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STABLE PARALLELIZABILITY OF PARTIALLY ORIENTED FLAG MANIFOLDS II

Dedicated to Professor K. Varadarajan on the occasion of his sixtieth birthday.

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ABSTRACT. In the first paper with the same title the authors were able to determine all partially oriented flag manifolds that are stably parallelizable or parallelizable, apart from four infinite families that were undecided. Here, using more delicate techniques (mainly K-theory), we settle these previously undecided families and show that none of the manifolds in them is stably parallelizable, apart from one 30-dimensional manifold which still remains undecided.

1. **Introduction.** Let $n_1, \ldots, n_s$ be any sequence of positive integers with $s \geq 2$, and let $n = \sum_{1 \leq i \leq s} n_i$. We regard $\mathbb{R}^n$ as an inner product space with its standard orientation in the usual way. Let $0 \leq r \leq s$. A sequence $(A_1, \ldots, A_s)$ of pairwise orthogonal vector subspaces of $\mathbb{R}^n$ with $\dim A_i = n_i$ for $1 \leq i \leq s$, and orientation on A_j for $1 \leq j \leq r$, is called a partially oriented flag of type $(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$, or simply a p.o. flag. The space $M = G(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$ of all p.o. flags of type $(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$ is a smooth compact manifold of dimension $\sum_{i<j} n_i n_j$, called a partially oriented (or p.o.) flag manifold. Indeed M can be identified with the homogeneous space $O(n) / \mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_r) \times O(n_{r+1}) \times \cdots \times O(n_s)$ in a well-known manner, from which it obtains its canonical differentiable structure. We stipulate that when $r = s$, each flag $(A_1, \ldots, A_s)$ be coherently oriented so that the direct sum orientation on the vector space $A_1 \dotplus \cdots \dotplus A_s = \mathbb{R}^n$ coincides with the standard orientation on $\mathbb{R}^n$. When $r = 0$, M is the usual flag manifold $G(n_1, \ldots, n_s)$, and when $r = s$, M is the oriented flag manifold $\tilde{G}(n_1, \ldots, n_s)$. The p.o. flag manifold M is a 2^r (resp. 2^{r-1})-fold covering of the usual flag manifold $G(n_1, \ldots, n_s)$ for $1 \leq r < s$ (resp. for $r = s$). The family of p.o. flag manifolds (and their tangent bundles) was first studied by K. Y. Lam [9], although many special cases such as ordinary flag manifolds (which include the classical flag manifolds $G(1, \ldots, 1)$ and the Grassmann manifolds), the oriented flag manifolds (which include the oriented Grassmann manifolds), the Stiefel manifolds $V_{n,r} = G(1, \ldots, 1 \mid n - r)$, and the projective Stiefel manifolds $X_{n,2} = G(n - 2 \mid 1, 1)$ have been extensively studied over the past half century.

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Work related to their parallelizability was done by R. Stong [13] and by the present authors [10], [11]. In particular, the problem of determining which of the p.o. flag manifolds are stably parallelizable was solved almost completely in [11], apart from four infinite families of p.o. flag manifolds which were left undetermined. In this paper, using mainly K-theoretic tools, we show that none of the cases left unsolved in [11] are stably parallelizable with the possible exception of $G(6,1,1 \mid 1,1)$. Our efforts to determine the stable parallelizability of this 30-dimensional manifold have failed, although we can show that its span (which equals its stable span) is at least 24.

We shall assume without loss of generality that $n_1 \geq \cdots \geq n_r$ and $n_{r+1} \geq \cdots \geq n_s$. We shall prove

THEOREM 1. *Let* $s \geq 3$. *With the above notation, the following p.o. flag manifolds are not stably parallelizable:*

(i) $G(1,\ldots,1 \mid 3,1)$,
(ii) $G(1,\ldots,1 \mid 7,1)$,
(iii) $G(6,1,\ldots,1 \mid 1,1)$, *with* $s \geq 6$,
(iv) $G(6,1,\ldots,1 \mid 1,1,1)$, *with* $s \geq 4$.

Together with Theorem 1.1 of [11], this leads to the following classification theorem for parallelizability of p.o. flag manifolds:

THEOREM 2.

(A) *Let* $s = 2$. *Assume that* $1 \leq k \leq n/2$. *Then* $\tilde{G}(n-k,k) \cong \tilde{G}_k(\mathbb{R}^n)$ *is stably parallelizable if and only if* $k = 1$, *or* $(n,k) = (4,2),(6,3)$. *Only* $\tilde{G}(1,1) \cong S^1, \tilde{G}(3,1) \cong S^3, \tilde{G}(7,1) \cong S^7, \tilde{G}(3,3)$ *are parallelizable.*

(B) *Let* $s \geq 3$. *With the above notation, the following p.o flag manifolds are stably parallelizable:*

  (i) $G(1,\ldots,1 \mid 1,\ldots,1)$,
  (ii) $\tilde{G}(n_1,1,\ldots,1)$,
  (iii) $G(3,\ldots,3,1,\ldots,1 \mid 1,\ldots,1)$,
  (iv) $G(2,\ldots,2,1,\ldots,1 \mid 1,\ldots,1)$,
  (v) $G(6 \mid 1,1), G(6,1 \mid 1,1,)$.

  Furthermore all of these are parallelizable except $\tilde{G}(2,\ldots,2)$ *and* $\tilde{G}(2,\ldots,2,1)$.

(C) *Let* $s \geq 3$. *Then* $M = G(n_1,\ldots,n_r \mid n_{r+1},\ldots,n_s)$ *is not stably parallelizable if* M *is not listed in (B) above and if* $M \neq G(6,1,1 \mid 1,1)$.

We remark first that new proofs for (A) in the above theorem have appeared recently, *cf.* [8], [12]. Secondly, the proof of Theorem 1 is greatly facilitated by use of the "inclusion method", which is now briefly recalled. As above let $M = G(n_1,\ldots,n_r|n_{r+1},\ldots,n_s)$, $n = \Sigma_{1\leq i\leq n_s}$. Also let $L = G(n_1,\ldots,n_{r-1}|n_{r+1},\ldots,n_s)$, $m = n - n_r$. The inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$, regarding $\mathbb{R}^m$ as the subspace $\mathbb{R}^{n_1} \dotplus \cdots \dotplus \mathbb{R}^{n_{r-1}} \dotplus 0 \dotplus \mathbb{R}^{n_{r+1}} \dotplus \cdots \dotplus \mathbb{R}^{n_s} \subset \mathbb{R}^n$, induces an evident inclusion $i\colon L \hookrightarrow M$. Furthermore, it is not hard to see that the normal bundle of this embedding is trivial, hence $i^*(\tau_M) \sim \tau_L$ where $\sim$ denotes stable equivalence (for details *cf.* [10], [11], or Section 6). In particular, if L is not stably parallelizable then M is also not stably parallelizable.

Using this inclusion method (*cf.* Section 6), we see that the proof of Theorem 1 is reduced to consideration of the "critical cases", namely $X = G(1 \mid 3, 1)$, $Y = G(1 \mid 7, 1)$, $Z = G(6, 1, 1, 1 \mid 1, 1)$ and $W = G(6 \mid 1, 1, 1)$, of dimensions respectively 7, 15, 40, and 21. To handle these critical cases, we use Lam's description [9] of the tangent bundle of p.o. flag manifolds to show that in each of these cases the tangent bundle τ is stably equivalent to a multiple $m\xi$ of a certain canonical bundle ξ over the manifold under consideration. We then use the Atiyah-Hirzebruch spectral sequence in the case of X and Y and the Hodgkin spectral sequence in the case of Z and W to compute enough of the K-ring in each case to show that the class of τ is nontrivial, and conclude that the manifolds under consideration are not stably parallelizable. Although the K-theoretic computations here are necessarily specific to the four flag manifolds X, Y, Z, W, and are also laborious, it is hoped that the techniques used may serve as a basis for computations on a much wider class of flag manifolds. In particular the calculation of $K(W)$ seems to involve more delicate use of the Hodgkin spectral sequence than has heretofore been made (compare [2], [3], [11]), and may point the way to further applications.

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2. **The tangent bundle.** Let $M = G(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$, and let τM denote its tangent bundle. For $1 \le i \le s$, let ξ_i denote the canonical n_i-plane bundle over M whose fibre over a p.o. flag $(A_1, \ldots, A_s)$ is the vector space A_i. One has the bundle isomorphism

$$\sum_{1 \le i \le s} \xi_i \approx n\varepsilon \tag{2.1}$$

where ε denotes a trivial line bundle. Note that ξ_j is orientable for $1 \le j \le r$. It follows from the above isomorphism that $\sum_{r<j\le s} \xi_j$ is also orientable. From Lam's formula [9], one has the bundle isomorphism

$$\tau M \approx \sum_{1 \le i < j \le s} \xi_i \otimes \xi_j. \tag{2.2}$$

We shall now focus on the p.o. flag manifolds $X = G(1 \mid 3, 1)$, $Y = G(1 \mid 7, 1)$, $Z = G(6, 1, 1, 1 \mid 1, 1)$, and $W = G(6 \mid 1, 1, 1)$. We shall use the following well known facts about vector bundles in the course of our proof of Theorem 4 (*cf.* [7], or using the fact that a line bundle is determined by its first Stiefel-Whitney class):

LEMMA 3. *Let ξ and η be real line bundles over a paracompact base space. Then*
(i) ξ is orientable if and only if it is isomorphic to a trivial line bundle,
(ii) $\xi \otimes \xi \approx \varepsilon$,
(iii) $\xi \oplus \eta$ is orientable if and only if $\xi \approx \eta$.

For a vector bundle α over a space S, its class in $K(S)$ (or in $\mathrm{KO}(S)$) will be denoted $[\alpha]$.

THEOREM 4. *With the above notation,*

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(i) $[\tau X] = 4[\xi_3] + 3 \in \mathrm{KO}(X)$,
(ii) $[\tau Y] = 8[\xi_3] + 7 \in \mathrm{KO}(Y)$,
(iii) $[\tau Z] = 16[\xi_6] + 24 \in \mathrm{KO}(Z)$,
(iv) $[\tau W] = 8[\xi_2 \oplus \xi_3 \oplus \xi_4] - 3 \in \mathrm{KO}(W)$.

Proof of (i). From (2.2), we get (noting $\xi_1 \approx \varepsilon$, so from 2.1 $[\xi_2] + [\xi_3] = 4$)

$$\tau X \approx \xi_1 \otimes (\xi_2 \oplus \xi_3) \oplus \xi_2 \otimes \xi_3 \approx \xi_2 \oplus \xi_3 \oplus \xi_2 \otimes \xi_3.$$

Therefore, using Lemma 3(ii), we get

$$[\tau X] = [\xi_2] + [\xi_3] + \big(4 - [\xi_3]\big)[\xi_3] = 4 + 4[\xi_3] - 1 = 4[\xi_3] + 3.$$

The proof of (ii) is similar.

Proof of (iii). From Lemma 3(iii), we see that $\xi_5 \approx \xi_6$. Using 2.1, we now get $[\xi_1] + 2[\xi_6] = 8$ in $\mathrm{KO}(Z)$. Also note that $\xi_2 \approx \xi_3 \approx \xi_4 \approx \varepsilon$. Substituting this in (2.2), we get

$$\begin{aligned}
[\tau Z] &= 3[\xi_1 \oplus \xi_5 \oplus \xi_6] + 3 + [\xi_1 \otimes \xi_5 \oplus \xi_5 \otimes \xi_6 \oplus \xi_1 \otimes \xi_6] \\
&= 3\big([\xi_1] + [2\xi_6]\big) + 3 + 2[\xi_1][\xi_6] + 1 \\
&= 3 \cdot 8 + 4 + 2[\xi_6]\big(8 - 2[\xi_6]\big) \\
&= 28 + 16[\xi_6] - 4 \\
&= 16[\xi_6] + 24.
\end{aligned}$$

Proof of (iv). Note that $\xi := \xi_2 \oplus \xi_3 \oplus \xi_4$ is an orientable 3-plane bundle over W, and $\xi_1 \oplus \xi \approx 9\varepsilon$. Hence $\xi \approx \lambda^2(\xi) \approx \xi_2 \otimes \xi_3 \oplus \xi_3 \otimes \xi_4 \oplus \xi_4 \otimes \xi_2$. In particular, $[\xi]^2 = [\xi_2]^2 + [\xi_3]^2 + [\xi_4]^2 + 2[\xi] = 3 + 2[\xi]$. Thus, from (2.2), we get

$$\begin{aligned}
[\tau W] &= [\xi_1 \otimes \xi] + \big[\lambda^2(\xi)\big] \\
&= \big[(9 - \xi)\big][\xi] + [\xi] \\
&= 10[\xi] - [\xi]^2 \\
&= 10[\xi] - 3 - 2[\xi] \\
&= 8[\xi] - 3.
\end{aligned}$$

This completes the proof. ∎

The proof that X, Y, Z, W are not stably parallelizable, and hence of Theorems 1 and 2, now can be completed by computation of the additive order of $[\xi] - \mathrm{rank}(\xi)$ in $\mathrm{KO}(M)$ (or in $K(M)$), where $\xi = \xi_3$ when $M = X, Y$, $\xi = \xi_6$ when $M = Z$, and $\xi = \xi_2 \oplus \xi_3 \oplus \xi_4$ when $M = W$. This is done in the following three sections.

3. **KO-groups of X and Y.** We preserve the notation of the previous section. In particular $X = G(1 \mid 3, 1)$ and $Y = G(1 \mid 7, 1)$. We shall compute the group $\widetilde{\mathrm{KO}}(X)$ using known facts about the projective space $\mathbb{R}P^4$ and the Atiyah-Hirzebruch spectral sequence, thus determining the additive order of $[\xi_3] - 1 \in \widetilde{\mathrm{KO}}(X)$. Computations for Y are generally quite similar and for the most part omitted. We will also use the notation, for a vector bundle α over a space S, $[\alpha] - \mathrm{rank}(\alpha) = \langle \alpha \rangle \in \widetilde{\mathrm{KO}}(S)$.

First note that one has the usual projection $p_3\colon X \longrightarrow \mathbb{R}P^4$, $(A_1, A_2, A_3) \longmapsto A_3$. Under this map the canonical line bundle ξ over $\mathbb{R}P^4$ pulls back to ξ_3. It follows from [1] that $8[\xi_3] = 8 \in \mathrm{KO}(X)$, or equivalently $8\langle\xi_3\rangle = 0$. We wish to show that the order of $\langle\xi_3\rangle$ is in fact 8. Of course this will imply $4\langle\xi_3\rangle \neq 0 \in \widetilde{\mathrm{KO}}(X)$ and consequently, using Theorem 3(i), that τX is not stably trivial.

To apply the Atiyah-Hirzebruch spectral sequence, we must first determine the cohomology of X with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients. Let $p_1\colon X \longrightarrow S^4$ be the bundle projection $(A_1, A_2, A_3) \longmapsto A_1$, whose fibre is $\mathbb{R}P^3$. Using the Leray-Serre spectral sequence ($\mathbb{Z}$ coefficients) and a dimension argument one shows that $H^1(X) = H^5(X) = 0$, $H^2(X) \cong H^6(X) \cong \mathbb{Z}/2$, and $H^7(X) \cong \mathbb{Z}$. In order to determine $H^3(X)$ and $H^4(X)$ we have to determine the differential $\mathbb{Z} \cong H^3(\mathbb{R}P^3) \cong E_4^{0,3} \xrightarrow{d_4} E_4^{4,0} \cong H^4(S^4) \cong \mathbb{Z}$. This is seen to be multiplication by 4 using the map between spectral sequences induced by the projection map $V_{5,2} \longrightarrow X$. Note that the Stiefel manifold $V_{5,2}$ is fibred by S^3 over S^4 and the projection map $V_{5,2} \longrightarrow X$ is a map of fibre bundles covering the identity map of S^4. Indeed one has the following commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \cong H^3(\mathbb{R}P^3) \cong E_4^{0,3} & \xrightarrow{\kappa^*} & \tilde{E}_4^{0,3} \cong H^3(S^3) \cong \mathbb{Z} \\
{\scriptstyle d_4}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{d}_4} \\
\mathbb{Z} \cong H^4(S^4) \cong E_4^{4,0} & = & \tilde{E}_4^{4,0} \cong H^4(S^4) \cong \mathbb{Z}
\end{array}$$

in which $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$ is the spectral sequence for $V_{5,2}$, and $\kappa\colon S^3 \longrightarrow \mathbb{R}P^3$ is the standard double cover. Both the transgression map $\tilde{d}_4$ and κ^* are well-known to be multiplication by 2. This shows that d_4 is multiplication by 4. Hence we see that $H^4(X) \cong \mathbb{Z}/4$, and $H^3(X) = 0$.

Having determined the cohomology groups of X, we now proceed to determine its ring structure. Let y_i denote the generator of $H^i(X)$, $i = 2, 4, 7$.

CLAIM. $y_2^2 = 2y_4$ in $H^4(X)$.

First note that the projection map $\tilde{X} = V_{5,2} \longrightarrow X$ is a universal double covering. Let $q\colon X \longrightarrow \mathbb{R}P^\infty$ denote the classifying map of the double covering $\tilde{X} \longrightarrow X$. Applying the Borel construction one can replace the space X by a homotopically equivalent space X' and the map q by a bundle projection $X' \longrightarrow \mathbb{R}P^\infty$ with fibre $V_{5,2}$. In the resulting spectral sequence, the local coefficients can be shown to constant. In any case the automorphism groups of $H^i(\tilde{X})$ are trivial for $1 \leq i \leq 6$ and hence the corresponding local coefficient systems are constant in this range. This is all that we need to prove our claim. The multiplicative property of the spectral sequence, and the fact that $H^4(X) \cong \mathbb{Z}/4$ now establishes our claim. In fact the spectral sequence also shows that y_2y_4 generates the group $H^6(X) \cong \mathbb{Z}/2$, and $y_2^3 = 0$ since $d_5\colon E_5^{1,4} \longrightarrow E_5^{6,0} = \mathbb{Z}/2$ is an isomorphism. We summarize this in the following proposition. The results for Y can be established in an entirely analogous manner.

PROPOSITION 5. *(i) The manifold X is 7-dimensional, its cohomology being given by $H^*(X) = \mathbb{Z}[y_2, y_4, y_7]/\sim$ where $\deg(y_i) = i$ and the ideal of relations is generated by the elements $2y_2$, $y_2^2 - 2y_4$, y_4^2, y_7^2, y_2y_7, y_4y_7,*

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(ii) $H^*(X;\mathbb{Z}/2) = (\mathbb{Z}/2)[y_1, y_4]/\sim$, $\deg(y_i) = i$ *and the ideal of relations is generated by* y_1^4, y_4^2,

(iii) *the manifold* Y *is* 15*-dimensional and* $H^*(Y) = \mathbb{Z}[y_2, y_8, y_{15}]/\sim$, *where* $\deg(y_i) = i$ *and the ideal of relations is generated by* $2y_2, y_2^4 - 2y_8, y_8^2, y_{15}^2, y_2y_{15}, y_8y_{15}$,

(iv) $H^*(Y;\mathbb{Z}/2) = (\mathbb{Z}/2)[y_1, y_8]/\sim$, $\deg(y_i) = i$ *and the ideal of relations is generated by* y_1^8, y_8^2.

PROOF. Parts (i) and (iii) have been proved above. Part (ii) follows from the result for the integral cohomology algebra and the Leray-Hirsch theorem using the fibration $\mathbb{R}P^3 \hookrightarrow X \longrightarrow S^4$. Similarly one proves (iv). ∎

REMARK. In each case of Proposition 5, the given relations of course imply many other relations in the ideal. For example, in (i) it is easy to deduce the further relations: $4y_4, y_2^3, y_2^2y_4$, and all classes with deg ≥ 8. Similarly for (ii), (iii), and (iv).

COROLLARY 6. *The canonical projection maps* $j: X \longrightarrow \mathbb{R}P^4$ *and* $k: Y \longrightarrow \mathbb{R}P^8$ *induce monomorphisms in integral cohomology. Furthermore,* $j^*: H^p(\mathbb{R}P^4;\mathbb{Z}/2) \longrightarrow H^p(X;\mathbb{Z}/2)$ *and* $k^*: H^q(\mathbb{R}P^8;\mathbb{Z}/2) \longrightarrow H^q(Y;\mathbb{Z}/2)$ *are isomorphisms for* $0 \leq p \leq 3$ *and* $0 \leq q \leq 7$.

We are now ready to apply the Atiyah-Hirzebruch spectral sequence to compute $\mathrm{KO}(X)$.

One sees that the non-zero terms along the diagonal in the Atiyah-Hirzebruch spectral sequence $\{E_r^{p,q}(X)\}$ for $\mathrm{KO}(X)$ are $E_2^{p,-p}(X), p = 0, 1, 2, 4$. The only possible non-zero differentials mapping into a term along the diagonal are $d_3: E_3^{1,-2}(X) \cong H^1(X;\mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow E_3^{4,-4}(X) \cong H^4(X) = \mathbb{Z}/4$ and $d_4: E_4^{0,-1}(X) \cong H^0(X;\mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow E_4^{4,-4}(X) \cong H^4(X) = \mathbb{Z}/4$. Using the map of the spectral sequences induced by j one obtains the following commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{R}P^4;\mathbb{Z}/2) \cong E^{0,-1}(\mathbb{R}P^4) & \xrightarrow{j*} & E^{0,1}(X) \cong H^0(X;\mathbb{Z}/2) \\ 0\big\downarrow & & \big\downarrow d_4 \\ H^4(\mathbb{R}P^4) \cong E^{4,-4}(\mathbb{R}P^4) & \xrightarrow{j*} & E^{4,-4}(X) \cong H^4(X), \end{array}$$

where the horizontal maps are isomorphisms from Corollary 6 and the differential under consideration for $\mathbb{R}P^n$ is well-known to be zero [1]. It follows that $d_4: E_4^{0,-1}(X) \longrightarrow E_4^{4,-4}(X)$ is also zero. Similarly one shows that $d_3: E_3^{1,-2}(X) \longrightarrow E_3^{4,-4}(X)$ is zero.

It follows that $E_\infty^{p,-p}(X) = E_2^{p,-p}(X)$ for all p. To complete our calculation of $\mathrm{KO}(X)$, we make use of the map j again. Proposition 4(i), (ii) shows that $j^*: E_2^{p,-p}(\mathbb{R}P^4) \cong \mathbb{Z}/2 \longrightarrow E_2^{p,-p}(X)$, $p = 1, 2, 4$ is injective. Then the same holds for E_∞ at these positions by the above remarks on differentials. Consequently $j^*: \widetilde{\mathrm{KO}}(\mathbb{R}P^4) \cong \mathbb{Z}/8 \longrightarrow \widetilde{\mathrm{KO}}(X)$ is also injective, whence $\langle \xi_3 \rangle = j^*\langle \xi \rangle$ has order 8.

The argument for Y is completely analogous, showing $j^*: \widetilde{\mathrm{KO}}(\mathbb{R}P^8) \cong \mathbb{Z}/16 \longrightarrow \widetilde{\mathrm{KO}}(Y)$ injective, and completes the proof of (i) in the following theorem. We remark that with a little extra effort in reassembling the short exact sequences relating the usual quotients

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of successive filtrations of $\widetilde{\mathrm{KO}}(X)$ with $E_\infty^{p,-p}$ one can also prove (ii) in this theorem. We omit this since only (i) is needed for our present purposes. We also remark that finding $\widetilde{\mathrm{KO}}(Y)$ seems more difficult, due to terms $E_2^{p,-p} \neq 0$ for $p > 8$.

THEOREM 7.
(i) The order of $\langle \xi_3 \rangle \in \widetilde{\mathrm{KO}}(X)$ *is 8, and the order of* $\langle \xi_3 \rangle \in \widetilde{\mathrm{KO}}(Y)$ *is 16.*
(ii) $\widetilde{\mathrm{KO}}(X) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$.

4. **Calculation of K(Z).** In this section we compute $K(Z)$, where Z is, as before, $G(6,1,1,1 \mid 1,1)$. It will be convenient to think of Z as consisting of flags $(A_1,\ldots,A_6)$ in $\mathbb{R}^{11}$ where A_1, A_4, A_5, A_6 are oriented, $\dim A_1 = 6$, $\dim A_i = 1,\ 2 \le i \le 6$. We can then readily identify Z with the homogeneous space $\mathrm{SO}(11)/\big(\mathrm{SO}(6)\times S\big(O(1)\times O(1)\big)\times\{1\}\big)$ where $\mathrm{SO}(6)\times S\big(O(1)\times O(1)\big)\times\{1\}$ is the subgroup of SO(11) which preserves the p.o. flag $(\mathbb{R}^6, \mathbb{R}e_7, \mathbb{R}e_8, \mathbb{R}e_9, \mathbb{R}e_{10}, \mathbb{R}e_{11})$ (the first, fourth, fifth, and sixth subspaces of this flag being oriented) in Z. We wish to apply the Hodgkin spectral sequence to compute the complex K-ring of the space Z. This method has been used in [2] to compute the K-ring of the projective Stiefel manifolds $X_{4n,k}$. More recently Barufatti and Hacon [3] have computed the K-ring for any projective Stiefel manifold using the Hodgkin spectral sequence. It turns out that our computation is very similar to the case of $X_{8,2}$ which is not surprising since as a homogeneous space $X_{8,2} = \mathrm{SO}(8)/\,\mathrm{SO}(6)\times S\big(O(1)\times O(1)\big)$, and there is a fibration $X_{8,2} \hookrightarrow Z \longrightarrow V_{11,3}$. The following calculation is arranged so that steps 4.1, 4.2 and 4.3 are the same as in the calculation of $K(X_{8,2})$ and we refer to [2], [3] for further details on these steps.

As a first step in applying the Hodgkin spectral sequence, we must express the space Z as a quotient G/H with $\pi_1(G)$ torsion free. This is achieved by using the universal (double) cover $\phi\colon \mathrm{Spin}(11) \longrightarrow \mathrm{SO}(11)$. Thus we see that $Z = \mathrm{Spin}(11)/H$, where $H = \mathrm{Spin}(6)\times\mathbb{Z}/2 \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(11)$, the factor $\mathbb{Z}/2$ being generated by the element $\omega = e_1\cdots e_8 \in \mathrm{Spin}(8)$. (For basic facts about spin groups see [7].)

The next step is to understand the structure of the complex representation ring RH as an $R\mathrm{Spin}(11)$-module via the restriction map $j^\#\colon R\mathrm{Spin}(11) \longrightarrow \mathrm{RH}$, where $j\colon H \hookrightarrow \mathrm{Spin}(11)$.

4.1. As in [2], [3] $\mathrm{RH} \cong R\mathrm{Spin}(6)\otimes R\mathbb{Z}/2 = \mathbb{Z}[p_1, \Delta_3^+, \Delta_3^-]\otimes\mathbb{Z}[y]/\langle y^2+2y\rangle$, where p_1 denotes the first Pontrjagin class, Δ_3^+, Δ_3^- are the half-spin representations each of degree 4. The class y is the degree zero class $x-1$ where x is the representation defined by the nontrivial character $H \longrightarrow \mathbb{Z}/2 \subset U(1)$. One has relations on the Pontrjagin classes p_i:

(i) $p_2 = \Delta_3^+\Delta_3^- - 4p_1 - 16$,
(ii) $p_3 = \Delta_3^2 - 4p_2 - 16p_1 - 64$,
where $\Delta_3 = \Delta_3^+ + \Delta_3^-$.

4.2. Let $i\colon H \hookrightarrow \mathrm{Spin}(8)$ and let $k\colon \mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(11)$. Then $j = k\circ i$. The map $i^\#$ can be calculated as in [2] or [3] to obtain

(i) $p_1' := i^\#(P_1) = 8y + (1+y)p_1$,
(ii) $p_2' := i^\#(P_2) = -(6yp_1 + 48y) + p_2$,

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(iii) $p_3' := i^\#(P_3) = 128y + 24yp_1 + 4yp_2 + (1+y)p_3$,
(iv) $i^\#(\delta_4) = (y+2)\delta_3 + 8y$,
where $\delta_r = \Delta_r - 2^r$.

4.3. It is well-known that

$$k^\#\colon R\mathrm{Spin}\,(11) = \mathbb{Z}[P_1, P_2, P_3, P_4, \Delta_5] \longrightarrow \mathbb{Z}[P_1, P_2, \Delta_4^+, \Delta_4^-] = R\mathrm{Spin}\,(8)$$

is given by
(i) $k^\#(P_i) = P_i, \quad 1 \leq i \leq 4$,
(ii) $k^\#(\Delta_5) = 2\Delta_4$.

4.4. Since $j^\# = i^\# \circ k^\#$, it follows that
(i) $j^\#(P_i) = p_i', \quad 1 \leq i \leq 3$,
(ii) $j^\#(\delta_5) = 2(y+2)\delta_3 + 16y$,
(iii) $j^\#(P_4) = -128y - 2y(p_3 + 4p_2 + 16p_1)$.
The first two are trivial, while (iii) can be computed from 4.2 and 4.3 using the relation $P_4 = \Delta_4^2 - 4P_3 - 16P_2 - 64P_1 - 256$ in $R\mathrm{Spin}\,(8)$.

4.5. The next step is to compute $\mathrm{Tor}^{\bullet}_{R\mathrm{Spin}\,(11)}(\mathrm{RH}, \mathbb{Z})$, which yields the E_2-term of the Hodgkin spectral sequence. This is achieved by applying the change of rings theorem ([4], p. 349). Let $\Lambda = \mathbb{Z}[P_1, P_2, P_3] \subset R\mathrm{Spin}\,(11)$, and let $A = R\mathrm{Spin}\,(11)/\langle P_1, P_2, P_3\rangle \cong \mathbb{Z}[P_4, \delta_5]$. We set $B = \mathrm{RH}\,/\langle p_1', p_2', p_3'\rangle$ so that $j^\#$ defines a map $\theta\colon A \longrightarrow B$ making B a module over A. One readily sees that RH is free as a $\mathbb{Z}[y, p_1, p_2, p_3]/\langle y^2 + 2y\rangle$-module with basis $\{1, \delta_3^-\} \cup \{(\delta_3^+)^m\}_{m\geq 1}$. From 4.2 and the fact that $(1+y)$ is a unit in RH, it follows that $\mathbb{Z}[y, p_1, p_2, p_3]/\langle y^2+2y\rangle = \mathbb{Z}[y, p_1', p_2', p_3']/\langle y^2+2y\rangle$ is free over Λ on basis $\{1, y\}$. Therefore it follows that RH is Λ-free. Now an application of the change of rings theorem shows that

$$\mathrm{Tor}^{\bullet}_{A}\big(\mathrm{Tor}^{\bullet}_{\Lambda}(\mathrm{RH}, \mathbb{Z}), \mathbb{Z}\big) \cong \mathrm{Tor}^{\bullet}_{A}(B, \mathbb{Z}) \cong \mathrm{Tor}^{\bullet}_{R\mathrm{Spin}\,(11)}(\mathrm{RH}, \mathbb{Z}).$$

4.6. We now describe the structure of B as an algebra over A. Let $\theta\colon A \cong \mathbb{Z}[P_4, \delta_5] \longrightarrow B \cong \mathbb{Z}[y, \delta_3^+, \delta_3^-]/\sim$ denote the map induced by $j^\#$. The relations in B are
(i) $y^2 = -2y$,
(ii) $\delta_3^2 + 16\delta_3 - 64y = 0$,
(iii) $\delta_3^+\delta_3^- + 4\delta_3 + 16y = 0$,
and θ is given by
(iv) $\theta(P_4) = 128y$,
(v) $\theta(\delta_5) = 2(y+2)\delta_3 + 16y$.

The relations (i)–(iii) above follow from 4.2 and the well-known expressions for $\Delta_3^+\Delta_3^-$ and $(\Delta_3^+ + \Delta_3^-)^2$ in terms of the Pontrjagin classes. Indeed one obtains the following relations from 4.2: $p_1 = 8y$, $p_2 = -48y$, and $p_3 = 128y$. The above relations (i)–(iii) can then be derived from 4.1 by substitution. Equations (iv) and (v) above follow from 4.4 (note that $p_3 + 4p_2 + 16p_1 = 64y$ in B).

One can now compute $\mathrm{Tor}^*_A(B, \mathbb{Z})$ using the Koszul resolution of $\mathbb{Z}$. This leads to

PROPOSITION 8. $\mathrm{Tor}_A^0(B,\mathbb{Z}) = B/\langle 2(y+2)\delta_3 + 16y\rangle$, $\mathrm{Tor}_A^1(B,\mathbb{Z}) = Bu/\big\langle\big(2(y+2)\delta_3 + 16y\big)u\big\rangle$, *and* $\mathrm{Tor}_A^q(B,\mathbb{Z}) = 0$, *for* $q \geq 2$.

Here $u = U + 4yV$, where U, V are the standard Koszul generators in degree 1 with $d_1(U) = 128y$, $d_1(V) = 2(y+2)\delta_3 + 16y$. As in [2] we can now conclude that the Hodgkin spectral sequence collapses and then apply [2] 6.1–6.5 to conclude that $K^*(Z)$ is isomorphic to $\mathrm{Tor}_A^*(B,\mathbb{Z})$. Therefore we obtain

THEOREM 9. $K^0(Z) = B/\langle 2(y+2)\delta_3 + 16y\rangle \cong \mathbb{Z}[y,\delta_3^+,\delta_3^-]/\sim$, *where the relations are*

(i) $y^2 + 2y = 0$,
(ii) $\delta_3^2 + 16\delta_3 - 64y = 0$,
(iii) $\delta_3^+\delta_3^- + 4\delta_3 + 16y = 0$,
(iv) $2(y+2)\delta_3 + 16y = 0$,

and $\delta_3 = \delta_3^+ + \delta_3^-$.

REMARK. From the above theorem it follows that

$$0 = y\big(2(y+2)\delta_3 + 16y\big) = 16y^2 = -32y.$$

That is $32y = 0$. Therefore (ii) above reduces to the relation
(ii)′ $\delta_3^2 + 16\delta_3 = 0$.

Taking into account Theorem 4(iii) above, our proof that Z is not stably parallelizable is completed by the next result.

PROPOSITION 10. *The additive order of* y *in* $K^0(Z)$ *is* 32.

PROOF. From the above remark $32y = 0$, so we shall complete the proof by showing that $16y \neq 0$ in a certain quotient ring R of $K^0(Z)$ (abusing notation slightly by writing y for the image of y in R). Indeed, we obtain R by adjoining, to $K^0(Z)$, the relations $\delta_3^+ = \delta_3^- =: d$ (whence $\delta_3 = 2d$), and $d^2 = yd = 0$. One easily checks that the resulting ring is

$$R := \mathbb{Z}[y,d]/\langle y^2 + 2y, 8d + 16y, d^2, yd\rangle.$$

To analyze this ring let us introduce an "intermediate" ring

$$T := \mathbb{Z}[y,d]/\langle y^2 + 2y, d^2, yd\rangle.$$

It is trivial to verify that as a ring T is the abelian group $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, with generators $1, y, d$ and multiplication given by 1 is the multiplicative identity, $y^2 = -2y$, $yd = d^2 = 0$.

Next, we clearly have $R \cong T/J$, where the ideal $J = \langle 8\delta + 16y\rangle \subset T$. Now let J' be the abelian subgroup of T generated by $8\delta + 16y$ and $32y$. It is clear that J' is an ideal so that $J \subset J'$. On the other hand it is easy to check that $J' \subseteq J$. Therefore we see that $J = J'$. It follows that as an abelian group, R is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, with generators $1, y, d$, modulo the abelian subgroup generated by $32y, 8d + 16y$. A simple exercise in elementary divisors now shows that (again as abelian groups) $R \cong \mathbb{Z}\oplus\mathbb{Z}/32\oplus\mathbb{Z}/8$ with respective generators $1, y, d + 2y$. ∎

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5. **Calculation of $K(W)$.** We first express W as $\mathrm{Spin}(9)/H$ where $H = \phi^{-1}\big(\mathrm{SO}(6)\times D\big)$, with $\phi\colon \mathrm{Spin}(9) \longrightarrow \mathrm{SO}(9)$ being the double covering; D denotes the diagonal subgroup of SO(3) and $\mathrm{SO}(6)\times D \subset \mathrm{SO}(6)\times\mathrm{SO}(3)\subset \mathrm{SO}(9)$. We let $Q = \phi^{-1}(D)$, $G = \mathrm{Spin}(9)$. We wish to apply the Hodgkin Spectral Sequence to compute $K^*(W)$.

5.1. *Computation of* RH. First note that $D \cong \mathbb{Z}/2\oplus\mathbb{Z}/2$. Denoting the standard basis of $\mathbb{R}^9$ by $e_1,\ldots,e_9$, it is easy to check that $Q = \{\pm 1, \pm e_7e_8, \pm e_8e_9, \pm e_7e_9\}$. One has

$$(e_7e_8)^2 = (e_7e_9)^2 = (e_8e_9)^2 = -1,\quad (e_7e_8)(e_8e_9) = e_9e_7,\quad etc.$$

Hence Q is isomorphic to the quaternion group $\{\pm1,\pm i,\pm j,\pm k\}$ of order 8. The calculation of RQ, its complex representation ring, is a fairly routine exercise in the use of characters (*cf.* [5], pp. 22–23); nevertheless it will be useful for us to give a few details. Clearly Q has an irreducible representation of dimension 2 over $\mathbb{C}$, obtained from the quaternion algebra $\mathbb{H}$. More precisely, we regard $\mathbb{H}$ as the left $\mathbb{C}$-vector space $V = \mathbb{C}\oplus\mathbb{C}$ where $(z_1,z_2)\in V$ is identified with z_1+z_2j, $z_1,z_2\in\mathbb{C}$. Now V becomes a complex Q-module where

$$e_7e_8\cdot(z_1,z_2) = (z_1i, -z_2i)\quad e_8e_9\cdot(z_1,z_2) = (-z_2,z_1).$$

We shall denote the class of V in RQ by q. Since $Q/\{\pm1\}\xrightarrow{\cong} D\cong \mathbb{Z}/2\oplus\mathbb{Z}/2$, we obtain (irreducible) 1 dimensional representations of Q induced from D on which $-1\in Q$ acts as the identity. They are the trivial representation 1, and X,Y,Z where e_7e_8 acts on X as -1 and e_8e_9 acts as 1 on X. On Y, e_7e_8 acts as 1, and e_8e_9 acts as -1. On Z both e_7e_8 and e_8e_9 act as -1. We shall denote their class in RQ by $1,x,y,z$ respectively. Clearly one has $x^2=y^2=z^2=1$, and $xy=z$. Using characters one readily finds that $q^2 = (1+x)(1+y)$ in RQ. The basic facts about representations of finite groups now show that $\mathrm{RQ} = \mathbb{Z}[x,y,q]/\sim$, the relations being

(i) $x^2=y^2=1$,
(ii) $q^2=(1+x)(1+y)$,
(iii) $qx=q=qy$.

To compute RH first note that $H\cong \mathrm{Spin}(6)\times_{\mathbb{Z}/2}Q$. Indeed one has the multiplication map $\mu\colon\mathrm{Spin}(6)\times Q\longrightarrow\mathrm{Spin}(9)$, which is a homomorphism of groups because $a\cdot b=b\cdot a$ for every $a\in\mathrm{Spin}(6)$ and $b\in Q$. It is obvious that $H=\mathrm{Im}\,\mu$ and that $\mathrm{Ker}\mu = \{(1,1),(-1,-1)\}$. It follows that $H\cong\mathrm{Spin}(6)\times_{\mathbb{Z}/2}Q$, where $\mathbb{Z}/2$ acts on $\mathrm{Spin}(6)\times Q$ via the involution $(a,b)\longmapsto(-a,-b)$. Therefore

$$\begin{aligned}\mathrm{RH} &= \Big(R\big(\mathrm{Spin}(6)\times Q\big)\Big)^{\mathbb{Z}/2}\\ &= \big(R\mathrm{Spin}(6)\otimes\mathrm{RQ}\big)^{\mathbb{Z}/2},\end{aligned}$$

and one also has injections $\mathrm{Spin}(6)\hookrightarrow H$, $Q\hookrightarrow H$.

LEMMA 11. *Let* $\theta\colon\mathrm{Spin}(6)\times Q\longrightarrow\mathrm{Spin}(6)\times Q$ *denote the involution* $\theta(a,b) = (-a,-b)$. *Then* θ *induces the automorphism* $\theta^{\#}\colon R\mathrm{Spin}(6)\otimes\mathrm{RQ}\longrightarrow R\mathrm{Spin}(6)\otimes\mathrm{RQ} = \mathbb{Z}[p_1,\Delta_3^+,\Delta_3^-,x,y,q]/\sim$, *where*

(i) $\theta^{\#}(p_1) = p_1$,
(ii) $\theta^{\#}(\Delta_3^{\pm}) = -\Delta_3^{\pm}$,
(iii) $\theta^{\#}(x) = x, \quad \theta^{\#}(y) = y$,
(iv) $\theta^{\#}(q) = -q$.

PROOF. We omit the proof, which involves a straightforward verification.

COROLLARY 12.

$$\begin{aligned}\mathrm{RH} &\cong \mathbb{Z}\big[p_1, p_2, (\Delta_3^+)^2, (\Delta_3^-)^2, \Delta_3^+ q, \Delta_3^- q, x, y\big]/\sim\\ &\cong \mathbb{Z}[p_1, p_2, p_3]\big[(\Delta_3^+)^2, \Delta_3^+ q, \Delta_3^- q, x, y\big]/\sim .\end{aligned}$$

The relations are, when expressed over the polynomial algebra $\mathbb{Z}[p_1, p_2, p_3]$, *as follows:*
(i) $x^2 = y^2 = 1$,
(ii) $(\Delta_3^+ q)^2 = (\Delta_3^+)^2 \cdot (1+x)(1+y)$,
(iii) $(\Delta_3^- q)^2 = \big(p_3 + 2p_2 + 8p_1 + 32 - (\Delta_3^+)^2\big) \cdot (1+x)(1+y)$,
(iv) $(\Delta_3^+ q)(\Delta_3^- q) = (p_2 + 4p_1 + 16) \cdot (1+x)(1+y)$,
(v) $(\Delta_3^+ q)x = (\Delta_3^+ q)y = \Delta_3^+ q$,
(vi) $(\Delta_3^- q)x = (\Delta_3^- q)y = \Delta_3^- q$,
(vii) $(\Delta_3^+)^4 = (\Delta_3^+)^2(p_3 + 2p_2 + 8p_1 + 32) - (p_2 + 4p_1 + 16)^2$,
(viii) $(\Delta_3^+)^2(\Delta_3^- q) = (\Delta_3^+ q) \cdot (p_2 + 4p_1 + 16)$,
(ix) $(\Delta_3^+)^2(\Delta_3^+ q) = p_3(\Delta_3^+ q) + (2\Delta_3^+ q - \Delta_3^- q) \cdot (p_2 + 4p_1 + 16)$.

We caution the reader that $\Delta_3^+ q$ and $\Delta_3^- q$ are not a product in RH, indeed q is not in RH although q^2 is in RH.

5.2. The restriction map $j^{\#}\colon R\mathrm{Spin}(9) \longrightarrow \mathrm{RH}$.

Recall that $R\mathrm{Spin}(9) = \mathbb{Z}[P_1, P_2, P_3, \Delta_4]$. It will be useful to define the class $\omega := x + y + xy - 3 = q^2 - 4 \in \mathrm{RH}$. Note that $\omega^2 = -4\omega$ and hence $\omega^n = (-4)^{n-1}\omega$ for any integer $n > 1$.

LEMMA 13. *The restriction map* $j^{\#}\colon R\mathrm{Spin}(9) \longrightarrow \mathrm{RH}$ *is given by*
(i) $j^{\#}(P_1) = p_1 + \omega =: p_1'$,
(ii) $j^{\#}(P_2) = p_2 + \omega p_1 =: p_2'$,
(iii) $j^{\#}(P_3) = p_3 + \omega p_2 =: p_3'$,
(iv) $j^{\#}(P_4) = p_3\omega$,
(v) $j^{\#}(\Delta_4) = \Delta_3 \cdot q = \Delta_3^+ q + \Delta_3^- q$.

PROOF. Let u_i, u_i^{-1} have their usual meaning as elements of the representation ring of the standard reference torus $\mathbb{T}^4$, so that the P_i are the elementary symmetric polynomials in

$$u_1^2 + u_1^{-2} - 2, \ldots, u_4^2 + u_4^{-2} - 2.$$

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Note that $u_4^2+u_4^{-2}-2$ is "concentrated" on the subgroup $S = \{\cos 2\pi\theta + \sin 2\pi\theta \cdot e_7e_8 : 0 \leq \theta \leq 1\} \subset T^4$, the 'standard' maximal torus of $\mathrm{Spin}\,(9)$. Also, the subgroup S is the standard maximal torus of $\mathrm{Spin}\,(3) \subset \mathrm{Spin}\,(6) \times_{\mathbb{Z}/2} \mathrm{Spin}\,(3)$ which is contained in $\mathrm{Spin}\,(9)$ and based on the last three coordinates, $e_7, e_8, e_9 \in \mathbb{R}^9$. Now

$$\begin{aligned}(u_4^2+u_4^{-2}-2)(e_7e_8) &= (u_4^2+u_4^{-2}-2)\left(\cos\left(2\pi\cdot\frac{1}{4}\right)+\sin\left(2\pi\cdot\frac{1}{4}\right)e_7e_8\right)\\ &= e^{2\pi i\cdot(\frac{1}{2})}+e^{2\pi i(-\frac{1}{2})}-2\\ &= -4.\end{aligned}$$

Also,

$$\begin{aligned}(q^2-4)(e_7e_8) &= (x+y+xy-3)(e_7e_8)\\ &= -1+1-1-3=-4.\end{aligned}$$

In $\mathrm{Spin}\,(3) \subset \mathrm{Spin}\,(9)$, the element e_8e_9 is conjugate to $e_7e_8 \in S$ (as S is the maximal torus of $\mathrm{Spin}\,(3)$). It follows that $(u_4^2+u_4^{-2}-2)$, which represents an element of $R\mathrm{Spin}\,(3)$, assumes the same value at e_8e_9 as at e_7e_8. Hence

$$(u_4^2+u_4^{-2}-2)(e_8e_9) = (q^2-4)(e_8e_9) = -4.$$

Also $(u_4^2+u_4^{-2}-2)(\pm 1) = 0 = (q^2-4)(\pm 1)$. Hence $(u_4^2+u_4^{-2}-2)$ corresponds to (q^2-4). Now Lemma 13(i)–(iv) follows immediately from this.

To establish (v), one first proceeds as above to show that $u_4+u_4^{-1} = 1\otimes\Delta_1 \in R\big(\mathrm{Spin}\,(6)\times\mathrm{Spin}\,(3)\big)$ restricts to $1\otimes q \in R\big(\mathrm{Spin}\,(6)\times Q\big)$; indeed both take values $0, 0, 2, -2$ respectively on $e_7e_8, e_8e_9, 1, -1$. The image of $\Delta_4 = \prod_{i=1}^4(u_i+u_i^{-1})$ under $j^\#$ can then be seen to be the element $\Delta_3^+q+\Delta_3^-q \in \mathrm{RH}$. This is because $\prod_{i=1}^4(u_i+u_i^{-1}) = \Delta_3\Delta_1 \in R\big(\mathrm{Spin}\,(6)\times\mathrm{Spin}\,(3)\big)$ maps to the element $(\Delta_3^++\Delta_3^-)q = \Delta_3^+q+\Delta_3^-q \in R\big(\mathrm{Spin}\,(6)\times Q\big)$ which is actually in RH. This gives (v). ∎

5.3. *Change of rings.* Let $\Lambda = Z[P_1, P_2, P_3]$, and let

$$A = R\mathrm{Spin}\,(9)/\langle\Lambda^+\rangle \cong \mathbb{Z}[\Delta_4].$$

Let $B = \mathrm{RH}/\langle P_1', P_2', P_3'\rangle$. Then B is an A-module.

THEOREM 14. $\mathrm{Tor}^\bullet_{RG}(\mathrm{RH}, \mathbb{Z}) \cong \mathrm{Tor}_A(B, \mathbb{Z})$, *as graded algebras.*

PROOF. We apply the change of rings theorem:

$$\mathrm{Tor}^\bullet_A\big(\mathrm{Tor}^\bullet_\Lambda(\mathrm{RH}, \mathbb{Z}), \mathbb{Z}\big) \Longrightarrow \mathrm{Tor}^\bullet_{RG}(\mathrm{RH}, \mathbb{Z}),$$

and show in fact that the spectral sequence (on the left) collapses (in fact lives in one line) and yields the required isomorphism.

The only non-trivial statement to verify is that RH is Λ-flat, and this is done in the next lemma (in fact it is Λ-free).

LEMMA 15. RH *is* Λ*-free.*

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PROOF. One has maps of Lie groups $\mathrm{Spin}(6) \hookrightarrow \mathrm{Spin}(7) \longrightarrow \mathrm{SO}(7)$, where the second map is the universal double covering map. As is well known, these maps induce monomorphisms $\mathrm{RSO}(7) \longrightarrow R\mathrm{Spin}(7) \longrightarrow R\mathrm{Spin}(6)$, which we regard as inclusions. Observe that $R\mathrm{Spin}(7) \cong \mathbb{Z}[p_1, p_2, \Delta_3] = \mathbb{Z}[p_1, p_2, p_3][\Delta_3]/\sim \cong \mathrm{RSO}(7)[\Delta_3]/\sim$ is free over $\mathrm{RSO}(7) \cong \mathbb{Z}[p_1, p_2, p_3]$, since Δ_3 satisfies a monic quadratic relation over $\mathbb{Z}[p_1, p_2, p_3]$ and this generates all relations in $R\mathrm{Spin}(7)$. Thus a basis for $R\mathrm{Spin}(7)$ over $\mathrm{RSO}(7)$ is $\{1, \Delta_3\}$. Similarly we note

$$R\mathrm{Spin}(6) \cong \mathbb{Z}[p_1, \Delta_3^+, \Delta_3^-] \cong \mathbb{Z}[p_1, p_2, \Delta_3][\Delta_3^+]/\sim \cong R\mathrm{Spin}(7)[\Delta_3^+]/\sim,$$

where all the relations in $R\mathrm{Spin}(6)$ over $R\mathrm{Spin}(7)$ arise from the single monic quadratic polynomial $(\Delta_3^+)^2 = (\Delta_3)(\Delta_3^+) - p_2 - 4p_1 - 16$. Again we conclude that $R\mathrm{Spin}(6)$ is free over $R\mathrm{Spin}(7)$ with basis $\{1, \Delta_3^+\}$. Combining the above we see that $R\mathrm{Spin}(6)$ is free over $\mathrm{RSO}(7)$ with basis $\{1, \Delta_3, \Delta_3^+, \Delta_3\Delta_3^+\}$. Using $\Delta_3^+ + \Delta_3^- = \Delta_3$, it is clear that an equivalent basis for $R\mathrm{Spin}(6)$ over $\mathrm{RSO}(7) \cong \mathbb{Z}[p_1, p_2, p_3]$ is $S := \{1, \Delta_3^+, \Delta_3^-, (\Delta_3^+)^2\}$.

Note that one has the relation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 0 & \omega & 1 \end{pmatrix} \begin{pmatrix} 1 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ p_1' \\ p_2' \\ p_3' \end{pmatrix}.$$

Now, observe that RH is generated as a $\mathbb{Z}[p_1, p_2, p_3]$-algebra by $(\Delta_3^+)^2, \Delta_3^+ q, \Delta_3^- q, x, y$. From Corollary 12, we see readily that $\{(\Delta_3^+)^2, (\Delta_3^+)^2 x, (\Delta_3^+)^2 y, (\Delta_3^+)^2 xy, \Delta_3^+ q, \Delta_3^- q, 1, x, y, xy\}$ is a generating set for RH as a module over $\mathbb{Z}[p_1, p_2, p_3]$. From the above relation between the p_1, p_2, p_3, and p_1', p_2', p_3', it follows easily that $\mathbb{Z}[p_1, p_2, p_3][x, y] = \mathbb{Z}[p_1', p_2', p_3'][x, y]$. Using this and the relations (i), (v), and (vi) of Corollary 12, one can show that the same set generates RH as a module over $\mathbb{Z}[p_1', p_2', p_3']$. We claim that in fact RH is a free $\mathbb{Z}[p_1', p_2', p_3']$-module with basis

$$\left\{(\Delta_3^+)^2, (\Delta_3^+)^2 x, (\Delta_3^+)^2 y, (\Delta_3^+)^2 xy, \Delta_3^+ q, \Delta_3^- q, 1, x, y, xy\right\}.$$

Suppose there exist elements $\alpha, \beta, \gamma, \delta, \varepsilon, \eta, f, g, h, k \in \mathbb{Z}[p_1', p_2', p_3']$ such that

$$(*) \qquad (f + gx + hy + kxy)(\Delta_3^+)^2 + \alpha + \beta x + \gamma y + \delta xy + \varepsilon \Delta_3^+ q + \eta \Delta_3^- q = 0 \text{ in RH.}$$

We must show that $\alpha = \beta = \gamma = \delta = \varepsilon = \eta = f = g = h = k = 0$.

Note that under the restriction map $\phi_1 \colon \mathrm{RH} \longrightarrow R\mathrm{Spin}(6)$, $x \longmapsto 1$, $y \longmapsto 1$, "$q \longmapsto 2$"(the quotes signifying a shorthand for $\Delta_3^+ q \longmapsto 2\Delta_3^+$, *etc.*). It follows that $\omega \longmapsto 0$, and consequently $\phi_1(p_i') = p_i$ for $1 \leq i \leq 3$. Since ϕ_1 is an algebra map, we see that for any polynomial $P(p') = P(p_1', p_2', p_3')$, $\phi_1\big(P(p')\big) = P(p_1, p_2, p_3) = P(p)$.

Applying ϕ_1 to $(*)$ we see that, in $R\mathrm{Spin}(6)$,

$$\big(f(p)+g(p)+h(p)+k(p)\big)(\Delta_3^+)^2 + \big(\alpha(p)+\beta(p)+\gamma(p)+\delta(p)\big) + 2\varepsilon(p)\Delta_3^+ + 2\eta(p)\Delta_3^- = 0.$$

Since S is a $\mathbb{Z}[p_1, p_2, p_3]$-basis for $R\mathrm{Spin}(6)$, it follows that $\varepsilon(p) = \eta(p) = 0$, and

$$(1a) \qquad f + g + h + k = 0, \quad \alpha + \beta + \gamma + \delta = 0$$

in $\mathbb{Z}[p_1, p_2, p_3]$, where we have written f for $f(p)$, *etc*. Since $\mathbb{Z}[p_1, p_2, p_3]$ is a polynomial algebra and hence a free object in the category of commutative rings with unit, $\varepsilon(p) = \eta(p) = 0$ implies that $\varepsilon(p') = \eta(p') = 0$.

Now let $\tilde{\phi}_x\colon \mathrm{RH} \longrightarrow R\mathrm{Spin}(6)$ be the ring homomorphism defined by $p_i \longmapsto p_i$, $(\Delta_3^+)^2 \longmapsto (\Delta_3^+)^2$, and setting $x = -1$, $y = +1$, "$q = 0$". Under this map, $p_1' \mapsto p_1 - 4, p_2' \mapsto p_2 - 4p_1, p_3' \mapsto p_3 - 4p_2$. Let $\rho\colon R\mathrm{Spin}(6) \longrightarrow R\mathrm{Spin}(6)$ be the abelian group homomorphism defined as follows: On the polynomial algebra $\mathbb{Z}[p_1, p_2, p_3]$, ρ is the ring homomorphism such that $p_1 \mapsto p_1 + 4, p_2 \mapsto p_2 + 4p_1 + 16, p_3 \mapsto p_3 + 4p_2 + 16p_1 + 64$. Then extend the map ρ to the whole of $R\mathrm{Spin}(6)$ by mapping each of the basis elements $1, \Delta_3^+, \Delta_3^-, (\Delta_3^+)^2$ to itself, so that $\rho(am) = \rho(a)\rho(m)$ for any $a \in \mathbb{Z}[p_1, p_2, p_3]$, and $m \in R\mathrm{Spin}(6)$. Finally, let $\phi_x = \rho \circ \tilde{\phi}_x$. We define ϕ_y, and ϕ_{xy} similarly. Note that for any $P(p') \in \mathbb{Z}[p_1', p_2', p_3']$, we have

$$\begin{aligned} \phi_x\big(P(p')\big) &= \rho\big(P(p_1 - 4, p_2 - 4p_1, p_3 - 4p_2)\big) \\ &= P\big(\rho(p_1 - 4), \rho(p_2 - 4p_1), \rho(p_3 - 4p_2)\big) = P(p) \in \mathbb{Z}[p_1, p_2, p_3]. \end{aligned}$$

Now using $\varepsilon = \eta = 0$ in $\mathbb{Z}[p_1', p_2', p_3']$, and applying ϕ_x to $(*)$ we obtain

$$(1b) \qquad f - g + h - k = 0, \qquad \alpha - \beta + \gamma - \delta = 0$$

in $\mathbb{Z}[p_1, p_2, p_3]$ (as before $f = f(p)$, *etc*.). Applying ϕ_y and ϕ_{xy}, we similarly obtain

$$(1c) \qquad f + g - h - k = 0, \qquad \alpha + \beta - \gamma - \delta = 0,$$

$$(1d) \qquad f - g - h + k = 0, \qquad \alpha - \beta - \gamma + \delta = 0.$$

$$\text{Therefore } A\begin{pmatrix} f \\ g \\ h \\ k \end{pmatrix} = O = A\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}, \text{ where } A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Since A is nonsingular, (in fact $AA^t = 4I$) we get $f = g = h = k = 0 = \alpha = \beta = \gamma = \delta$ in $\mathbb{Z}[p_1, p_2, p_3]$. Since $\mathbb{Z}[p_1, p_2, p_3]$ is a polynomial algebra, it follows that $f = g = h = k = 0 = \alpha = \beta = \gamma = \delta$ in $\mathbb{Z}[p_1', p_2', p_3']$. Hence RH is free over $\mathbb{Z}[p_1', p_2', p_3']$. ∎

5.4. *Calculation of* $\mathrm{Tor}_A^\bullet(B, \mathbb{Z})$. Let $\eta\colon A \longrightarrow B$ be the map induced by the restriction homomorphism $j^\#\colon R\mathrm{Spin}(9) \longrightarrow \mathrm{RH}$. One has $B = \mathbb{Z}[(\Delta_3^+)^2, \Delta_3^+ q, \Delta_3^- q, x, y]/\sim$ where the relations are as in Corollary 12 and with $p_1 = -\omega$, $p_2 = -4\omega$, $p_3 = -16\omega$. The map $\eta\colon A \cong \mathbb{Z}[\Delta_4] \longrightarrow B$ is given by $\eta(\Delta_4) = \Delta_3^+ q + \Delta_3^- q$. To compute $\mathrm{Tor}_A^\bullet(B, \mathbb{Z})$ we make use of the Koszul resolution.

$$0 \longrightarrow AD \xrightarrow{d_1} A \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

where d_1 is defined by $D \longmapsto \Delta_4 - 16$, and d_0 is the augmentation map.

Thus $\mathrm{Tor}_A^{\bullet}(B, \mathbb{Z})$ is the homology of the chain complex

$$0 \longrightarrow BD \longrightarrow B \longrightarrow 0.$$
$$D \longmapsto \Delta_3^+ q + \Delta_3^- q - 16.$$

H_0 is $B/\langle \Delta_3^+ q + \Delta_3^- q - 16\rangle \cong \mathbb{Z}[(\Delta_3^+)^2, \Delta_3^+ q, x, y]/\sim$ where the relations are those coming from Corollary 12 (with $p_1 = -\omega, p_2 = -4\omega, p_3 = -16\omega$), together with $\Delta_3^- q = 16 - \Delta_3^+ q$. We write them out explicitly:

(i) $x^2 = y^2 = 1$,

(ii) $(\Delta_3^+ q)^2 = (\Delta_3^+)^2(1+x)(1+y) = (\Delta_3^+)^2(\omega+4)$,

(iii) $(\Delta_3^+ q)x = (\Delta_3^+ q)y = \Delta_3^+ q$,

(iv) $(\Delta_3^+)^2(\Delta_3^+ q) = 32\Delta_3^+ q - (16 - \Delta_3^+ q)(16 - 8\omega)$
$= 32\Delta_3^+ q + 16\Delta_3^+ q - 8\Delta_3^+ q\omega - 16^2$
$= 48\Delta_3^+ q - 256$,

(v) $16\big((\Delta_3^+)^2 - 4\Delta_3^+ q + 16\big) = 0$,

(vi) $16x = 16y = 16, \quad 16\omega = 0$,

(vii) $\big((\Delta_3^+)^2 - 16\big)^2 = 0$.

PROOF. It is clear that $p_1 = -\omega$, $p_2 = \omega^2 = -4\omega$, $p_3 = -\omega^3 = -16\omega$ in H_0 and that, since $j^{\#}(\Delta_4) = \Delta_3^+ q + \Delta_3^- q$ in RH, one has $\Delta_3^+ q + \Delta_3^- q = 16$ in H_0.

Using $\Delta_3^+ q + \Delta_3^- q = 16$, and the relations $\Delta_3^+ qx = \Delta_3^+ q$, $\Delta_3^- qx = \Delta_3^- q$; we get $16x = \Delta_3^+ qx + \Delta_3^- qx = \Delta_3^+ q + \Delta_3^- q = 16$ in H_0. That is, $16x = 16$. Similarly $16y = 16$. It follows that $16\omega = 16(x+y+xy-3) = 0$. In particular $p_3 = 0$. We shall only verify relation (vii), the rest of them are similarly established.

By Corollary 12(vii) and substituting for p_1, p_2, p_3, we get

$$\begin{aligned}(\Delta_3^+)^4 &= (\Delta_3^+)^2(p_3 + 2p_2 + 8p_1 + 32) - (p_2 + 4p_1 + 16)^2 \\ &= (\Delta_3^+)^2(0 - 8\omega - 8\omega + 32) - (-4\omega - 4\omega + 16)^2 \\ &= 32(\Delta_3^+)^2 - (16)^2.\end{aligned}$$

Therefore $(\Delta_3^+)^4 - 32(\Delta_3^+)^2 + 16^2 = 0$, *i.e.*, $\big((\Delta_3^+)^2 - 16\big)^2 = 0$.

As in [2] we conclude that

$$K^0(W) = H_0 = B/\langle \Delta_3 q - 16\rangle \cong \mathbb{Z}\big[(\Delta_3^+)^2, \Delta_3^+ q, x, y\big]/\sim\,. \qquad \blacksquare$$

5.5. *Order of* ω.

PROPOSITION 16. $8\omega \neq 0$ *in* $K^0(W)$.

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PROOF. As in the proof of Proposition 10, we will produce a homomorphic image R of $K(W)$ and show that the image of 8ω is non-zero in R. Indeed we have an epimorphism

$$r: K(W) \longrightarrow R := (\mathbb{Z}/16)[u, v]/\langle u^2 + 2u, v^2 + v\rangle,$$

where $r\big((\Delta_3^+)^2\big) = 0$, $r(\Delta_3^+ q) = 0$, $r(x) = u + 1$, $r(y) = v + 1$. It is a routine matter to verify that r respects the relations in $K(W)$ and is therefore well-defined. Also note that $r(\omega) = uv + 2u + 2v$, so to complete the proof it suffices to show that $8uv \neq 0$. This is accomplished by representing R as a subring of the ring of 4×4 matrices over $\mathbb{Z}/16$. Indeed let

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

and let

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}.$$

Then one checks that $U^2 + 2U = 0$, $V^2 + 2V = 0$, and

$$VU = UV = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 4 \end{pmatrix}.$$

It follows that one obtains a representation of R by $u \mapsto U, v \mapsto V$. Clearly $8UV \neq 0$, completing the proof. ∎

6. **Proof of Theorem 1.** We are now ready to prove Theorem 1, using the information obtained about the spaces X, Y, Z, W and the inclusion method mentioned in the Introduction. To apply this method, note that any flag manifold in Theorem 1(i) is fibred by $X = G(1 \mid 3, 1)$, *i.e*, the map $G(1, \ldots, 1 \mid 3, 1) \longrightarrow \tilde{G}(5, 1, \ldots, 1)$ obtained by combining the last three orthogonal subspaces of any p.o. flag $x \in G(1, \ldots, 1 \mid 3, 1)$ to a single (oriented) 5-dimensional subspace, has fibre X. As mentioned in the Introduction, the resulting inclusion $X \hookrightarrow G(1, \ldots, 1 \mid 3, 1)$ has trivial normal bundle (*cf.* [10], p. 456, or simply note that this is clearly true for the fibre inclusion of any locally trivial smooth fibration of smooth manifolds). Now from Theorem 4(i) and Theorem 7(i), it follows that X is not stably parallelizable. Hence none of the flag manifolds in 1(i) can be stably parallelizable. Parts (ii) and (iii) are proved exactly the same way, using Theorem 4(ii), (iii), Theorem 7(ii), and Proposition 10.

To prove part (iv), as in the other cases, we need only show W is not stably parallelizable. Using Theorem 4(iv),

$$\langle \tau W\rangle = [\tau W] - 21 = 8\big([\xi_2 \oplus \xi_3 \oplus \xi_4] - 3\big).$$

Next observe that the complexification of $\xi_2 \oplus \xi_3 \oplus \xi_4$ represents the element $x + y + xy = \omega + 3$ in $K(W)$ by the α-construction. It follows, in $K^0(W)$ (indeed in $\tilde{K}^0(W)$), that $\langle c(\tau W)\rangle = 8\omega$, and the proof is now a direct consequence of Proposition 16. ∎

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