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The Spectrum and Isometric Embeddings of Surfaces of Revolution

For Gus and Sonia

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Abstract. A sharp upper bound on the first S^1 invariant eigenvalue of the Laplacian for S^1 invariant metrics on S^2 is used to find obstructions to the existence of S^1 equivariant isometric embeddings of such metrics in $(\mathbb{R}^3, \text{can})$. As a corollary we prove: If the first four distinct eigenvalues have even multiplicities then the metric cannot be equivariantly, isometrically embedded in $(\mathbb{R}^3, \text{can})$. This leads to generalizations of some classical results in the theory of surfaces.

1 Introduction

The problem of isometrically embedding (S^2, g) in $(\mathbb{R}^3, \text{can})$ has a long history which goes back at least as far as 1916. In that year, Weyl [22], and in the years since, Lewy [17], Nirenberg [19], Heinz [11], Alexandrov [2], and Pogorelov [20, 21], to name a few, proved embedding theorems of various orders of differentiability (or analyticity) in case the Gauss curvature is positive. The recent results of Guan and Li [10], and Hong and Zuily [13] address the case of non-negative curvature. But, of course, not every metric on S^2 admits such an isometric embedding. The reader may refer to Greene [9], wherein one finds examples of smooth metrics on S^2 for which there is no C^2 isometric embedding in $(\mathbb{R}^3, \text{can})$.

In the presence of examples such as Greene's, one might naturally ask if there exist intrinsic geometric conditions on metrics which obstruct such isometric embeddings. Inasmuch as the above mentioned embedding theorems require, at least, non-negativity of the Gauss curvature, one must look for embedding obstructed metrics among those with some negative curvature. Of course, having some negative curvature is not enough, but one might hope that some stronger condition, associated with the existence of some negative curvature, might satisfy our requirements. The purpose of this paper is to provide, in a special case, conditions on the spectrum of the Riemannian manifold which are intrinsic obstructions to the above isometric embedding problem.

It is no surprise that the spectrum might make an appearance in this subject. There is an extensive literature which associates the spectrum with (more generally)

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isometric immersions (see [4, 18]). Much of this work relates the spectrum to the mean curvature, and is associated with the Willmore conjecture. By way of comparison, the embedding problem of this paper is almost trivial, but it is the intrinsic relation between the spectrum and embeddability which we hope the reader will find interesting.

In the case of S^1 invariant metrics on S^2, one can prove that, while the first eigenvalue must be bounded above by $8\pi/\text{area}$ (Hersch's theorem, [12]), the first S^1 invariant eigenvalue can be arbitrarily large. At the same time, however, there is a sharp upper bound, depending on the metric, for the first S^1 invariant eigenvalue. We will prove that it is this upper bound which, upon exceeding a certain critical value, becomes an obstruction to the existence of equivariant isometric embeddings into $(\mathbb{R}^3, \text{can})$. As a result, if the first S^1 invariant eigenvalue becomes too large then the surface cannot be isometrically embedded in $(\mathbb{R}^3, \text{can})$ as a classical surface of revolution. (See also Abreu and Freitas [1].)

Another characteristic of the spectrum of a surface of revolution is that the eigenspaces are even dimensional unless the eigenvalue happens to correspond to an S^1 invariant eigenvalue. As a result, one way of increasing the first S^1 invariant eigenvalue is to insist that the multiplicities be even up to a certain point. This leads to a result, proved in Section 5, that even multiplicity for the first 4 distinct eigenvalues is an obstruction to isometric embeddability.

In the last section, we will remark on how these results give a generalization of a well known corollary of the Gauss–Bonnet Theorem regarding the existence of non-positive integral curvature.

The author is indebted to Andrew Hwang for a brief but enlightening conversation about momentum coordinates.

2 Isometric Embeddings and Momentum Coordinates

The results that we obtain are based on a well known and quite elementary result regarding S^1 invariant metrics on S^2 which can be found in Besse [3, pp. 95–106]. The reader will find the results of this section to be, essentially, nothing but a reformulation of Besse's treatment. We use the terminology *surface of revolution* when referring to a classical surface of revolution, *i.e.*, a surface generated by rotation of a curve about a line. This is an example of an S^1 *equivariant embedding*. An S^1 invariant metric on S^2 is sometimes called an *abstract surface of revolution*. See the definition in Hwang [14].

We will, henceforth, assume $\mathbb{R}^3$ to be endowed with its standard flat metric and therefore suppress any further mention of its metric. Let (M, g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2 and whose area is 4π. We will assume the metric to be C^∞. Since (M, g) has an effective S^1 isometry group there are exactly two fixed points. We call the fixed points np and sp and let U be the chart $M \setminus \{np, sp\}$. On U the metric has the form $ds \otimes ds + a^2(s)\, d\theta \otimes d\theta$ where s is the arclength along a geodesic connecting np to sp and $a(s)$ is a function $a\colon [0, L] \to \mathbb{R}^+$ satisfying $a(0) = a(L) = 0$ and $a'(0) = 1 = -a'(L)$.

One can always isometrically embed such a metric into $\mathbb{R}^2 \times \mathbb{C}$ as follows:

$$\psi^1(s,\theta) = a(s)\cos\theta$$
$$\psi^2(s,\theta) = a(s)\sin\theta \qquad (2.1)$$
$$\psi^3(s,\theta) = \int_c^s \sqrt{1-(a')^2(t)}\,dt$$

for some $c \in [0, L]$. Using this formula, it can be shown that (M, g) can be isometrically C^1 embedded as a surface of revolution in $\mathbb{R}^3$ if and only if

$$|a'(s)| \le 1 \quad \text{for all } s \in [0, L]. \qquad (2.2)$$

We will find it convenient to make a change of variables to *action-angle coordinates*. These are given by a diffeomorphism $(s, \theta) \to (x, \theta)$ where $x \equiv \phi\colon [0, L] \to [-1, 1]$ is defined by:

$$x \equiv \phi(s) \equiv \int_b^s a(t)\,dt \qquad (2.3)$$

for a suitably chosen value of the constant b.

These are also called *symplectic coordinates* since ϕ is a moment map of the S^1 action. These coordinates play an important rôle in the explicit construction of a large class of complete Kähler metrics of constant curvature. (See Hwang and Singer [15].)

If we let $f(x) \equiv (a^2 \circ \phi^{-1})(x)$, then in the new coordinates the metric on the chart U takes the form

$$g = \frac{1}{f(x)}\,dx \otimes dx + f(x)\,d\theta \otimes d\theta \qquad (2.4)$$

where $(x, \theta) \in (-1, 1) \times [0, 2\pi)$. In these coordinates the conditions at the endpoints translate to $f(-1) = 0 = f(1)$ and $f'(-1) = 2 = -f'(1)$. In this form, it is easy to see that the Gauss curvature of this metric is given by $K(x) = (-1/2)f''(x)$. It is also worth observing that the function $f(x)$ is the square of the length of the Killing field (infinitesimal isometry) $\partial/\partial\theta$ on the chart U. The canonical (*i.e.,* constant curvature) metric is obtained by taking $f(x) = 1 - x^2$ and is denoted by can.

Using these coordinates, Hwang [14] proves a general proposition which includes the following version of Besse's result.

Proposition 2.1 *Let (M, g), with metric g as in (2.4), be diffeomorphic to S^2. (M, g) can be isometrically C^1 embedded in $\mathbb{R}^3$ as a surface of revolution if and only if $|f'(x)| \le 2$ for all $x \in [-1, 1]$.* ■

Furthermore, since f' is simply -2 times an antiderivative of the curvature, this result can be restated in terms of integral curvatures as follows. (The details appear in [8].)

Corollary 2.2 *Let (M, g) be an S^1-invariant Riemannian manifold that is diffeomorphic to S^2. Let np and sp denote the fixed points of the S^1-action, and let K be the Gauss curvature of (M, g). Then (M, g) has an isometric C^1-embedding in $\mathbb{R}^3$ as a surface of revolution if and only if*

$$\int_\Omega K \geq 0$$

for all geodesic disks Ω, of constant radii, centered at np or sp.

3 Some Properties of the Spectrum

In the interest of presenting a self-contained exposition we will review some of the relevant facts about the spectrum (eigenvalues) of a surface of revolution in this section. The interested reader may consult [5, 6, 7] for further details.

Let Δ denote the scalar Laplacian on a surface of revolution (M, g), where g is given by (2.4) and let λ be any eigenvalue of $-\Delta$. We will use the symbols E_λ and $\dim E_\lambda$ to denote the eigenspace for λ and it's multiplicity respectively. In this paper the symbol λ_m will always mean the m-th *distinct* eigenvalue. We adopt the convention $\lambda_0 = 0$. Since S^1 (parametrized here by $0 \leq \theta < 2\pi$) acts on (M, g) by isometries and because $\dim E_{\lambda_m} \leq 2m + 1$ (see [7] for the proof), the orthogonal decomposition of E_{λ_m} has the special form

$$E_{\lambda_m} = \bigoplus_{k=-m}^{k=m} e^{ik\theta} W_k$$

in which $W_k (= W_{-k})$ is the "eigenspace" (it might contain only 0) of the ordinary differential operator

$$L_k = -\frac{d}{dx}\Big(f(x) \frac{d}{dx} \Big) + \frac{k^2}{f(x)}$$

with suitable boundary conditions. It should be observed that $\dim W_k \leq 1$, a value of zero for this dimension occuring when $\lambda_m \notin \operatorname{Spec} L_k$.

It is easy to see that $\operatorname{Spec}(-\Delta) = \bigcup_{k \in \mathbb{Z}} \operatorname{Spec} L_k$ and consequently the non-zero part of the spectrum of $-\Delta$ can be studied via the spectra $\operatorname{Spec} L_k = \{0 < \lambda_k^1 < \lambda_k^2 < \cdots < \lambda_k^j < \cdots\} \, \forall k \in \mathbb{Z}$. The eigenvalues λ_0^j in the case $k = 0$ above are called the S^1 *invariant eigenvalues* since their eigenfunctions are invariant under the action of the S^1 isometry group. If $k \neq 0$ the eigenvalues are called k *equivariant* or simply *of type* $k \neq 0$. Each L_k has a Green's operator, $\Gamma_k \colon \big(H^0(M)\big)^\perp \to L^2(M)$, whose spectrum is $\{1/\lambda_k^j\}_{j=1}^\infty$, and whose trace is defined by, $\operatorname{tr} \Gamma_k \equiv \sum 1/\lambda_k^j$.

Proposition 3.1 (See [5, 6]) *With the notations as above:*

(i)

$$\operatorname{tr} \Gamma_k = \begin{cases} \frac{1}{2} \int_{-1}^{1} \frac{1-x^2}{f(x)} \, dx & \text{if } k = 0, \\ \frac{1}{|k|} & \text{if } k \neq 0. \end{cases}$$

(ii) *For all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_k^j = \lambda_{-k}^j$.*

(iii) $\forall k \geq 1$ *and* $\forall j \geq 0$, $\lambda_{k+j} \leq \lambda_k^{j+1}$; *and* $\lambda_1 \leq \lambda_0^1$.
(iv) $\dim E_{\lambda_m}$ *is odd if and only if* λ_m *is an* S^1 *invariant eigenvalue.* ∎

Remark One must be careful with the definition of $\operatorname{tr}\Gamma_0$ since $\lambda_0 = 0$ is an S^1 invariant eigenvalue of $-\Delta$. To avoid this difficulty we studied the S^1 invariant spectrum of the Laplacian on 1-forms in [6] and then observed that the non-zero eigenvalues are the same for functions and 1-forms.

4 A Sharp Upper Bound for the First Eigenvalue

In [7] we derived sharp upper bounds for all of the distinct eigenvalues on an abstract surface of revolution diffeomorphic to S^2. These estimates were obtained using the the k-type eigenvalues for $k \neq 0$. In this section we will obtain a sharp bound for λ_1 using the S^1 invariant spectrum. In contrast with the more general result of Hersch [12], the reader will find that this bound exhibits, more explicitly, its dependence on the metric. This fact will play an important rôle in embedding problems.

Proposition 4.1 *Let* (M, g) *be an* S^1 *invariant Riemannian manifold of area* 4π *which is diffeomorphic to* S^2 *with metric* (2.4). *Let* λ_0^1 *be the first non-zero* S^1 *invariant eigenvalue for this metric; then*

$$\lambda_0^1 \leq \frac{3}{2}\int_{-1}^{1} f(x)\,dx$$

and equality holds if and only if (M, g) *is isometric to* (S^2, can).

Proof The minimum principle associated with the first non-zero S^1 invariant eigenvalue problem,

$$L_0 u = -\frac{d}{dx}\Big(f(x)\frac{du}{dx}\Big) = \lambda_0^1 u, \tag{4.1}$$

states that

$$\lambda_0^1 \leq \frac{\int_{-1}^{1} f(x)(\frac{du}{dx})^2\,dx}{\int_{-1}^{1} u^2\,dx} \tag{4.2}$$

for all S^1 invariant functions $u \in C^\infty(M)$ with $u \perp \ker L_0$. Equality holds if and only if u is an eigenfunction for λ_0^1. Since $\ker L_0$ consists of constant functions and $\int_{-1}^{1} x \cdot 1\,dx = 0$, we see that $u(x) = x$ is an admissible solution of (4.2) and therefore $\lambda_0^1 \leq \frac{3}{2}\int_{-1}^{1} f(x)\,dx$. Equality holds if and only if $u(x) = x$ is the first S^1 invariant eigenfunction. In this case, upon substitution of $u(x) = x$ into (4.1) we obtain the equivalent equation $-f'(x) = \lambda_0^1 x$. Recalling that $f(x)$ and $f'(x)$ must satisfy certain boundary conditions forces $\lambda_0^1 = 2$ and yields the unique solution $f(x) = 1 - x^2$. In other words, $g = \mathrm{can}$. ∎

Because of Proposition 3.1(iii), we have the immediate corollary:

Corollary 4.2 *Let (M, g) be an S^1 invariant Riemannian manifold of area 4π, which is diffeomorphic to S^2 with metric (2.4). Let λ_1 be the first, non-zero, distinct eigenvalue for this metric, then*

$$\lambda_1 \leq \frac{3}{2} \int_{-1}^{1} f(x)\, dx$$

and equality holds if and only if (M, g) is isometric to (S^2, can). ∎

5 Spectral Obstructions to Equivariant Isometric Embeddings

In [6] we used the trace formula of Proposition 3.1(i) to show that there exist abstract surfaces of revolution with arbitrarily large first S^1 invariant eigenvalue. This fact, together with Proposition 4.1 of the last section, shows that as λ_0^1 increases so does the integral $\int_{-1}^{1} f(x)\, dx$. This fact is the key to the results of this section, but first we will prove a lemma which gives lower bounds for our eigenvalues.

Lemma 5.1 *Let $f(x)$ and λ_k^m be defined as above then for all $m \in \mathbb{N}$*

$$\lambda_k^m > \begin{cases} 2m\left[\int_{-1}^{1} \frac{1-x^2}{f(x)}\, dx\right]^{-1} & \text{if } k = 0, \\ m|k| & \text{if } k \neq 0. \end{cases}$$

Proof From Proposition 3.1(i)

$$\frac{1}{2} \int_{-1}^{1} \frac{1-x^2}{f(x)}\, dx = \sum_{j=1}^{\infty} \frac{1}{\lambda_0^j} \quad \text{and} \quad \frac{1}{|k|} = \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j}.$$

Each of the sequences $\{\lambda_k^j\}_{j=1}^{\infty}$ is positive and strictly increasing so by truncating the above series after m terms and then replacing each term with the smallest one we obtain

$$\frac{1}{2} \int_{-1}^{1} \frac{1-x^2}{f(x)}\, dx > \frac{m}{\lambda_0^m} \text{ and } \frac{1}{|k|} > \frac{m}{\lambda_k^m}.$$

This produces the desired inequalities. ∎

As was observed in [6], the $k = 0$, $m = 1$ case of this inequality, together with the minimal restrictions on the function f is enough to ensure that there exist surfaces of revolution with arbitrarily large λ_0^1. Because of this, we can be confident that the next two results are non-vacuous.

Proposition 5.2 *Let (M, g) be an S^1 invariant Riemannian manifold of area 4π which is diffeomorphic to S^2 and let λ_0^1 be it's first non-zero S^1 invariant eigenvalue. If $\lambda_0^1 > 3$ then (M, g) cannot be isometrically C^1 embedded in $\mathbb{R}^3$ as a surface of revolution.*

Proof By Proposition 4.1, since $\lambda_0^1 > 3$, then $\int_{-1}^1 f(x)\,dx > 2$. Upon integrating by parts we have $-\int_{-1}^1 xf'(x)\,dx > 2$ so that

$$2 < \left| -\int_{-1}^1 xf'(x)\,dx \right| \leq \int_{-1}^1 |x|\,|f'(x)|\,dx \leq \max_{x\in[-1,1]} |f'(x)|.$$

So there exists $x_0 \in [-1,1]$ with $|f'(x_0)| > 2$, thus, by Proposition 2.1, precluding the possibility of an isometric embedding as a surface of revolution. ■

An immediate consequence of Corollary 2.2 is:

Corollary 5.3 *Let (M,g) be an S^1 invariant Riemannian manifold of area 4π which is diffeomorphic to S^2, let K be it's Gauss curvature, and let λ_0^1 be it's first S^1 invariant eigenvalue. If $\lambda_0^1 > 3$ then there exists a pole-centered disk $\Omega \subset M$ such that $\int_\Omega K < 0$.* ■

Remark Rafe Mazzeo and Steve Zelditch have brought to our attention a recent result of Abreu and Freitas, [1], which is a significant improvement of Proposition 5.2. They prove, with the same hypothesis as Proposition 5.2 and using the notation of this paper, that, for a metric isometrically embedded in $\mathbb{R}^3$ as a surface of revolution, $\lambda_0^j < \xi_j^2/2$, for all j where ξ_j is a positive zero of a certain Bessel function or its derivative. In particular, $\lambda_0^1 < \xi_1^2/2 \approx 2.89$. We have left Proposition 5.2 in the paper since its proof is so easy, and because the eigenvalue bound contained therein is sufficient for proving the main theorem (Theorem 5.5) below.

As we allow the first S^1 invariant eigenvalue to increase one might suspect that, so to speak, some small eigenvalues with even multiplicity are "left behind". This suggests that we might find an obstruction to embedding if the first few eigenvalues have even multiplicities. We will soon see that even multiplicities for the first four eigenvalues will constitute such an obstruction, but first it would be a good idea to know if metrics with this property exist. This is the subject of:

Theorem 5.4 *There exist metrics on S^2 whose first four distinct non-zero eigenvalues have even multiplicity.*

Proof To prove this theorem we will find an S^1 invariant metric of area 4π with this property.

By Proposition 3.1(iv), $\dim E_{\lambda_m}$ is even if and only if λ_m is not an S^1 invariant eigenvalue, *i.e.,* if and only if $\lambda_m \neq \lambda_0^j$ for any j. It is now clear that the first four multiplicities are even if and only if $\lambda_4 < \lambda_0^1$, and, by Proposition 3.1(iii), this will occur if our metric satisfies $\lambda_4^1 < \lambda_0^1$. Using a variational principle, as in [7], for the operator L_4, we obtain the upper bound

$$\lambda_4^1 \leq \frac{\int_{-1}^1 [f(x)(\frac{du}{dx})^2 + \frac{4^2}{f(x)}u^2]\,dx}{\int_{-1}^1 u^2\,dx}$$

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$\forall u \in C^\infty(-1,1)$ such that $u(-1) = u(1) = 0$.

Comparing this upper bound with the lower bound on λ_0^1 provided by Lemma 5.1, the proof of this theorem may now be reduced to finding a function f and a suitable test function u such that

$$\text{(5.1)} \qquad \frac{\int_{-1}^{1}[f(x)(\frac{du}{dx})^2 + \frac{16}{f(x)}u^2]\,dx}{\int_{-1}^{1} u^2\,dx} < 2\Big[\int_{-1}^{1} \frac{1-x^2}{f(x)}\,dx\Big]^{-1}.$$

We claim that $f(x) = \frac{10(1-x^2)}{1+9x^{36}}$ and $u(x) = \sqrt{1-x^2}$ will satisfy the inequality (5.1).

It is not difficult to see that $2[\int_{-1}^{1} \frac{1-x^2}{f(x)}\,dx]^{-1} = \frac{185}{23} > 8$ for this choice of $f(x)$. So the right hand side of (5.1) is greater than 8. Calculating the left hand side of (5.1) for this choice of $f(x)$ and $u(x)$ yields:

$$\begin{aligned}\frac{\int_{-1}^{1}[f(x)(\frac{du}{dx})^2 + \frac{16}{f(x)}u^2]\,dx}{\int_{-1}^{1} u^2\,dx} &= \frac{3}{4}\Big[10\int_{-1}^{1} \frac{x^2}{1+9x^{36}}\,dx + \frac{8}{5}\int_{-1}^{1}(1+9x^{36})\,dx\Big] \\ &< \frac{3}{4}\Big[10\cdot\frac{2}{3} + \frac{16}{5}\cdot\frac{46}{37}\Big] = \frac{1477}{185} < 8,\end{aligned}$$

where the first integral in brackets has been approximated in the obvious way. Since the left hand side is less than 8 and the right hand side is greater than 8, the proof is finished. ■

The proof of this theorem is hardly optimal since there are, certainly, many such metrics. We also believe that using a similar technique, one should be able to find metrics whose first m distinct eigenvalues have even multiplicity for arbitrary m, but we will not address these problems here.

Theorem 5.5 *Let (M,g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2 and let λ_m be its m-th distinct eigenvalue. If $\dim E_{\lambda_m}$ is even for $1 \le m \le 4$ then (M,g) cannot be isometrically C^1 embedded in $\mathbb{R}^3$ as a surface of revolution.*

Proof Without loss of generality, we may assume the area of the metric is 4π. As seen in the proof of Theorem 5.4, the first four eigenvalues have even multiplicity if and only if $\lambda_4 < \lambda_0^1$. This result will then follow from Proposition 5.2 as long as we can prove that $\lambda_4 > 3$. This is most easily accomplished by contradiction.

Assume $\lambda_4 \le 3$ so that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \le 3$. Now each λ_i for $1 \le i \le 4$ must satisfy $\lambda_i = \lambda_k^l$ for some $k \ne 0$ and $l \ge 1$. However, by Lemma 5.1, $\lambda_k^l > l|k|$ so if $\lambda_i = \lambda_k^l \le 3$ it must be the case that $l|k| \le 2$. By Proposition 3.1ii $\lambda_k^l = \lambda_{-k}^l$ so there are only three (possibly) distinct eigenvalues with these properties and their values coincide with λ_1^1, λ_2^1, and λ_1^2. There are, therefore, at most three distinct values for the four distinct eigenvalues λ_i for $1 \le i \le 4$, but this contradicts the pigeonhole principle. ■

Again there is an immediate corollary:

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Corollary 5.6 *Let (M,g) be an S^1 invariant Riemannian manifold which is diffeomorphic to S^2, let K be it's Gauss curvature, and let λ_m be it's m-th distinct eigenvalue. If $\dim E_{\lambda_m}$ is even for $1 \leq m \leq 4$ then there exists a pole-centered disk $\Omega \subset M$ such that $\int_\Omega K < 0$.* ■

One cannot help but ponder the possibility that one can remove the, a priori, assumption of S^1 invariance since, according to legend, only S^1 invariant metrics would have a lot of even multiplicities anyway.

6 Remarks on Classical Surface Theory

In this final section we leave behind the question of embeddability and focus our attention on the way in which Corollary 5.6 can be viewed as an extension of some of the consequences of the Gauss–Bonnet theorem.

Let (M,g) be any compact, orientable, boundaryless surface with metric g. We recall that the Euler characteristic, $\chi(M)$, and curvature K are related by the Gauss–Bonnet theorem:

$$2\pi\chi(M) = \int_M K,$$

so that one has the obvious consequence:

Proposition 6.1 *If $\chi(M) \leq 0$ then $\int_M K \leq 0$.* ■

Via the Hodge–DeRham isomorphism, one can restate the Gauss–Bonnet Theorem as follows:

Let $\lambda_{q,j}$ be the j-th distinct eigenvalue of the Laplacian acting on q-forms and $E_{\lambda_{q,j}}$ its "eigenspace" (this vector space may consist of the zero vector only). Then

$$\frac{1}{2\pi}\int_M K = 2 - \dim E_{\lambda_{1,0}}. \tag{6.1}$$

Of course $\dim E_{\lambda_{1,0}}$ is simply twice the genus of the surface since $\lambda_{1,0} = 0$. But this form of the Gauss–Bonnet formula does allow us to observe that: *If $\dim E_{\lambda_{1,0}}$ is even (this is automatic) and positive then $\int_M K \leq 0$.*

In case $\dim E_{\lambda_{1,0}} > 0$, M is not a sphere. So these results tell us how non-positive integral curvature arises from adding handles to the sphere. Corollary 5.6 gives a philosophically similar result without changing the topology of the sphere.

Collecting the forgoing ideas together, one can state a result which gives a unified, if not quite complete, answer to the question of the existence of non-positive integral curvature, in other words: a generalization of Proposition 6.1 which includes surfaces with Euler characteristic 2.

Corollary 6.2 *Let (M,g) be an orientable, compact, boundaryless surface with metric g, isometry group $\Im(M,g)$ and j-th distinct q-form eigenvalue $\lambda_{q,j}$. If, for some $q \in \{|\dim \Im(M,g) - 1|, 1\}$, $\dim E_{\lambda_{q,|1-q|\cdot j}}$ is even and positive for all j such that $1 \leq j \leq 4$, then there exists a non-empty open $\Omega \subseteq M$ such that $\int_\Omega K \leq 0$.*

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Proof If (M, g) satisfies the hypothesis for $q = 1$ then the statement of this result is simply Proposition 6.1 with $\Omega = M$ as can be seen from Equation (6.1).

If (M, g) satisfies the hypothesis for $q = |\dim \Im(M, g) - 1|$, then, necessarily, $\dim \Im(M, g) \leq 2$ since, for all such surfaces, $\dim \Im(M, g) \leq 3$ (see [16], p. 46, 47) and (S^2, can) is the only one with $\dim \Im(M, g) = 3$; but all of its 2-form eigenspaces are odd dimensional and $\dim E_{\lambda_{1,0}} = 0$. If $\dim \Im(M, g) = 0$ or 2 then, again, $q = 1$ and, again, this is Proposition 6.1. Finally, if $\dim \Im(M, g) = 1$ then $q = 0$ or 1. If, in this case, the hypothesis holds for $q = 0$ only, then M is, topologically, the sphere, and thus the statement of this theorem reduces to Corollary 5.6 with Ω a non-empty pole-centered disk. ■

The reader may have noticed that for the case $\dim \Im(M, g) = 0$, if one relaxes the positivity condition of this theorem and allows $\dim E_{\lambda_{1,0}} = 0$, then the resulting statement is the contrapositive of the 2-dimensional version of a conjecture of Yau [23] that all compact manifolds of positive curvature must have an effective S^1 isometry group. The author does not yet know how to prove this.

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