PRIME RINGS WITH FINITENESS PROPERTIES
ON ONE-SIDED IDEALS

TSIU-KWEN LEE

Department of Mathematics, National Taiwan University,
Taipei 106, Taiwan (tklee@math.ntu.edu.tw)

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Abstract Let $R$ be a prime ring with extended centroid $C$, $\rho$ a non-zero right ideal of $R$ and let $f(X_1,\ldots,X_t)$ be a polynomial, having no constant term, over $C$. Suppose that $f(X_1,\ldots,X_t)$ is not central-valued on $RC$. We denote by $f(\rho)$ the additive subgroup of $RC$ generated by all elements $f(x_1,\ldots,x_t)$ for $x_i \in \rho$. The main goals of this note are to prove two results concerning the extension properties of finiteness conditions as follows.

(I) If $f(\rho)$ spans a non-zero finite-dimensional $C$-subspace of $RC$, then $\dim_C RC$ is finite.

(II) If $f(\rho) \neq 0$ and is a finite set, then $R$ itself is a finite ring.

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1. Introduction and results

Throughout this note, $R$ always denotes a prime ring with extended centroid $C$ and $\rho$ a non-zero right ideal of $R$. The goal of this note is to study the extension properties of certain finiteness conditions from one-sided ideals of a prime ring to the whole prime ring. We were motivated by two elementary observations: (I) if $\dim_C RC$ is finite, then so is $\dim_C RC$; and (II) if $\rho$ is a finite subset of $R$, then so is $R$. For (I) see [3, Lemma 1]. For (II), since, by the primeness of $R$, $R$ can be embedded in $\text{End}(\rho,+)$ via right multiplications and $\text{End}(\rho,+)$ is a finite set, $R$ itself is a finite ring. In a recent paper Bell proved the following theorem.

Theorem. Suppose that $\rho$ is of finite index in $R$ and $[\rho,\rho]$ is finite. Then $R$ is either finite or commutative (see [1, Theorem 2.2]).

Since, in Bell’s Theorem, $[\rho,\rho]$ contains all elements $xy - yx$ for all $x, y \in \rho$, we want to extend these results above to more generalized forms. In particular, we shall see that, in Bell’s Theorem, the assumption that $\rho$ is of finite index in $R$ is superfluous. Our point of view in this note is different from that of [1]. To state our results we require some notation. For a polynomial $f(X_1,\ldots,X_t)$ over $C$, where the $X_i$ are non-commuting indeterminates,
we denote by \( f(\rho) \) the additive subgroup of \( RC \) generated by all elements \( f(x_1, \ldots, x_t) \) for \( x_i \in \rho \). For \( x, y \in R \), set \([x, y]_0 = x, [x, y]_1 = [x, y] = xy - yx \) and \([x, y]_k = [[x, y]_{k-1}, y] \) for \( k > 1 \). Also, \([\rho, \rho]_k \) denotes the additive subgroup of \( R \) generated by all elements \([x, y]_k \) for \( x, y \in \rho \). We are now ready to state our results.

**Theorem 1.1.** Let \( R \) be a prime ring with extended centroid \( C \), \( \rho \) a non-zero right ideal of \( R \) and \( f(X_1, \ldots, X_t) \) a polynomial, having no constant term, over \( C \). Suppose that \( f(X_1, \ldots, X_t) \) is not central-valued on \( RC \).

(I) Suppose that \( f(\rho) \) spans a non-zero finite-dimensional \( C \)-subspace of \( RC \). Then \( \dim_C RC \) is finite.

(II) Suppose that \( f(\rho) \neq 0 \) and is a finite set. Then \( R \) itself is a finite ring.

**Corollary 1.2.** Let \( R \) be a prime ring with extended centroid \( C \), \( \rho \) a non-zero right ideal of \( R \), and \( k \) a non-negative integer.

(I) Suppose that \([\rho, \rho]_k \) spans a finite-dimensional \( C \)-subspace of \( RC \). Then \( \dim_C RC \) is finite.

(II) Suppose that \([\rho, \rho]_k \) is a finite set. Then \( R \) is either commutative or finite.

**Proof.** In view of Theorem 1.1, it suffices to prove that \([\rho, \rho]_k \nsubseteq C \) unless \( R \) is commutative. Indeed, suppose that \([\rho, \rho]_k \subseteq C \). Then \([x, y]_{k+1} = 0 \) for all \( x, y \in \rho \). In view of [5, Lemma 1], \( R \) is commutative. This proves the corollary.

We remark that Corollary 1.2 (II) gives a generalization of Bell’s Theorem [1, Theorem 2.2].

**2. Proof of Theorem 1.1**

To prove Theorem 1.1 we need the following two theorems [2, Theorem 1 (II) (i) and Theorem 2]. Recall that a prime ring \( R \) is called centrally closed if \( R = RC \) and that a right ideal of a ring is called a polynomial identity (PI) right ideal if the right ideal is itself a PI-ring. We write \( M_2(GF(2)) \) to stand for the 2 by 2 matrix ring over \( GF(2) \), the Galois field of two elements.

**Theorem 2.1.** Let \( R \) be a centrally closed prime \( C \)-algebra and \( f(X_1, \ldots, X_t) \) a non-zero polynomial over \( C \). Suppose that \( \rho \) is a PI right ideal of \( R \) such that \( f(\rho) \neq 0 \). Then there exists an idempotent \( e \) in the socle of \( R \) such that \( \rho = eR \) and \( eR(1-e) \subseteq f(\rho) \).

**Theorem 2.2.** Let \( R \) be a prime ring with extended centroid \( C \) and \( I \) a non-zero ideal of \( R \). Suppose that \( f(X_1, \ldots, X_t) \) is a polynomial over \( C \) which is not central-valued on \( RC \). Then \([M, R] \subseteq f(I) \) for some non-zero ideal \( M \) of \( R \), except when \( R \cong M_2(GF(2)) \) and \( f(R) = \{0, e_{12} + e_{21}, 1 + e_{12}, 1 + e_{21}\} \) or \( \{0, 1, e_{11} + e_{12} + e_{21}, e_{22} + e_{12} + e_{21}\} \).
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From now on, $R$ is always a prime ring with extended centroid $C$, $\rho$ a non-zero right ideal and $f(X_1, \ldots, X_t)$ a non-zero polynomial, having no constant term, over $C$. We denote by $d_n(X_1, \ldots, X_n; Y_1, \ldots, Y_{n-1})$ the Capelli polynomial of degree $2n - 1$, that is

$$d_n(X_1, \ldots, X_n; Y_1, \ldots, Y_{n-1}) = \sum_{\sigma \in S_n} (-1)^{\sigma} X_{\sigma(1)} Y_1 X_{\sigma(2)} \cdots Y_{n-1} X_{\sigma(n)},$$

where $X_i$ and $Y_j$ are non-commuting indeterminates. To apply Theorem 2.1 to our case, we require the following lemma.

**Lemma 2.3.** Suppose that $f(\rho) \neq 0$. Then

(I) if the additive subgroup $f(\rho)$ spans a finite-dimensional $C$-space, then $f(\rho)C = f(\rho)C$; and

(II) if $f(\rho)$ is finite, then so are both $C$ and $f(\rho)C$.

**Proof.** (I) Suppose that $\dim_C f(\rho)C = n < \infty$. Then

$$d_{n+1}(f(x_{11}, \ldots, x_{1t}), \ldots, f(x_{n+1}, \ldots, x_{n+1t}); y_1, \ldots, y_n) = 0$$

(2.1)

for all $x_{ij} \in \rho$ and all $y_i \in R$. Since $\rho$ and $\rho C$ satisfy the same generalized polynomial identities (GPIs) [4, Lemma 2], equation (2.1) still holds for all $x_{ij} \in \rho C$ and all $y_i \in RC$. In view of [7, Theorem 7.6.16], for $x_{ij} \in \rho C$ the $n + 1$ elements $f(x_{11}, \ldots, x_{1t}), \ldots, f(x_{n+1}, \ldots, x_{n+1t})$ are $C$-dependent and so $\dim_C f(\rho)C \leq n$. But since $f(\rho)C \leq f(\rho)C$, we conclude that $f(\rho)C = f(\rho)C$ as asserted.

(II) Suppose that $f(\rho)$ is a finite set. Write

$$f(X_1, \ldots, X_t) = \sum_{i=1}^{m} f_i(X_1, \ldots, X_t),$$

(2.2)

where $f_i(X_1, \ldots, X_t)$ is the homogeneous part of $f(X_1, \ldots, X_t)$ of degree $i$ for $1 \leq i \leq m$. Suppose on the contrary that $C$ is infinite. Choose $m$ distinct elements $\beta_1, \ldots, \beta_m$ in $C$ and then a non-zero ideal $I$ of $R$ such that $\beta_i I \subseteq R$. Then $\beta_i \rho I \subseteq \rho$ for each $i$. Applying a standard determinant argument to equation (2.2) we see that $f_i(\rho I)$ is finite for each $i$. Since $f(\rho) \neq 0$, we see that $f_j(\rho) \neq 0$ for some $j$. Thus we may assume from the start that $f(X_1, \ldots, X_t)$ is homogeneous of degree $m \geq 1$. Suppose that $f(\rho)$ consists of $\ell$ elements. Since $C$ is an infinite field, we can choose $\mu_1, \ldots, \mu_{\ell+1}$ in $C$ such that

$$\mu_i^m \neq \mu_j^m$$

for $i \neq j$. Let $J$ be a non-zero ideal of $R$ such that $\mu_i J \subseteq R$ for each $i$. In view of [4, Lemma 2], there exist $x_i \in \rho J$, $1 \leq i \leq t$, such that $f(x_1, \ldots, x_t) \neq 0$. Now, we see that $\mu_i^m f(x_1, \ldots, x_t) = f(\mu x_1, \ldots, \mu x_t) \in f(\rho)$ for each $i$. This derives a contradiction, as the set $\{\mu_i^m f(x_1, \ldots, x_t) | 1 \leq i \leq \ell + 1\}$ consists of $\ell + 1$ elements. Thus $C$ is a finite field. By (I) we see that $f(\rho)C = f(\rho)C$. But since $f(\rho)$ and $C$ are finite sets, this implies that $f(\rho)C$ is a finite set, proving (II). □
Lemma 2.4. Let $R$ be a centrally closed prime $C$-algebra with non-zero socle $H$. Suppose that $e$ is a non-trivial idempotent in $H$.

(I) If $\dim_C eR(1-e) < \infty$, then $R$ is finite dimensional over $C$.

(II) If $eR(1-e)$ is a finite set, then $R$ is a finite ring.

Proof. (I) Suppose that $\dim_C eR(1-e) = n < \infty$. Note that $e$ and $1-e$ are $C$-independent. Thus we see that

$$S_{n+1}(eX_1(1-e)Y, \ldots, eX_{n+1}(1-e)Y)$$

is a non-trivial GPI for $R$, where $S_{n+1}(X_1, \ldots, X_{n+1})$ is the standard polynomial of degree $n + 1$. Since $R$ is centrally closed, it follows from Martindale’s Theorem (6) that $R$ is a strongly primitive ring. By assumption, $e \in H$ and so $\dim_C eRe < \infty$. Thus $\dim_C eR = \dim_C eRe + \dim_C eR(1-e) < \infty$. That is, $R$ contains a non-zero right ideal $eR$, which is finite dimensional over $C$. In view of [3, Lemma 1], $\dim_C R < \infty$ follows. This proves (I).

(II) Since $eR(1-e) \neq 0$, we choose $x_0 \in R$ such that $ex_0(1-e) \neq 0$. But $Cex_0(1-e) \subseteq eR(1-e)$, so the finiteness of $eR(1-e)$ implies that $C$ is a finite field. By (I), $R$ is finite dimensional over $C$ and so $R$ is a finite ring, proving (II).

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (I) Suppose that $f(\rho)$ spans a non-zero finite-dimensional $C$-subspace of $RC$. It follows from Lemma 2.3 (I) that $0 \neq f(\rho C)$ is finite dimensional over $C$. Clearly, $\rho C$ is a PI right ideal of $RC$. Note that $RC$ is a centrally closed prime $C$-algebra. In view of Theorem 2.1, there exists an idempotent $e$ in the socle of $RC$ such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$ as $f(\rho C) \neq 0$. If $e \neq 1$, applying Lemma 2.4 we conclude that $\dim_C RC$ is finite. Suppose next that $e = 1$. Then $\rho C = RC$. Since $f(X_1, \ldots, X_1)$ is not central-valued on $RC$, by Theorem 2.2 there exists a non-zero ideal $I$ of $RC$ such that $[I, RC] \subseteq f(\rho C)$ unless $R = RC \cong M_2(GF(2))$. There is nothing to prove for the exceptional case. Therefore, we may assume that $[I, RC] \subseteq f(\rho C)$ and so $\dim_C [I, RC] < \infty$, implying that $R$ itself is a PI-ring. Applying Posner’s Theorem yields that $\dim_C RC$ is finite.

(II) Suppose that $f(\rho) \neq 0$ and is a finite set. By Lemma 2.3, $f(\rho C)$ is a finite set. Clearly, in this case $\rho C$ must be a PI right ideal of $R$. In view of Theorem 2.1, there exists an idempotent $e$ in the socle of $RC$ such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$. If $e \neq 1$, Lemma 2.4 implies that $R$ is a finite ring. Otherwise, $e = 1$ follows and so $\rho C = RC$. As in the argument given in (I), we may assume that $R \neq M_2(GF(2))$. By Theorem 2.2 there exists a non-zero ideal $I$ of $RC$ such that $[I, RC] \subseteq f(\rho C)$ and so $[I, RC]$ is a finite set. But since $R$ is not commutative, $[I, RC] \neq 0$ and so $C$ must be finite. Now, $RC$ is a PI-ring with $C$ a finite field. It is now clear that $R$ itself is a finite ring, proving (II).
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References