the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ commutes with the L-action:*

$$[\rho(E), \rho(F)] \colon \mathcal{O}_m(\Xi) \to \mathcal{O}_m(\Xi), \quad \psi(z)w^m \mapsto (P_m\psi)(z)w^m,$$

where P_m is an L-invariant differential operator on V of degree ≤ 4. It is given by

$$P_m = \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).$$

Proof Restricted to $\mathcal{O}_m(\Xi)$,

$$\mathcal{M}^\sigma\mathcal{D} = \mathbf{D}_0, \quad \mathcal{D}\mathcal{M}^\sigma = \mathbf{D}_{-1}, \quad \mathcal{M}\mathcal{D}^\sigma = \mathbf{D}^*_{-m}, \quad \mathcal{D}^\sigma\mathcal{M} = \mathbf{D}^*_{-m-1}.$$

It follows that the restriction of the operator $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ is given by

$$\begin{aligned}
[\rho(E), \rho(F)] &= [\mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma, \mathcal{M} - \delta \circ \mathcal{D}] \\
&= [\mathcal{M}, \delta \circ \mathcal{D}^\sigma] + [\delta \circ \mathcal{D}, \mathcal{M}^\sigma] \\
&= \mathcal{M}\delta\mathcal{D}^\sigma - \delta\mathcal{D}^\sigma\mathcal{M} + \delta\mathcal{D}\mathcal{M}^\sigma - \mathcal{M}^\sigma\delta \circ \mathcal{D} \\
&= \delta_m(\mathcal{D}\mathcal{M}^\sigma - \mathcal{D}^\sigma\mathcal{M}) + \delta_{m-1}(\mathcal{M}\mathcal{D}^\sigma - \mathcal{M}^\sigma\mathcal{D}) \\
&= \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).
\end{aligned}$$ ∎

By the Harish-Chandra isomorphism, the operator P_m corresponds to the polynomial $p_m = \gamma(P_m)$,

$$p_m(\lambda) = \delta_m\big(\gamma_{-1}(\lambda) - \gamma_{-m-1}(-\lambda)\big) + \delta_{m-1}\big(\gamma_{-m}(-\lambda) - \gamma_0(\lambda)\big).$$

The question is now whether it is possible to choose the sequence (δ_m) in such a way that $[\rho(E), \rho(F)] = \rho(H)$. Recall that restricted to $\mathcal{O}_m(\Xi)$, $\rho(H) = \mathcal{E} - 2m$, where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w, z) = \frac{d}{dt}\Big|_{t=0}\phi(w, e^t z).$$

Then, by Proposition 5.3, it amounts to checking that for every m,

$$p_m(\lambda) = \gamma(\mathcal{E})(\lambda) - 2m.$$

Theorem 5.4 *It is possible to choose the sequence (δ_m) such that*

$$[\rho(H), \rho(E)] = 2\rho(E), \quad [\rho(H), \rho(F)] = -2\rho(F), \quad [\rho(E), \rho(F)] = \rho(H),$$

if and only if (V, Q) has property (T), and then

$$\delta_m = \frac{A}{(m+\eta)(m+\eta+1)},$$

where A is a constant depending on (V, Q).

(This corresponds to [7, Theorem 6.3].)

Proof (a) Let us assume first that the Jordan algebra V is simple of rank 4. In such a case

$$\gamma_\alpha(\lambda) = \prod_{j=1}^{4}\Big(\lambda_j - \alpha + \frac{1}{2}(\eta - 1)\Big) \quad \Big(\eta = \frac{n}{r}\Big)$$

(Proposition 5.2) . With $X_j = \lambda_j + \frac{1}{2}(\eta - 1)$, the polynomial p_m can be written

$$p_m(\lambda) = \delta_m\Big(\prod_{j=1}^{4}(X_j+1) - \prod_{j=1}^{4}(X_j - m - \eta)\Big) + \delta_{m-1}\Big(\prod_{j=1}^{4}(X_j - m + 1 - \eta) - \prod_{j=1}^{4} X_j\Big).$$

Furthermore

$$\gamma(\mathcal{E})(\lambda) - 2m = \sum_{j=1}^{4}\lambda_j - 2m = \sum_{j=1}^{4} X_j - 2(m + \eta - 1).$$

Lemma 5.5 *The identity in the four variables X_j,*

$$\alpha\Big(\prod_{j=1}^{4}(X_j + 1) - \prod_{j=1}^{4}(X_j - b_j - 1)\Big) + \beta\Big(\prod_{j=1}^{4}(X_j - b_j) - \prod_{j=1}^{4} X_j\Big) = \sum_{j=1}^{4} X_j + c$$

holds if and only if there is a constant b such that

$$b_1 = b_2 = b_3 = b_4 = b,\ c = -2b,$$
$$\alpha = \frac{1}{(b+1)(b+2)},\ \beta = \frac{1}{b(b+1)}.$$

Hence we apply the lemma and get $b = m + \eta - 1$.
(b) In the general case,

$$\begin{aligned}\gamma_\alpha(\lambda) &= \prod_{i=1}^{s}\prod_{j=1}^{r_i}\Big[\lambda_j^{(i)} - k_i\alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big)\Big]_{k_i}\\ &= \prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}\Big(\lambda_j^{(i)} - k_i\alpha + \frac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big) - (k-1)\Big)\\ &= A\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}\Big(\frac{\lambda_j^{(i)}}{k_i} - \alpha + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i}\Big),\end{aligned}$$

with $A = \prod_{i=1}^{s} k_i^{k_i r_i}$. We introduce the notation

$$X_{jk}^{(i)} = \frac{\lambda_j^{(i)}}{k_i} + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i}, \quad b_m^{(i)} = m + \frac{n_i}{k_i r_i} - 1.$$

Then we obtain

$$p_m(\lambda) = A\delta_m\Big(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}+1) - \prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}-b_m^{(i)}-1)\Big) + A\delta_{m-1}\Big(\prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)}-b_m^{(i)}) - \prod_{i=1}^{s}\prod_{j=1}^{r_i}\prod_{k=1}^{k_i}(X_{jk}^{(i)})\Big)$$

and

$$\gamma(\mathcal{E})(\lambda) = \sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i} X_{jk}^{(i)} - \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{r_i}\sum_{k=1}^{k_i} b_m^{(i)}.$$

If the rank of V is equal to 4, then the k_i are equal to 1, and the four variables $X_{j1}^{(i)}$ are independent. By Lemma 5.5, Theorem 5.4 is proven in that case.

If the rank r of V is < 4, then

$$X_{jk}^{(i)} = X_{j1}^{(i)} - \frac{k-1}{k_i},$$

and there are only r independent variables: $X_{j1}^{(i)}$. In that case Theorem 5.4 is proven by using an alternative form of Lemma 5.5. ■

Lemma 5.6 *With a partition $k = (k_1, \dots, k_\ell)$ of 4 and length ℓ, $k_1 + \cdots + k_\ell = 4$, and the numbers γ_{ij} $(1 \le i \le \ell, 1 \le j \le k_i - 1)$, one associates the polynomial F in the ℓ variables $T_1, \dots, T_\ell$:*

$$F(T_1, \dots, T_\ell) = \prod_{i=1}^{\ell} T_i \prod_{j=1}^{k_i-1}(T_i + \gamma_{ij}).$$

Given $\alpha, \beta, c \in \mathbb{R}$, and $b_1, \dots, b_\ell \in \mathbb{R}$, then

$$\alpha\Big(F(T_1+1, \dots, T_\ell+1) - F(T_1-b_1-1, \dots, T_\ell-b_\ell-1)\Big) + \beta\Big(F(T_1-b_1, \dots, T_\ell-b_\ell) - F(T_1, \dots, T_\ell)\Big) = \sum_{i=1}^{\ell} T_i + c$$

is an identity in the variables $T_1, \dots, T_\ell$ if and only if there exists b such that

$$b_1 = \cdots = b_\ell = b,\ \alpha = \frac{1}{(b+1)(b+2)},\ \beta = \frac{1}{b(b+1)},$$

and

$$c = \sum_{i=1}^{\ell}\sum_{j=1}^{k_i-1}\gamma_{ij} - 2b.$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

For $p \in \mathfrak{p}$, define the multiplication operator $\mathcal{M}(p)$ given by

$$\big(\mathcal{M}(p)\phi\big)(w,z) = wp(z)\phi(w,z).$$

Observe that $\mathcal{M}(1) = \mathcal{M}$. Then, for $g \in K$,

$$\mathcal{M}\big(\kappa(g)p\big) = \pi(g)\mathcal{M}(p)\pi(g^{-1}).$$

In fact

$$\big(\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z) = wp(z)\phi\big(\mu(g,z)w, g\cdot z\big),$$

and

$$\begin{aligned}\big(\pi(g)\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z)\\ &= \mu(g^{-1},z)wp(g^{-1}\cdot z)\phi\big(\mu(g^{-1},z)\mu(g,g^{-1}\cdot z)w, g^{-1}g\cdot z\big)\\ &= w\big(\kappa(z)p\big)(z)\phi(w,z) = \mathcal{M}\big(\kappa(g)p\big)\phi(w,z).\end{aligned}$$

Proposition 5.7 *There is a unique map*

$$\mathfrak{p} \to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big), \quad p \mapsto \mathcal{D}(p)$$

such that $\mathcal{D}(1) = \mathcal{D}$, and, for $g \in K$,

$$\mathcal{D}\big(\kappa(g)p\big) = \pi(g)\mathcal{D}(p)\pi(g^{-1}).$$

(This corresponds to part of [7, Theorem 6.1].)

Proof Recall that for $g \in P_{\max}$,

$$\big(\kappa(g)p\big)(z) = \chi(g)p(g^{-1}\cdot z),$$

and

$$\big(\pi(g)\phi\big)(w,z) = \phi\big(\chi(g)w, g^{-1}\cdot g\big).$$

Let us show that for $g \in P_{\max}$,

$$\pi(g)\mathcal{D}\pi(g^{-1}) = \chi(g)\mathcal{D}.$$

Observe first that, for $\ell \in L$ and a smooth function ψ on V,

$$Q\Big(\frac{\partial}{\partial z}\Big)\big(\psi(\ell\cdot z)\big) = \gamma(\ell)\Big(Q\big(\frac{\partial}{\partial z}\big)\psi\Big)(\ell\cdot z).$$

Therefore, for $g \in P_{\max}$,

$$\begin{aligned}\mathcal{D}\pi(g^{-1})\phi(w,z) &= \frac{1}{w}Q\Big(\frac{\partial}{\partial z}\Big(\phi\big(\chi(g^{-1})w, g\cdot z\big)\Big)\\ &= \frac{1}{w}\chi(g)^2\Big(Q\big(\frac{\partial}{\partial z}\big)\phi\Big)\big(\chi(g^{-1})w, g\cdot z\big),\end{aligned}$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

and

$$\Big(\pi(g)\mathcal{D}\pi(g^{-1})\phi\Big)(w,z) = \frac{1}{\chi(g)w}\chi(g)^2\Big(Q\Big(\frac{\partial}{\partial z}\Big)\phi\Big)(w,z) = \chi(g)\mathcal{D}\phi(w,z).$$

It follows that the vector subspace in $\mathrm{End}(\mathcal{O}_{\mathrm{fin}}(\Xi))$ generated by the endomorphisms $\pi(g)\mathcal{D}\pi(g^{-1})$ $(g \in K)$ is a representation space for K equivalent to $\mathfrak{p}$. (See [8, Theorem 3.10].) Hence there exists a unique K-equivariant map $p \mapsto \mathcal{D}(p)$ such that $\mathcal{D}(1) = \mathcal{D}$. ■

For $p \in \mathfrak{p}$, define $\rho(p) = \mathcal{M}(p) - \delta\mathcal{D}(p)$. Observe that this definition is consistent with the definition of $\rho(E)$ and $\rho(F)$. Recall that for $X \in \mathfrak{k}$, $\rho(X) = d\pi(X)$. Hence we get a map

$$\rho \colon \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\mathrm{fin}}\big).$$

Theorem 5.8 *Assume that Property* (T) *holds. Fix* (δ_m) *as in Theorem* 5.4.

(i) *ρ is a representation of the Lie algebra $\mathfrak{g}$ on $\mathcal{O}(\Xi)_{\mathrm{fin}}$.*
(ii) *The representation ρ is irreducible.*

Proof (i) Since π is a representation of K, for $X, X' \in \mathfrak{k}$,

$$[\rho(X), \rho(X')] = \rho([X, X']).$$

It follows from Proposition 5.7 that for $X \in \mathfrak{k}$, $p \in \mathfrak{p}$,

$$[\rho(X), \rho(p)] = \rho([X, p]).$$

It remains to show that for $p, p' \in \mathfrak{p}$,

$$[\rho(p), \rho(p')] = \rho([p, p']).$$

By Theorem 5.4, $[\rho(E), \rho(F)] = \rho(H)$. Then this follows from [9, Lemma 3.6]. Consider the map

$$\tau \colon \bigwedge^2 \mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\mathrm{fin}}$$

defined by

$$\tau(p \wedge p') = [\rho(p), \rho(p')] - \rho([p, p']).$$

We know that $\tau(E \wedge F) = 0$. It follows that, for $g \in K$,

$$\tau\big(\kappa(g)E \wedge \kappa(g)F\big) = 0.$$

Since the representation κ is irreducible, and E and F are highest and lowest vectors with respect to P, the vector $E \wedge F$ is cyclic in $\bigwedge^2\mathfrak{p}$ for the action of K. Therefore $\tau \equiv 0$.

(ii) Let $\mathcal{V} \neq \{0\}$ be a $\rho(\mathfrak{g})$-invariant subspace of $\mathcal{O}(\Xi)_{\mathrm{fin}}$. Then $\mathcal{V}$ is $\rho(\mathfrak{k})$-invariant. As $\mathcal{O}(\Xi)_{\mathrm{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi)$ and as the subspaces $\mathcal{O}_m(\Xi)$ are $\rho(\mathfrak{k})$-irreducible, then there exists $\mathcal{I} \subset \mathbb{N}$ $(\mathcal{I} \neq \varnothing)$ such that $\mathcal{V} = \sum_{m\in\mathcal{I}} \mathcal{O}_m(\Xi)$. Observe that if $\mathcal{V}$ contains

$\mathcal{O}_m(\Xi)$, then it contains $\mathcal{O}_{m+1}(\Xi)$. In fact denote by ϕ_m the function in $\mathcal{O}_m(\Xi)$ defined by $\phi_m(w,z) = w^m$. As $\mathcal{D}\phi_m = 0$, it follows that

$$\rho(F)\phi_m = \mathcal{M}\phi_m = \phi_{m+1},$$

and $\rho(F)\phi_m$ belongs to $\mathcal{O}_{m+1}(\Xi)$; therefore $\mathcal{O}_{m+1}(\Xi) \subset \mathcal{V}$. Denote by m_0 the minimum of the m such that $\mathcal{O}_m(\Xi) \subset \mathcal{V}$, then

$$\mathcal{V} = \bigoplus_{m=m_0}^{\infty} \mathcal{O}_m(\Xi).$$

The function $\phi(w,z) = Q(z)^m w^m$ belongs to $\mathcal{O}_m(\Xi)$, and

$$\rho(F)\phi(w,z) = Q(z)^m w^{m+1} - \delta_{m-1} Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m w^{m-1}.$$

By the Bernstein identity (Proposition 3.1)

$$Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m = B(m) Q(z)^{m-1},$$

and since $B(m) > 0$ for $m > 0$, it follows that, if $\mathcal{O}_m(\Xi) \subset \mathcal{V}$ with $m > 0$, then $\mathcal{O}_{m-1}(\Xi) \subset \mathcal{V}$. Therefore $m_0 = 0$ and $\mathcal{V} = \mathcal{O}(\Xi)_{\text{fin}}$. ∎

6 The Unitary Representation of the Kantor–Koecher–Tits Group

We consider, for a sequence (c_m) of positive numbers, an inner product on $\mathcal{O}(\Xi)_{\text{fin}}$ such that

$$\|\phi\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|\psi_m\|_m^2$$

for

$$\phi(w,z) = \sum_{m=0}^{\infty} \psi_m(z) w^m.$$

This inner product is invariant under $K_{\mathbb{R}}$. We assume that Property (T) holds, and we will determine the sequence (c_m) such that this inner product is invariant under the representation ρ restricted to $\mathfrak{g}_{\mathbb{R}}$. We denote by $\mathcal{H}$ the Hilbert space completion of $\mathcal{O}(\Xi)_{\text{fin}}$ with respect to this inner product. We will assume $c_0 = 1$.

The Bernstein polynomial B is of degree 4 and vanishes at 0 and $\alpha_1 = 1 - \eta$. Let α_2 and α_3 be the two remaining roots:

$$B(\alpha) = A\alpha(\alpha - \alpha_1)(\alpha - \alpha_2)(\alpha - \alpha_3).$$

(1) $V = \mathbb{C}^n$, $Q(z) = (z_1^2 + \cdots + z_n^2)^2$. Then

$$B(\alpha) = A\alpha\Big(\alpha - \frac{1}{2}\Big)\Big(\alpha + \frac{n-4}{4}\Big)\Big(\alpha + \frac{n-2}{4}\Big).$$

$A = 2^4$ if $n \geq 2$, $A = 4^4$ if $n = 1$.

(2) $V = \mathbb{C}^{2p}$, $Q(z) = (z_1^2 + \cdots + z_p^2)(z_{p+1}^2 + \cdots + z_{2p}^2)$. Then

$$B(\alpha) = \alpha^2 \Big(\alpha + \frac{p-2}{2} \Big)^2.$$

(3) V is simple of rank 4, complexification of $V_{\mathbb{R}} = \mathrm{Herm}(4, \mathbb{F})$, $Q(z) = \Delta(z)$, the determinant polynomial. Then

$$B(\alpha) = \alpha \Big(\alpha + \frac{d}{2} \Big) \Big(\alpha + 2\frac{d}{2} \Big) \Big(\alpha + 3\frac{d}{2} \Big),$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$.

Here are the non zero roots of the Bernstein polynomial:

	η	α_1	α_2	α_3
(1)	$\frac{n}{4}$	$-\frac{n-4}{4}$	$\frac{1}{2}$	$-\frac{n-2}{4}$
(2)	$\frac{p}{2}$	$-\frac{p-2}{2}$	0	$-\frac{p-2}{2}$
(3)	$1 + 3\frac{d}{2}$	$-3\frac{d}{2}$	$-\frac{d}{2}$	$-2\frac{d}{2}$

Theorem 6.1

(i) *The inner product of $\mathcal{H}$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if*

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m} \frac{1}{m!}.$$

(ii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi, \xi') = {}_1F_2\big(\eta + 1; \eta + \alpha_2, \eta + \alpha_3; H(z, z') w\overline{w'}\big),$$

for $\xi = (w, z)$, $\xi' = (w', z')$.

(This corresponds to [7, Theorems 6.6 and 8.1].)

Proof (i) Recall that $\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\}$, where β is the conjugation of $\mathfrak{p}$ we introduced at the end of Section 4. Recall also that

$$\beta(\kappa(g)p) = \kappa\big(\alpha(g)\big)\beta(p).$$

The inner product of $\mathcal{H}$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if and only if, for every $p \in \mathfrak{p}$,

$$\rho(p)^* = -\rho\big(\beta(p)\big).$$

But this is equivalent to the single condition $\rho(E)^* = -\rho(F)$. In fact, assume that this condition is satisfied. Then, for $p = \kappa(g)E$, $(g \in K)$,

$$\rho(p) = \pi(g)\rho(E)\pi(g^{-1}), \quad \rho(p)^* = -\pi(g^{-1})^*\rho(F)\pi(g)^*.$$

https://doi.org/10.4153/CJM-2012-011-9 Published online by Cambridge University Press

Since $\pi(g)^* = \pi\big(\alpha(g)\big)^{-1}$, we get

$$\rho(p)^* = -\pi\big(\alpha(g)\big)\,\rho(F)\pi\big(\alpha(g^{-1})\big) = -\rho\big(\kappa(\alpha(g))F\big)$$
$$= -\rho\big(\kappa(\alpha(g))\beta(E)\big) = -\rho\big(\beta(\kappa(g)E)\big) = -\rho\big(\beta(p)\big).$$

Finally, observe that the vector E is cyclic in $\mathfrak{p}$ for the K-action.

For $m \geq 0$, $\phi \in \mathcal{O}_{m+1}(\Xi)$, $\phi' \in \mathcal{O}_m(\Xi)$, the condition $\rho(E)^* = -\rho(F)$ is equivalent to

$$\frac{1}{c_{m+1}}(\phi \mid \mathcal{M}^\sigma \phi')_{m+1} = \frac{1}{c_m}\delta_m(\mathcal{D}\phi \mid \phi')_m.$$

Recall that $m_0(dz) = H_0(z)m(dz)$ with $H_0(z) = H(z)^{-2\eta}$, and the norm of $\widetilde{\mathcal{O}}_m(V)$ can be written

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m-2\eta} m(dz).$$

Then the required condition of invariance becomes

$$\frac{1}{c_{m+1}a_{m+1}}\int_V \psi(z)\overline{Q(z)\psi'(z)}H(z)^{-(m+1)-2\eta}m(dz) = \frac{\delta_m}{c_m a_m}\int_V \Big(Q\Big(\frac{\partial}{\partial z}\Big)\psi\Big)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz).$$

By integrating by parts

$$\int_V \Big(Q\Big(\frac{\partial}{\partial z}\Big)\psi\Big)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz) = \int_V \psi(z)\overline{\psi'(z)}\Big(Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta}\Big)m(dz),$$

and, by the relation

$$Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta} = B(-m-2\eta)\overline{Q(z)}H(z)^{-(m+1)-2\eta},$$

the condition can be written

$$\frac{1}{c_{m+1}} = \frac{a_{m+1}}{a_m}\delta_m B(-m-2\eta)\frac{1}{c_m}.$$

From Proposition 3.2 it follows that

$$\frac{a_{m+1}}{a_m} = \frac{B(-m-\eta)}{B(-m-2\eta)}.$$

We obtain finally

$$\frac{c_{m+1}}{c_m} = \frac{m+\eta+1}{(m+\eta+\alpha_2)(m+\eta+\alpha_3)(m+1)},$$

and since $c_0 = 1$,

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m}\frac{1}{m!}.$$

(ii) By Theorem 2.5 the reproducing kernel of $\mathcal{H}$ is given by

$$\begin{aligned}\mathcal{K}(\xi,\xi') &= \sum_{m=0}^{\infty} c_m H(z,z')^m w^m \overline{w'}^m \\ &= {}_1F_2\big(\eta+1;\eta+\alpha_2,\eta+\alpha_3;H(z,z')w\overline{w'}\big),\end{aligned}$$

with $\xi = (w,z)$, $\xi' = (w',z')$. ■

We will see that the Hilbert space $\mathcal{H}$ is a pseudo-weighted Bergman space. By this we mean that the norm is given by an integral of $|\phi|^2$ with respect to a weight taking both positive and negative values. The weight involves a Meijer G-function

$$G(u) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)} u^{-s}ds,$$

where $\alpha, \beta_1, \beta_2, \beta_3$ are real numbers, and $c > \sigma = -\inf\{\beta_1, \beta_2, \beta_3\}$. This function is denoted by

$$G(u) = G^{3,0}_{1,3}\left(x \Big| \begin{matrix} \alpha & & \\ \beta_1 & \beta_2 & \beta_3 \end{matrix}\right)$$

(see for instance [17]). By the inversion formula for the Mellin transform

$$\int_0^{\infty} G(u)u^{s-1}du = \frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)},$$

for $\operatorname{Re} s > \sigma$, and the integral is absolutely convergent. If the numbers $\beta_1, \beta_2, \beta_3$ are distinct, then

$$G(u) = \varphi_1(u)u^{\beta_1} + \varphi_2(u)u^{\beta_2} + \varphi_3(u)u^{\beta_3},$$

where $\varphi_1, \varphi_2, \varphi_3$ are holomorphic near 0. (Note that $\varphi_1, \varphi_2, \varphi_3$ are ${}_1F_2$ hypergeometric functions.)

The function G may not be positive on $]0,\infty[$, but is positive for u large enough. In fact

$$G(u) \sim \sqrt{\pi}u^{\theta}e^{-2\sqrt{u}} \quad (u \to \infty),$$

where $\theta = \beta_1 + \beta_2 + \beta_3 - \alpha - \frac{1}{2}$. ([18, Theorem 3, p. 32].)

Now take

$$\alpha = \eta - 1,\ \beta_1 = 2\eta - 1,\ \beta_2 = 2\eta + a - 1,\ \beta_3 = 2\eta + b - 1:$$

	α	β_1	β_2	β_3
(1)	$\frac{n}{4} - 1$	$\frac{n-2}{2}$	$\frac{n-1}{2}$	$\frac{n-2}{4}$
(2)	$\frac{p}{2} - 1$	$p - 1$	$p - 1$	$\frac{p}{2}$
(3)	$3\frac{d}{2}$	$3d + 1$	$5\frac{d}{2} + 1$	$2d + 1$

The Mellin transform of G vanishes at $-\alpha$, with changing sign. One can check that $-\alpha > \sigma$ in all cases. Therefore there are real values $s > \sigma$ for which the integral

$$\int_0^\infty G(u)u^{s-1}du < 0.$$

This implies that the function G takes negative values on $]0, \infty[$.

Theorem 6.2 *For $\phi \in \mathcal{H}$,*

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w) m(dw) m_0(dz),$$

with

$$p(w,z) = CG\big(|w|^2 H(z)\big) H(z).$$

The integral is absolutely convergent.

Proof We will follow the proof of [6, Theorem 5.7].

(a) From the proof of Theorem 6.1 it follows that

$$\begin{aligned}\frac{1}{a_m c_m} &= \frac{(2\eta)_m(2\eta+\alpha_2)_m(2\eta+\alpha_3)_m}{(\eta)_m}\\ &= C\frac{\Gamma(2\eta+m)\Gamma(2\eta+\alpha_2+m)\Gamma(2\eta+\alpha_3+m)}{\Gamma(\eta+m)}\\ &= C\int_0^\infty G(u)u^m du.\end{aligned}$$

(One checks that $\sigma < 1$, *i.e.,* G is integrable.) By the computation we did for the proof of Theorem 2.6, we obtain, for $\phi(w,z) = w^m\psi(z) \in \mathcal{O}_m$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w) m(dw) m_0(dz) = \|\phi\|^2.$$

Furthermore, if $\phi \in \mathcal{O}_m$, $\phi' \in \mathcal{O}_{m'}$, with $m \neq m'$,

$$\int_{\mathbb{C}\times V} \phi(w,z)\overline{\phi'(w,z)} m(dw) m_0(dz) = 0.$$

It follows that, for $\phi \in \mathcal{O}_{\text{fin}}$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w) m(dw) m_0(dz) = \|\phi\|^2.$$

The computation is justified by the fact that, for $s > \sigma$,

$$\int_0^\infty |G(u)|u^{s-1}du < \infty.$$

(b) Let us consider the weighted Bergman space $\mathcal{H}^1$ whose norm is given by

$$\|\phi\|_1^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 |p(w,z)| m(dw) m_0(dz).$$

By Theorem 2.6,

$$\|\phi\|_1^2 = \sum_{m=0}^{\infty} \frac{1}{c_m^1} \|\psi_m\|_m^2,$$

with

$$\frac{1}{a_m c_m^1} = C \int_0^\infty |G(u)| u^m du.$$

Obviously $c_m^1 \leq c_m$, therefore $\mathcal{H}^1 \subset \mathcal{H}$. We will show that $\mathcal{H} \subset \mathcal{H}^1$. For that we will prove that there is a constant A such that $c_m \leq A c_m^1$. As observed above there is $u_0 \geq 0$ such that $G(u) \geq 0$, for $u \geq u_0$, and then

$$\int_0^\infty |G(u)| u^m \leq \int_0^\infty G(u) u^m du + 2 \int_0^{u_0} |G(u)| u^m du.$$

Hence

$$\frac{1}{c_m^1} \leq \frac{1}{c_m} + 2 a_m u_0^m \int_0^{u_0} |G(u)| du.$$

By the formula we gave at the beginning of (a), the sequence $a_m c_m u_0^m$ is bounded. Therefore there is a constant A such that $\frac{1}{c_m^1} \leq A \frac{1}{c_m}$, and this implies that $\mathcal{H} \subset \mathcal{H}_1$. ■

Let $\widetilde{G_{\mathbb{R}}}$ be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and denote by $\widetilde{K}_{\mathbb{R}}$ the subgroup of $\widetilde{G}_{\mathbb{R}}$ with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. It is a covering of $K_{\mathbb{R}}$. We denote by $s\colon \widetilde{K}_{\mathbb{R}} \to K_{\mathbb{R}}, g \mapsto s(g)$ the canonical surjection.

Theorem 6.3

(i) *There is a unique unitary irreducible representation $\widetilde{\pi}$ of $\widetilde{G}_{\mathbb{R}}$ on $\mathcal{H}$ such that $d\widetilde{\pi} = \rho$. And, for all $k \in \widetilde{K}_{\mathbb{R}}$, $\widetilde{\pi}(k) = \pi(s(k))$.*

(ii) *The representation $\widetilde{\pi}$ is spherical.*

Proof (i) Notice that if the operators $\rho(E+F)$ and $\rho(i(E-F))$ are skew-symmetric, then for each $p \in \mathfrak{p}_{\mathbb{R}}$, the operator $\rho(p)$ is skew-symmetric. In fact, since the $\mathfrak{sl}_2$-triple (E,F,H) is strictly normal (see [22]), which means that $H \in i\mathfrak{k}_{\mathbb{R}}, E+F \in \mathfrak{p}_{\mathbb{R}}, i(E-F) \in \mathfrak{p}_{\mathbb{R}}$, and since $\mathfrak{p} = \mathcal{U}(\mathfrak{k})E$, hence $\mathfrak{p}_{\mathbb{R}} = \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(E+F) + \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(i(E-F))$, and the assertion follows.

Now, by Nelson's criterion, it is enough to prove that the operator $\rho(\mathcal{L})$ is essentially self-adjoint where $\mathcal{L}$ is the Laplacian of $\mathfrak{g}_{\mathbb{R}}$. Let us consider a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{k}_{\mathbb{R}}$ and a basis $\{p_1, \dots, p_l\}$ of $\mathfrak{p}_{\mathbb{R}}$, orthogonal with respect to the Killing form. As $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ is the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, then the Laplacian and the Casimir operators of $\mathfrak{g}_{\mathbb{R}}$ are given by

$$\mathcal{L} = X_1^2 + \cdots + X_k^2 + p_1^2 + \cdots + p_l^2,$$
$$\mathcal{C} = X_1^2 + \cdots + X_k^2 - p_1^2 - \cdots - p_l^2.$$

It follows that $\mathcal{L} = 2(X_1^2 + \cdots + X_k^2) - \mathcal{C}$ and $\rho(\mathcal{L}) = 2\rho(X_1^2 + \cdots + X_k^2) - \rho(\mathcal{C})$. Since $\rho(X_1^2 + \cdots + X_k^2) = d\pi(X_1^2 + \cdots + X_k^2)$ and as π is a unitary representation of $K_{\mathbb{R}}$, hence the image $\rho(X_1^2 + \cdots + X_k^2)$ of the Laplacian of $\mathfrak{k}_{\mathbb{R}}$ is essentially self-adjoint. Moreover, since the dimension of $\mathcal{O}(\Xi)_{\mathrm{fin}}$ is countable, then the commutant of ρ, which is a division algebra over $\mathbb{C}$, also has a countable dimension, and is equal to $\mathbb{C}$ (see [10, p. 118]). It follows that $\rho(\mathcal{C})$ is scalar. We deduce that $\rho(\mathcal{L})$ is essentially self-adjoint and that the irreducible representation ρ of $\mathfrak{g}_{\mathbb{R}}$ integrates to an irreducible unitary representation of $\widetilde{G}_{\mathbb{R}}$, on the Hilbert space $\mathcal{H}$.

(ii) The space $\mathcal{O}_0(\Xi)$ reduces to the constant functions that are the K-fixed vectors. ■

We do not know whether the representation $\widetilde{\pi}$ goes down to a representation of a real Lie group $G_{\mathbb{R}}$ with $K_{\mathbb{R}}$ as a maximal compact subgroup.

References

[1] D. Achab, *Algèbres de Jordan de rang 4 et représentations minimales.* Adv. Math. **153**(2000), no. 1, 155–183. http://dx.doi.org/10.1006/aima.1999.1874

[2] ———, *Construction process for simple Lie algebras.* J. Algebra **325**(2011), 186–204. http://dx.doi.org/10.1016/j.jalgebra.2010.10.002

[3] B. N. Allison, *Models of isotropic simple Lie algebras.* Comm. Algebra **7**(1979), no. 17, 1835–1875. http://dx.doi.org/10.1080/00927877908822432

[4] ———, *Simple structurable algebras of skew-dimension one.* Comm. Algebra **18**(1990), no. 4, 1245–1279. http://dx.doi.org/10.1080/00927879008823963

[5] B. N. Allison and J. R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras.* Trans. Amer. Math. Soc. **283**(1984), no. 1, 185–210. http://dx.doi.org/10.1090/S0002-9947-1984-0735416-2

[6] R. Brylinski, *Quantization of the 4-dimensional nilpotent orbit of $SL(3, \mathbb{R})$.* Canad. J. Math. **49**(1997), no. 5, 916–943. http://dx.doi.org/10.4153/CJM-1997-048-0

[7] ———, *Geometric quantization of real minimal nilpotent orbits. Symplectic geometry.* Differential Geom. Appl. **9**(1998), no. 1–2, 5–58. http://dx.doi.org/10.1016/S0926-2245(98)00017-5

[8] R. Brylinski and B. Kostant, *Minimal representations, geometric quantization, and unitarity.* Proc. Nat. Acad. Sci. U.S.A. **91**(1994), no. 13, 6026–6029. http://dx.doi.org/10.1073/pnas.91.13.6026

[9] ———, *Lagrangian models of minimal representations of E_8, E_7 and E_8.* In: Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993). Progr. Math., 131, Birkhäuser Boston, Boston, MA, pp. 13–63.

[10] P. Cartier, *Representations of p-adic groups. a survey* In: Automorphic forms, representations and L-functions, (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., 31, American Mathematical Society, Providence, RI, 1979, pp. 111–155.

[11] J.-L. Clerc, *Special prohomogeneous vector spaces associated to F_4, E_6, E_7, E_8 and simple Jordan algebras of rank 3.* J. Algebra **264**(2006), no. 1, 98–128. http://dx.doi.org/10.1016/S0021-8693(03)00115-7

[12] J. Faraut and S. Gindikin, *Pseudo-Hermitian symmetric spaces of tube type.* In: Topics in geometry, Progr. Nonlinear Differential Equations Appl., 20, Birkhäuser Boston, Boston, MA, 1996, pp. 123–154.

[13] J. Faraut and A. Korányi, *Analysis on symmetric cones.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

[14] R. Goodman, *Harmonic analysis on compact symmetric spaces: the legacy of Elie Cartan and Hermann Weyl.* In: Groups and analysis, London Math. Soc. Lecture Note Ser., 354, Cambridge University Press, Cambridge, 2008, pp. 1–23.

[15] T. Kobayashi and G. Mano, *The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$.* Mem. Amer. Math. Soc. **213**(2011), no. 1000.

[16] T. Kobayashi and B. Ørsted, *Analysis on the minimal representation of $O(p, q)$. I. Realization via conformal geometry.* Adv. Math. **180**(2003), no. 2, 486–512. http://dx.doi.org/10.1016/S0001-8708(03)00012-4

[17] A. M. Mathai, *A Handbook of generalized special functions for statistical and physical sciences.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[18] R. B. Paris and A. D. Wood, *Asymptotics of high order differential equations.* Pitman Research Notes in Mathematics Series, 129, Longman Scientific and Technical, Harlow; John Wiley & Sons, new York, 1986.

[19] M. Pevzner, *Analyse conforme sur les algèbres de Jordan.* J. Aust. Math. Soc. **73**(2002), no. 2, 279–299. http://dx.doi.org/10.1017/S1446788700008831

[20] J. Rawnsley and S. Sternberg, *On representations associated to the minimal nilpotent coadjoint orbit of* $SL(3,\mathbb{R})$. Amer. J. Math. **104**(1982), no. 6, 1153–1180. http://dx.doi.org/10.2307/2374055

[21] I. Satake, *Algebraic structures of symmetric domains.* Kanoô Memorial Lectures, 4, Iwanami Shoten, Tokyo; Princeton University Press, Princeton, NJ, 1980.

[22] J Sekiguchi, *Remarks on nilpotent orbits of a symmetric pair.* J. Math. Soc. Japan **39**(1987), no. 1, 127–138. http://dx.doi.org/10.2969/jmsj/03910127

[23] P. Torasso, *Quantification géométrique, opérateurs d'entrelacement et representations de* $SL_3(\mathbb{R})$. Acta Math. **150**(1983), no. 3–4, 153–242. http://dx.doi.org/10.1007/BF02392971

Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu, case 247, 75252 Paris cedex 05

e-mail: achab@math.jussieu.fr faraut@math.jussieu.fr