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TRUNCATIONS OF L-FUNCTIONS IN RESIDUE CLASSES

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Abstract. Let $\chi(n)$ be a quadratic character modulo a prime p. For a fixed integer $s \neq 0$, we estimate certain exponential sums with truncated L-functions

$$L_{s,p}(n) = \sum_{j=1}^{n} \frac{\chi(j)}{j^s} \qquad (n = 1, 2, \ldots).$$

Our estimate implies certain uniformly of distribution properties of reductions of $L_{s,p}(n)$ in the residue classes modulo p.

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1. Introduction. Let p be an odd prime and let $\chi(n)$ be a quadratic character modulo p. For a fixed positive integer $s \neq 0$ we define the truncated L-functions

$$L_{s,p}(n) = \sum_{j=1}^{n} \frac{\chi(j)}{j^s}, \qquad n = 1, 2, \ldots.$$

Various properties of such sums, especially for $s = 1$, have been considered in the literature, see [**2**, **5**, **8**, **9**] and references therein.

Here we consider the behaviour of these sums in the residue classes modulo p. More precisely, in this paper we obtain nontrivial bounds on exponential sums

$$T_s(a; p, M, N) = \sum_{n=M+1}^{M+N} \mathbf{e}_p(aL_{s,p}(n)),$$

where

$$\mathbf{e}_p(z) = \exp(2\pi iz/p),$$

and $L_{s,p}(n)$ is computed modulo p for $1 \leq n < p$. Then, in a standard fashion, we obtain a uniformity of distribution result for the sequence of fractional parts $\{L_{s,p}(n)/p\}$, $n = M+1, \ldots, M+N$.

Here we use an approach which is similar to that of [**4**] however it also needs some additional arguments.

Hereafter, the implied constants in symbols 'O' and '$\ll$' may depend on the integer parameter s and the real parameter ε (we recall that $A \ll B$ is equivalent to $A = O(B)$).

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2. Exponential sums.

THEOREM 1. *Let $\varepsilon > 0$ be a fixed real number. Let M and N be integers with $0 \le M < M + N < p$ and $N \ge p^{1/2+\varepsilon}$. Then, for every fixed integer $s \ge 1$, the following bound holds:*

$$\max_{\gcd(a,p)=1} |T_s(a; p, M, N)| \ll N(\log p)^{-1/2}.$$

Proof. We define $0^{-s} \equiv 0 \pmod p$; thus, $i^{-s} \pmod p$ is defined for all integer i. Then, for any integer $k \ge 0$, we have

$$T_s(a; p, M, N) = \sum_{n=M+1}^{M+N} \mathbf{e}_p(aL_{s,p}(n+k)) + O(k).$$

Therefore, for any integer $K \ge 1$,

$$T_s(a; p, M, N) = \frac{1}{K} W + O(K), \tag{1}$$

where

$$\begin{aligned} W &= \sum_{k=0}^{K-1} \sum_{n=M+1}^{M+N} \mathbf{e}_p(aL_{s,p}(n+k)) \\ &= \sum_{n=M+1}^{M+N} \sum_{k=0}^{K-1} \mathbf{e}_p\left(aL_{s,p}(n) + a\sum_{i=1}^{k} \chi(n+i)(n+i)^{-s}\right) \\ &= \sum_{n=M+1}^{M+N} \mathbf{e}_p(aL_{s,p}(n)) \sum_{k=0}^{K-1} \mathbf{e}_p\left(a\sum_{i=1}^{k} \chi(n+i)(n+i)^{-s}\right). \end{aligned}$$

Applying the Cauchy inequality, we derive

$$|W|^2 \le N \sum_{n=M+1}^{M+N} \left|\sum_{k=0}^{K-1} \mathbf{e}_p\left(a\sum_{i=1}^{k} \chi(n+i)(n+i)^{-s}\right)\right|^2. \tag{2}$$

For each K-dimensional ± 1-vector $= (\vartheta_1, \dots, \vartheta_K) \in \{-1, 1\}^K$ we see that for $1 \le n < p - K$,

$$\frac{1}{2^K} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n+i)) = \begin{cases} 1, & \text{if } \chi(n+i) = \vartheta_i, \quad i = 1, \dots K, \\ 0, & \text{otherwise,} \end{cases}$$

Therefore we derive from (2) (estimating the contribution of each of the at most K possible terms with $p - K \le n \le p$ as K^2),

$$\begin{aligned} |W|^2 \le \frac{N}{2^K} & \sum_{(\vartheta_1,\dots,\vartheta_K)\in\{-1,1\}^K} \sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n+i)) \\ & \times \left|\sum_{k=0}^{K-1} \mathbf{e}_p\left(a\sum_{i=1}^{k} \vartheta_i (n+i)^{-s}\right)\right|^2 + NK^2. \end{aligned}$$

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For every vector $(\vartheta_1, \dots, \vartheta_K) \in \{-1, 1\}^K$, one easily verifies that

$$\sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n+i)) \left| \sum_{k=0}^{K-1} \mathbf{e}_p \left(a \sum_{i=1}^{k} \vartheta_i (n+i)^{-s} \right) \right|^2$$
$$= \sum_{0 \le m,k \le K-1} \sum_{n=M+1}^{M+N} \prod_{i=1}^{K} (1 + \vartheta_i \chi(n+i))$$
$$\times \mathbf{e}_p \left(a \sum_{i=1}^{k} \vartheta_i (n+i)^{-s} - a \sum_{i=1}^{m} \vartheta_i (n+i)^{-s} \right).$$

We observe that each sum over n splits into at most 2^K sums of the form

$$\sigma_{\rho,g,f}(M, N) = \rho \sum_{n=M+1}^{M+N} \chi(g(n)) \mathbf{e}_p (f(n)),$$

where $\rho = \pm 1$, $g(X) \in \mathbb{Z}[X]$, $f(X) \in \mathbb{Z}(X)$ and $\deg g, \deg f = O(K)$. We observe that if $|k - m| \ge 2$ then $f(X)$ is a nonlinear rational function modulo p, and also for every k and m, there is only one sums for which the corresponding polynomial $g(X) = 1$ (otherwise $g(X)$ has no multiple roots modulo p). Thus, using the standard reduction between complete and incomplete sums (see [**1**]) we derive from the Weil bound see [**7**, Theorem 3, Chapter 6], that

$$\sigma_{\rho,g,f}(M, N) \ll K p^{1/2} \log p, \tag{3}$$

if either f is a nonlinear rational function modulo p or g is a nonconstant squarefree polynomial modulo p. Thus (3) applies for all $O(2^K K^2)$ sums $\sigma_{\rho,g,f}(M, N)$, except at most $O(K)$ such sums (as we have seen, at most one such sum may occur for $O(K)$ pairs of k and m with $|k - m| \le 1$). Estimating the exceptional sums $\sigma_{\rho,g,f}(M, N)$ trivially as $\sigma_{\rho,g,f}(M, N) \ll N$, and putting everything together, we obtain

$$W^2 \ll \frac{N}{2^K} \sum_{(\vartheta_1,\dots,\vartheta_K) \in \{-1,1\}^K} \left(2^K K^3 p^{1/2} \log p + KN\right) + NK^2$$
$$\ll 2^K K^3 N p^{1/2} \log p + KN^2.$$

Therefore, by (1), we derive

$$T_s(a; p, M, N) \ll K^{-1/2} N + 2^{K/2} K^{1/2} N^{1/2} p^{1/4} (\log p)^{1/2} + K.$$

Taking $K = \lfloor 0.5\varepsilon \log p \rfloor$, we finish the proof. □

3. Discrepancy. We recall that the *discrepancy* D of a sequence of M points $(\gamma_j)_{j=1}^M$ of the unit interval $[0, 1]$ is defined as

$$D = \sup_{\mathcal{I}} \left| \frac{A(\mathcal{I})}{M} - |\mathcal{I}| \right|,$$

where the supremum is taken over intervals $\mathcal{I} = [\alpha, \beta] \subseteq [0, 1]$ of length $|\mathcal{I}| = \beta - \alpha$ and $A(\mathcal{I})$ is the number of points of this set which belong to $\mathcal{I}$ (see [**3**, **6**]).

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For an integer a with $\gcd(a, p) = 1$, we denote by $D_{s,p}(M, N)$ the discrepancy of the sequence of fractional parts

$$\left\{ \frac{L_{s,p}(n)}{p} \right\}, \qquad M + 1 \le n \le M + N.$$

Using the *Erdős–Turán* bound (see [**3**, **6**]), which gives a discrepancy bound in terms of exponential sums, we derive:

THEOREM 2. *Let $\varepsilon > 0$ be a fixed real number. Let M and N be integers with $0 \le M < M + N < p$ and $N \ge p^{1/2+\varepsilon}$. Then, for every fixed integer $s \ge 1$, the following bound holds:*

$$D_{s,p}(M, N) \ll N(\log p)^{-1/2} \log\log p.$$

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