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EQUIVALENTS OF EKELAND'S PRINCIPLE

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In this note we present a new result which is equivalent to the celebrated Ekeland's variational principle, and a set of implications which includes a new non-convex minimisation principle due to Takahashi.

It is the purpose of this note to give a result which is equivalent to Ekeland's variational principle, to a fixed point theorem of Caristi and Kirk, and to a recent result of Takahashi about the existence of certain minima. The motivation for still another equivalent stems from the observation that the present proposition gives the previous results in the most direct way and seems to be somehow the "barycenter" of these results [**12**]. The previous results have been slightly extended to obtain a uniform level of generality.

Let (V, d) be a complete metric space. Let $f : V \times V \to (-\infty, +\infty]$ be a function which is lower semicontinuous in the second argument and satisfies

$$f(v,v) = 0 \text{ for all } v \in V, \tag{1}$$

$$f(u,v) \leqslant f(u,w) + f(w,v) \text{ for all } u,v,w \in V. \tag{2}$$

Assume that there exists $v_0 \in V$ such that

$$\inf_{v \in V} f(v_0, v) > -\infty. \tag{3}$$

Let

$$S_0 := \{v \in V \mid f(v_0,v) + d(v_0,v) \leqslant 0\}. \tag{4}$$

From (1) it follows that $v_0 \in S_0 \neq \emptyset$.

Under these specifications the following results are true:

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Theorem 1. (Ekeland [5]). *There exists* $v^* \in S_0$ *such that* $f(v^*,v)+d(v^*,v) > 0$ *for all* $v \in V$, $v \neq v^*$.

Theorem 2. (Takahashi [11]). *Assume that*

$$(5) \qquad \begin{cases} \text{for every } \overline{v} \in S_0 \text{ with } \inf\limits_{v\in V} f(\overline{v},v) < 0 \text{ there exists} \\ v \in V \text{ such that } v \neq \overline{v} \text{ and } f(\overline{v},v) + d(\overline{v},v) \leqslant 0. \end{cases}$$

Then there exists $v^* \in S_0$ *such that* $f(v^*,v) \geqslant 0$ *for all* $v \in V$.

Theorem 3. (Caristi - Kirk [2]). *Let* $T : V \rightrightarrows V$ *be a multivalued mapping such that*

$$(6) \qquad \begin{cases} \text{for every } \overline{v} \in S_0 \text{ there exists} \\ v \in T(\overline{v}) \text{ satisfying } f(\overline{v},v) + d(\overline{v},v) \leqslant 0. \end{cases}$$

Then there exists $v^* \in S_0$ *such that* $v^* \in T(v^*)$.

We add to these:

Theorem 4. *Let* $\Psi \subset V$ *have the property that*

$$(7) \qquad \begin{cases} \text{for every } \overline{v} \in S_0 \setminus \Psi \text{ there exists} \\ v \in V \text{ such that } v \neq \overline{v} \text{ and } f(\overline{v},v) + d(\overline{v},v) \leqslant 0. \end{cases}$$

Then there exists $v^* \in S_0 \cap \Psi$.

For the sake of completeness we give a self-contained proof of Theorem 4 which is similar to the proof of Ekeland's Theorem given in [4, p.16], [9].

Proof of Theorem 4: We shall construct inductively a sequence of points $v_n \in V$ $(n = 0, 1, \ldots)$. To each v_n we adjoin the closed set

$$S_n := \{v \in V \mid f(v_n,v) + d(v_n,v) \leqslant 0\},$$

and define the number

$$\gamma_n := \inf_{v \in S_n} f(v_n,v).$$

From (1) it follows that $v_n \in S_n \neq \emptyset$, and that $\gamma_n \leqslant 0$. The starting point v_0 is the same as in (3). S_0 coincides then with the set introduced in (4), and from (3) it follows that $\gamma_0 \geqslant \inf\limits_{v\in V} f(v_0,v) > -\infty$. Let $n \geqslant 1$ and suppose that v_{n-1} with $\gamma_{n-1} > -\infty$ is known. Then choose $v_n \in S_{n-1}$ such that

$$(8) \qquad f(v_{n-1},v_n) \leqslant \gamma_{n-1} + \frac{1}{n}\,.$$

Using (2) and the fact that $v_n \in S_{n-1}$, it follows readily that $S_{n-1} \supset S_n$. As a result, by virtue of (2) and (8) we obtain:

$$\begin{aligned}\gamma_n = \inf_{v \in S_n} f(v_n, v) &\geqslant \inf_{v \in S_n} \left(f(v_{n-1}, v) - f(v_{n-1}, v_n)\right)\\ &\geqslant \inf_{v \in S_{n-1}} f(v_{n-1}, v) - f(v_{n-1}, v_n)\\ &= \gamma_{n-1} - f(v_{n-1}, v_n) \geqslant -\frac{1}{n}.\end{aligned}$$

If $v \in S_n$, then $d(v_n, v) \leqslant -f(v_n, v) \leqslant -\gamma_n \leqslant 1/n$. This implies that the diameter of the sets S_n tends to zero. Moreover for all $k \geqslant n$ one has $v_k \in S_k \subset S_n$, hence $d(v_n, v_k) \leqslant 1/n$. Thus the sequence $\{v_n\}$ is Cauchy and tends to a limiting point $v^* \in V$. It is clear that $v^* \in \cap_{n=0}^{\infty} S_n$. Since the diameter of the sets S_n tends to zero, it follows that $\cap_{n=0}^{\infty} S_n = \{v^*\}$. We claim that $v^* \in \Psi$. If this was not true, then from (7) there would exist $v \neq v^*$ with $f(v^*, v) + d(v^*, v) \leqslant 0$. Since $v^* \in \cap_{n=0}^{\infty} S_n$, we have $f(v_n, v^*) + d(v_n, v^*) \leqslant 0$ for all n. Using (2) we would then obtain $f(v_n, v) + d(v_n, v) \leqslant 0$ for all n, so that $v \in \cap_{n=0}^{\infty} S_n$. This would contradict $v \neq v^*$. Thus $v^* \in \Psi$. ∎

Theorem 5. *Theorems 1 through 4 are equivalent.*

Proof:

(1) "Theorem 4 ⇒ Theorem 1".

Let Theorem 4 hold. For all $\overline{v} \in V$ let $\Gamma(\overline{v}) := \{v \in V \mid v \neq \overline{v}, f(\overline{v}, v) + d(\overline{v}, v) \leqslant 0\}$. Choose $\Psi := \{\overline{v} \in V \mid \Gamma(\overline{v}) = \emptyset\}$. If $\overline{v} \notin \Psi$, then from the definition of Ψ there exists $v \in \Gamma(\overline{v})$. Hence (7) is satisfied, and by Theorem 4 there exists $v^* \in S_0 \cap \Psi$. Then $\Gamma(v^*) = \emptyset$, that is, $f(v^*, v) + d(v^*, v) > 0$ for all $v \neq v^*$. Hence Theorem 1 holds.

(2) "Theorem 4 ⇒ Theorem 2".

Suppose that both Theorem 4 and the hypothesis of Theorem 2 hold. Choose $\Psi := \{\overline{v} \in V \mid \inf_{v \in V} f(\overline{v}, v) \geqslant 0\}$. Then (7) follows from (5), and Theorem 4 furnishes some $v^* \in S_0 \cap \Psi$. From the definition of Ψ follows then $\inf_{v \in V} f(v^*, v) \geqslant 0$. Hence Theorem 2 holds.

(3) "Theorem 4 ⇒ Theorem 3".

Suppose that both Theorem 4 and the hypothesis of Theorem 3 hold. Choose $\Psi := \{\overline{v} \in V \mid \overline{v} \in T(\overline{v})\}$. Then (7) follows from (6), and Theorem 4 furnishes some $v^* \in S_0 \cap \Psi$ which, from the definition of Ψ, necessarily belongs to $T(v^*)$. Hence Theorem 3 holds.

(4) "Theorem 1 ⇒ Theorem 4".

Let Theorem 1 and the hypothesis of Theorem 4 hold. Theorem 1 gives $v^* \in S_0$ such that $f(v^*, v) + d(v^*, v) > 0$ for all $v \neq v^*$. From (7) follows then $v^* \in \Psi$. Hence

$v^* \in S_0 \cap \Psi$, and Theorem 4 holds.

(5) "Theorem 2 $\Rightarrow$ Theorem 4".

Let Theorem 2 and the hypothesis of Theorem 4 hold. Assume, for contradiction, that $\overline{v} \notin \Psi$ for all $\overline{v} \in S_0$. Then by (7) for all $\overline{v} \in S_0$

$$(*) \qquad \text{there exists } v \neq \overline{v} \text{ with } f(\overline{v}, v) + d(\overline{v}, v) \leqslant 0.$$

Hence (5) is satisfied. By Theorem 2 there exists $v^* \in S_0$ such that $f(v^*, v) \geqslant 0$ for all $v \in V$. This implies that $f(v^*, v) + d(v^*, v) > 0$ for all $v \in V$, $v \neq v^*$, a contradiction with $(*)$. Hence $\overline{v} \in \Psi$ for some $\overline{v} \in S_0$, and Theorem 4 holds.

(6) "Theorem 3 $\Rightarrow$ Theorem 4".

Let Theorem 3 and the hypothesis of Theorem 4 hold. Define $T : V \rightrightarrows V$ by $T(\overline{v}) := \{v \in V \mid v \neq \overline{v}\}$. Assume, for contradiction, that $\overline{v} \notin \Psi$ for all $\overline{v} \in S_0$. Then (6) follows from (7), and by Theorem 3 there exists $v^* \in T(v^*)$. But this is clearly impossible from the definition of T. Hence $\overline{v} \in \Psi$ for some $\overline{v} \in S_0$, and Theorem 4 holds. $\square$

REMARKS.

1. Assumptions (1) and (2) are satisfied for instance if $f(u, v) := \varphi(Tv - Tu)$, where T maps V into a linear topological space X and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is subadditive on X and satisfies $\varphi(0) = 0$.

2. The primitive version of Ekeland's Theorem is obtained from Theorem 1 by choosing $f(u, v) := F(v) - F(u)$, where $F(.)$ is lower semicontinuous on V. With this choice, one obtains from Theorem 1: If

$$-\varepsilon := \inf_{v \in V} F(v) - F(v_0) > -\infty,$$

then there exists $v^* \in V$ such that

$$F(v) > F(v^*) - d(v^*, v) \text{ for all } v \in V,\ v \neq v^*,$$
$$F(v^*) \leqslant F(v_0), \quad d(v_0, v^*) \leqslant \varepsilon$$

(the last two inequalities come from $F(v^*) - F(v_0) + d(v_0, v^*) \leqslant 0$). Slight variants are obtained by replacing d with the equivalent metric $(\varepsilon/c) \cdot d$ for some $c > 0$.

3. As a byproduct of Theorem 1 we can obtain a result about perturbed equilibria: Let $f, g : V \times V \to \mathbb{R}$ be lower semicontinuous in the second argument and satisfy (1), (2). Let $E(f)$ denote the equilibrium points of f, in the sense that $E(f) := \{\overline{v} \in V \mid \inf_{v \in V} f(\overline{v}, v) \geqslant 0\}$. Likewise define $E(f + g)$ and $E(f + d)$. Assume that

$g(u,v) \leqslant \varepsilon$ for all $u, v \in V$. Let $v_0 \in E(f+g)$. This implies $\inf_{v \in V} f(v_0, v) \geqslant -\varepsilon$. By Theorem 1, there exists $v^* \in E(f+d)$ such that $v^* \in S_0$. From $v^* \in S_0$ follows $d(v_0, v^*) \leqslant -f(v_0, v^*) \leqslant \varepsilon$. Therefore we have obtained the following

PROPOSITION. *If $v_0 \in E(f+g)$ and if $g(u,v) \leqslant \varepsilon$ for all $u, v \in V$, then there exists $v^* \in E(f+d)$ such that $d(v_0, v^*) \leqslant \varepsilon$.*

We may interpret this result as follows: If we can control the changes in equilibria for the specific perturbation $f \to f + d$, then we can also for general perturbations $f \to f + g$.

4. Let $-\varepsilon := \inf_{v \in V} f(v_0, v)$. Replacing d by the equivalent metric $(1/n) \cdot d$ we obtain from Theorem 1 for arbitrary $n \in \mathbb{N}$ the existence of $v^* \in V$ such that

$$
\begin{aligned}
&f(v^*, v) + \frac{1}{n} d(v^*, v) \geqslant 0 \quad \forall v \in V, \\
&f(v_0, v^*) + \frac{1}{n} d(v_0, v^*) \leqslant 0.
\end{aligned} \tag{9}
$$

The second inequality implies in particular

$$f(v_0, v^*) \leqslant 0 \tag{10}$$

and, since $-f(v_0, v^*) \leqslant \varepsilon$,

$$d(v_0, v^*) \leqslant n\varepsilon. \tag{11}$$

As an application of Theorem 1 we consider a globalised version of [**6**, Theorem 4.7 and Theorem 5.10], which does not need a linear structure. The Palais-Smale condition employed in [**6**] will be replaced here by the following Condition (C).

DEFINITION: We say that $v_0 \in V$ satisfies Condition (C) if and only if every sequence $\{v_n\} \subset V$ satisfying $f(v_0, v_n) \leqslant 1/n \ \forall n$ and $0 \leqslant f(v_n, v) + (1/n) d(v_n, v)$ $\forall v \in V$, $\forall n$ has a convergent subsequence.

THEOREM 6. *Let (V, d) be a complete metric space. Let $f : V \times V \to \mathbb{R}$ be lower semicontinuous in the second argument, upper semicontinuous in the first argument, and satisfy (1), (2).*

(a) *If for some $v_0 \in V$, $\inf_{v \in V} f(v_0, v) > -\infty$, and v_0 satisfies Condition (C), then there exists $v^* \in V$ such that $f(v^*, v) \geqslant 0 \ \forall v \in V$.*

(b) *If for some $v^* \in V$, $f(v^*, v) \geqslant 0 \ \forall v \in V$, and v^* satisfies Condition (C), then for every $\alpha > 0$: either $\inf\{f(v^*, v) \mid v \in V,\ d(v^*, v) = \alpha\} > 0$, or there exists $u_\alpha \in V$ with $d(v^*, u_\alpha) = \alpha$ and $f(v^*, u_\alpha) = 0$ (implying $f(u_\alpha, v) \geqslant 0 \ \forall v \in V$).*

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PROOF:

(a) By Remark 4 above for all $n \in \mathbb{N}$ there exists $v_n \in V$ such that, correspondingly with (9), (10),

$$f(v_n, v) + \frac{1}{n} d(v_n, v) \geqslant 0 \quad \forall v \in V, \tag{12}$$
$$f(v_0, v_n) \leqslant 0.$$

From Condition (C) there exists a subsequence of $\{v_n\}$ converging towards some $v^* \in V$. Then from (12) and the upper semicontinuity of $f(\cdot, v)$ we have

$$f(v^*, v) \geqslant 0 \quad \forall v \in V.$$

(b) Let $\alpha > 0$ and assume that $\inf\{f(v^*, v) \mid v \in V,\ d(v^*, v) = \alpha\} = 0$. Then for all $n \in \mathbb{N}$ there exists $v_n \in V$ with $d(v^*, v_n) = \alpha$ such that $f(v^*, v_n) \leqslant 1/n^2$. Then for all $v \in V$,

$$0 \leqslant f(v^*, v) \leqslant f(v^*, v_n) + f(v_n, v) \leqslant \frac{1}{n^2} + f(v_n, v).$$

Hence $\inf\limits_{v \in V} f(v_n, v) \geqslant -1/n^2 > -\infty$. By Remark 4 above we obtain then, for all $n \in \mathbb{N}$, some $u_n \in V$ such that correspondingly with (9), (10), (11),

$$f(u_n, v) + \frac{1}{n} d(u_n, v) \geqslant 0 \quad \forall v \in V, \tag{13}$$
$$f(v_n, u_n) \leqslant 0, \tag{14}$$
$$d(v_n, u_n) \leqslant \frac{1}{n}. \tag{15}$$

From (14) and $f(v^*, v_n) \leqslant 1/n^2$ follows $f(v^*, u_n) \leqslant f(v^*, v_n) + f(v_n, u_n) \leqslant 1/n^2 \leqslant 1/n$. From this and (13) follows, since v^* satisfies Condition (C), the existence of a subsequence of $\{u_n\}$ converging towards some $u_\alpha \in V$. From $f(v^*, u_n) \leqslant 1/n$ follows $f(v^*, u_\alpha) \leqslant 0$, hence $f(v^*, u_\alpha) = 0$. Moreover $d(v^*, v_n) = \alpha$ and $\lim\limits_{n \to \infty} d(v_n, u_n) = 0$ imply that $d(v^*, u_\alpha) = \alpha$. It is clear that $f(v^*, u_\alpha) = 0$ and $f(v^*, v) \geqslant 0 \quad \forall v \in V$ imply

$$0 \leqslant f(v^*, v) \leqslant f(v^*, u_\alpha) + f(u_\alpha, v) = f(u_\alpha, v) \quad \forall v \in V. \qquad \square$$

The next result is a metric variant of the "Drop Theorem" [**3**], [**6**, Theorem 7.3]. Here we return to the setting described in the beginning. Thus we assume that V, d, f, v_0 are as specified in the paragraph preceding Theorem 1. Moreover, for $\alpha > 0$ and $w \in V$ we define

$$S_\alpha(w) := \{v \in V \mid f(w, v) + \alpha d(w, v) \leqslant 0\}.$$

Then we have:

Theorem 7. *Let $A \subset V$ be a closed set such that $v_0 \in A$. Let $B \subset V$ be bounded. Assume that*

$$\sup_{u \in A,\, v \in B} f(u,v) =: \rho < 0. \tag{16}$$

For given $b \in V$ let $R \geqslant d(b, v_0)$ and $A_R := \{v \in A \mid d(b,v) \leqslant R\}$. Then there exists $\alpha > 0$ and $v^* \in S_\alpha(v_0) \cap A_R$ such that

$$S_\alpha(v^*) \supset B, \; S_\alpha(v^*) \cap A_R = \{v^*\}.$$

Proof: Since $v_0 \in A_R$ we can apply Theorem 1 with V replaced by A_R. We replace d by the equivalent metric $\alpha \cdot d$, where

$$0 < \alpha \leqslant -\rho/(R+r), \; r := \sup_{v \in B} d(b,v).$$

From Theorem 1 we obtain $v^* \in S_\alpha(v_0) \cap A_R$ such that $f(v^*, v) + \alpha d(v^*, v) > 0$ for all $v \in A_R$, $v \neq v^*$. Then $v \in A_R$ with $v \neq v^*$ cannot be an element of $S_\alpha(v^*)$. If $v \in B$, then

$$\begin{aligned} f(v^*,v) + \alpha d(v^*,v) &\leqslant \rho + \alpha(d(v^*,b) + d(b,v)) \\ &\leqslant \rho + \alpha(R+r) \leqslant 0, \end{aligned}$$

and therefore $v \in S_\alpha(v^*)$. □

If V is in addition a linear space and the functions $f(u,\cdot)$ and $d(u,\cdot)$ are convex, then $S_\alpha(v^*)$ is a convex set, hence contains with v^* and B also the "drop" $D(v^*, B) :=$ cl conv$(\{v^*\} \cup B)$. The conclusion of Theorem 7 gives then $D(v^*, B) \cap A_R = \{v^*\}$. If furthermore $f(u,v) := d(b,v) - d(b,u)$, then (16) implies that $\sup_{v \in B} d(b,v) \leqslant \inf_{u \in A} d(b,u) \leqslant d(b,v_0) \leqslant R$. Hence v^* and B and therefore also $D(v^*, B)$ are contained in the ball around b with radius R, and therefore every point of $D(v^*, B) \cap A$ must be in A_R. The conclusion of Theorem 7 gives then $D(v^*, B) \cap A = \{v^*\}$, the classical result. See also [**1**, **7**, **8**, **10**].

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