ON THE PRODUCT OF $L(1, \chi)$

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Let $k(\geq 3)$ be a positive integer and $\varphi(k)$ be the Euler function. We denote by $\chi$ one of the $\varphi(k)$ characters formed with modulus $k$, and by $\chi_0$ the principal character. Let $L(s, \chi)$ be the $L$-series corresponding to $\chi$. Throughout the paper we use $c$ and $c(\varepsilon)$ to denote respectively an absolute positive constant and a positive constant depending on parameter $\varepsilon(>0)$ alone, not necessarily the same at their various occurrences. We use the symbol $Y = O(X)$ for positive $X$ when there exists a $c$ satisfying $Y \leq cX$ in the full domain of existence of $X$ and $Y$.

It is well known that

(1) $L(1, \chi) = O(\log k)$, for $\chi \neq \chi_0$.

On the other hand, we know from [7] and [53] (numbers in square brackets refer to the references at the end of the paper) that

(2) $L(1, \chi)^{-1} = O(\log k)$, for $\chi \neq \chi_0$,

with one possible exception, and if such a exceptional character exists, it is a real one. Let us denote it by $\chi_i$. If there exists no exceptional character, we take any non-principal character as $\chi_i$. Then, by Siegel's theorem (see [3] and [8]),

(3) $c(\varepsilon)k^{-1} < |L(1, \chi)|$

for any positive $\varepsilon$.

The object of this paper is to estimate $\prod_{\chi \leq 1} L(1, \chi)$ and $\prod_{\chi(1) = -1} L(1, \chi)$ as precisely as possible, and make some additions to the results of R. Brauer [2], Ankeny and Chowla [1],

1. In what follows, we denote by $p$ and $p_i$ the primes.

**Lemma 1.** $\sum_{\nu \leq x} \nu^{-1} = O(\log \log x)$, for $x \geq 3$.

**Lemma 2.** $c(\log x)^{-1} < \prod_{\nu \leq x} (1 - \nu^{-1}) < c(\log x)^{-1}$.

These are obtained by Theorem 7 of [4].

Let $7 \leq p_i < p_2 < \ldots < p_m$. We denote by $F(x; k; \nu_1, \nu_2, \ldots, \nu_m)$ the

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number of \( z \) satisfying

\[
0 < z \leq x, \quad z \equiv l \pmod{k}, \quad \text{and} \quad p_i \mid z \quad \text{for} \quad \nu = 1, 2, \ldots, m.
\]

If we take positive integers \( m = m_0, m_1, \ldots, m_{h-1} \) such that

\[
m = m_0 > m_1 > m_2 > \ldots > m_{h-1} > m_h = 0,
\]

\[
L_\nu = \prod_{\nu = 1}^{h_\nu} (1 - p_i^{-1}) \geq \frac{5}{4} \quad \text{for} \quad \nu = 1, 2, \ldots, h,
\]

(4) \[
(1 - p_i^{-1}) L_\nu < \frac{4}{5}, \quad \text{for} \quad \nu = 1, 2, \ldots, h - 1,
\]

then we have

**Lemma 3.** \( F(x; k, \ell; p_1, p_2, \ldots, p_m) < 2 x k^{-1} \prod_{\nu = 1}^{h} L_\nu + \prod_{\nu = 0}^{h-1} (2 m_\nu)^2, \) for \((k, \ell) = 1.\)

The proof is similar to that of Theorem 79 and Theorem 86 of [6].

**Lemma 4.** If \( x = 2 \psi(k), \) then

\[
\pi(x; k, \ell) < \frac{x}{\psi(k) \log(x/\psi(k))},
\]

for \((k, \ell) = 1,\) where \( \pi(x; k, \ell) \) denotes the number of primes \( p \) satisfying \( p \equiv x \)
and \( p \equiv \ell \pmod{k}. \)

Suppose that \( x \geq 7^2, a(>1) \) being a positive number to be determined later. We arrange all primes between 7 and \( \sqrt{x} \) except the prime factors of \( k \) such that

\[
7 \leq p_1 < p_2 < \ldots < p_m \equiv \sqrt{x}.
\]

If we write

\[
D(w, k) = \prod_{\nu = 1}^{h} (1 - p_i^{-1})^{-1},
\]

then

(5) \[
\prod_{\nu = 1}^{h} L_\nu = \prod_{r = 1}^{m} (1 - p_i^{-1}) = \prod_{r = 1}^{m} (1 - p_i^{-1}) D(\sqrt{x}, k).
\]

By Lemma 2,

\[
\prod_{r = 1}^{s} (1 - p_i^{-1}) < c(\log p_i)^{-1} D(p_1, k) < c(\log 2 s)^{-1} D(\sqrt{x}, k)
\]

for \( s = 1, 2, \ldots, m, \) it follows therefore that

\[
\log (2 m_\nu) < c D(\sqrt{x}, k) \prod_{r = 1}^{m_\nu} (1 - p_i^{-1})^{-1}
\]
\[ cD(\sqrt{x}, k) \prod_{r=1}^{n-1} (1 - p_r^{-1})^{-1}L_1L_2 \ldots L_n \]
\[ = c \prod_{p \equiv 1 \pmod{3}} (1 - p^{-1})^{-1}L_1L_2 \ldots L_n \]

(6) \[ < (c/a)(\log x)L_1L_2 \ldots L_n < (c/a)(14/15)^x \log x \]

by (4), whence follows

(7) \[ \prod_{r=1}^{n-1} (2m_r)^2 < x^{c/a}. \]

On the other hand

(8) \[ \prod_{r=1}^{n} L_r < ca(\log x)^{-1}D(\sqrt{x}, k) \leq ca(\log x)^{-1}\prod_{p \mid k} (1 - p^{-1})^{-1} \]
\[ = ca(\log x)^{-1}k\varphi(k)^{-1}, \]

by (5) and Lemma 2.

Inserting (7) and (8) for the right of Lemma 3, we obtain

\[ F(x; k, l; \beta_1, \beta_2, \ldots, \beta_m) < ca \frac{x}{\varphi(k)} \log x + x^{c/a} \]

for \((k, l) = 1\), provided that \(x \geq 7^a\). This, together with the inequality

\[ \pi(x; k, l) \leq F(x; k, l; \beta_1, \beta_2, \ldots, \beta_m) + m, \]

gives

(9) \[ \pi(x; k, l) < ca \frac{x}{\varphi(k)} \log x + 2x^{c/a} \]

by (6), where we may suppose that the constants \(c\) in both terms of the right are the same and \(c > 1\).

Suppose first that \(x \geq 7^{2\varphi(k)}\). Then we have

(10) \[ d = \frac{x}{\varphi(k)} \log \left( \frac{x}{\varphi(k)} \right) \geq \sqrt{\frac{x}{\varphi(k)}} \geq 7^a. \]

we can easily verify from (10) that the restriction \(x \geq 7^a\) is satisfied, if we put

\[ a = \frac{c \log x}{\log d}. \]

Inserting this in (9), and using the first inequality of (10), we get

\[ \pi(x; k, l) < c^a \frac{x}{\varphi(k)} \log d + 2d < 2(c^a + 1)d, \]

which is just what the Lemma claims.

Next we consider the trivial case, \(2 \varphi(k) \leq x < 7^{2\varphi(k)}\). Then
Thus the Lemma is completely proved.

2. We write for simplicity

\[ Q(x) = \sum_{\alpha \in \mathcal{A}_r} \sum_{n \in \mathcal{M}} \mathcal{M}(n) \lambda(n), \]

where \( \mathcal{M}(n) \) is \( \log p \) if \( n \) is a positive power of a prime \( p \) and is 0 otherwise.

**Lemma 5.** If \( k \leq \exp(\sqrt[3]{\log x}) \) and \( \mathcal{M} \neq \mathcal{Z}_0, \mathcal{Z}_1 \), then

\[ \sum_{n \in \mathcal{M}} \mathcal{M}(n) \lambda(n) = O(x \exp(-c \sqrt[3]{\log x})). \]

This is the result of Page [7].

**Lemma 6.** If \( x \geq \exp(\log k)^3 \), then \( Q(x) = O(x (\log x)^{-1}) \).

By Lemma 5,

\[ Q(x) = O(kx \exp(-c \sqrt[3]{\log x})) = O(x \exp(\log k, c \sqrt[3]{\log x})) \]

\[ = O(x \exp((\log x)^{1/3} - c(\log x)^{1/3})) = O(x (\log x)^{-1}), \]

since \( \log k \leq \frac{2}{3}(\log x) \).

3. Let \( a, b \) and \( n \) be positive integers.

**Lemma 7.** If \( (a, k) = 1 \), then the number of solutions of

\[ x^n \equiv a \pmod{k} \]

is at most \( n^{\omega(k)+1} \) where \( \omega(k) \) denotes the number of prime factors of \( k \).

This follows from the fact that the number of solutions of \( x^n \equiv a \pmod{p^r} \), \( (a, p) = 1 \), is at most \( n \) if \( p \) is an odd prime, and \( n^2 \) if \( p = 2 \).

**Lemma 8.**

\[ \omega(k) = O(\log k (\log \log k)^{-1}). \]

Suppose that

\[ \omega(k) = r, \quad k = p_1^{b_1}p_2^{b_2} \cdots p_r^{b_r} \quad \text{and} \quad p_1 < p_2 < \ldots < p_r. \]

By the prime number theorem,

\[ cr \log p'_r < \sum_{r=1}^{r} \log p'_r \leq \sum_{r=1}^{r} \log p_r \leq \log k, \]

where \( p'_r \) denotes the \( r \)-th prime. If \( r > \log k (\log \log k)^{-1} \), then

\[ p'_r > cr \log r > c \log k. \]

This combined with (11) gives \( r < c(\log k)(\log \log k)^{-1} \).

**Lemma 9.**

\[ \sum_{n \leq x} \sum_{\substack{p \leq k \\ p^n \equiv 1 \pmod{k}}} p^{-n} = O(\omega(k) k^{-1}). \]
By Lemma 7

\[ \sum_{n=2}^{\infty} n^{-1} \sum_{p \equiv k \mod k} p^{-n} = O(\sum_{n=1}^{\infty} n^{-1} \sum_{m \equiv (k+1) \mod k} (km+1)^{-1}), \]

where \( \Sigma \) means that the number of terms of the sum is \( \leq * \). If we put

\[ a_0 = 0, \quad a_n = \sum_{i=1}^{n} \nu_i^{(k)+1} \]

for \( n = 1, 2, \ldots \), then

\[ \sum_{n=2}^{\infty} n^{-1} \sum_{m \equiv k \mod k} m^{-1} = O(\sum_{n=1}^{\infty} n^{-1} \sum_{\nu_i \equiv \nu \mod \nu_n} m^{-1}). \]

Since

\[ a_{n-1} < \sum_{m \equiv \nu} m^{-1} < \log(a_n/a_{n-1}) \]

\[ < \log \left( \int_{0}^{n+1} x^{\nu} dx / \int_{0}^{n-1} x^{\nu} dx \right) \]

\[ = (\omega(k) + 2)\log((n+1) / (n-1)) = O(\omega(k)n^{-1}) \]

for \( n \gg 2 \), it follows from (12) that

\[ \sum_{n=2}^{\infty} n^{-1} \sum_{p \equiv k \mod k} p^{-n} = O(\omega(k)k^{-1} \sum_{n=1}^{\infty} n^{-1}) = O(\omega(k)k^{-1}). \]

4. We write

\[ \prod_{X \times X \subset X, X_1 n \equiv 2} L(1, \mathcal{I}) = \exp(\sum_{X \times X \subset X, X_1 n \equiv 2} \mathcal{I}(n, l(n)(n \log n))^{-1} \]

\[ = \exp(\sum_{l} + \sum_{n} + \sum_{i} + \sum_{n} + \sum_{s} + \sum_{k}), \]

where

\[ \sum_{l} = \sum_{p < k} p^{-1}(\sum_{X} \mathcal{I}(p)), \quad \sum_{n} = \sum_{l \equiv p < k} l^{-1}(\sum_{X} \mathcal{I}(p)), \]

\[ \sum_{s} = -\sum_{p < k} p^{-1}(\mathcal{I}(p) + Z_{l}(p)), \]

\[ \sum_{l} = \sum_{p \equiv k \mod k} \sum_{n \equiv 2} (n \nu^{n})^{-1}(\sum_{X} \mathcal{I}(p^{n})), \]

\[ \sum_{s} = -\sum_{p \equiv k \mod k} \sum_{n \equiv 2} (n \nu^{n})^{-1}(\mathcal{I}(p^{n}) + Z_{l}(p^{n})), \]

\[ \sum_{k} = \sum_{h \equiv k \mod k} \sum_{X \times X \subset X, X_1 n \equiv 2} \mathcal{I}(n, l(n)(n \log n))^{-1} \]

and

\[ A = 2k, \quad B = [\exp(\log k)^{3}]. \]

Now we shall estimate \( \sum_{n} \) using the above lemmas,

\[ \sum_{n} = \mathcal{I}(k) \sum_{p < k, \nu \equiv k^{-1} \nu \equiv k^{-1}} p^{-1} = \mathcal{I}(k) \sum_{m \equiv k^{-1}} (mk+1)^{-1} = O(1). \]
\[\sum_n = \varphi(k) \sum_{d \mid n, p < R, p \equiv 1 \mod k} p^{-1} = \varphi(k) \sum_{d \mid n, k < R} (\pi(n; k, 1) - \pi(n-1; k, 1))n^{-1} = \varphi(k) \left( \sum_{d \mid n, k < R} n^{-1} - (n+1)^{-1} - \pi(A-1; k, 1)A^{-1} + \pi(B-1; k, 1)B^{-1} \right) = O\left( \sum_{d \mid n, k < R} (n+1)^{-1} \left( \log \frac{n}{\varphi(k)} \right)^{-1} + O(1) \right) \text{ (by Lemma 4)} = O\left( \int_R^n \frac{du}{u \log \left( u / \varphi(k) \right)} \right) + O(1) = O(\log \log k).\]

\[\sum_3 = O(\sum_{p < R} p^{-1}) = O(\log \log B) = O(\log \log k). \text{ (by Lemma 1)}\]

\[\sum_1 = \varphi(k) \sum_{p \leq R} \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} (np^n)^{-1} = \varphi(k) \sum_{p \leq R} \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} (np^n)^{-1} = \varphi(k) \sum_{p \leq R} \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} (np^n)^{-1} + \varphi(k) \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} n^{-1} \sum_{p \leq R} p^{-n} = O(\varphi(k) \sum_{p \leq R} (p(p-1))^{-1}) + O(\omega(k)) \text{ (by Lemma 9)} = O(\omega(k)).\]

\[\sum_3 = O(\sum_{p \leq R} (np^n)^{-1}) = O(\sum_{p \leq R} (p(p-1))^{-1}) = O(1).\]

\[\sum_5 = \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} (Q(n) - Q(n-1))(n \log n)^{-1} = \sum_{n \equiv k \pmod{\text{gcd}(k, 1)}} Q(n)((n \log n)^{-1} - ((n+1)\log(n+1))^{-1}) - Q(B-1)(B \log B)^{-1} = O\left( \int_{n \equiv k \pmod{\text{gcd}(k, 1)}} \log x + 1 \int_{n \equiv k \pmod{\text{gcd}(k, 1)}} dx + \frac{1}{\log^2 B} \right) \text{ (by Lemma 6)} = O(1).\]

Collecting these results, we obtain

\[\exp(-c(\log \log k + \omega(k))) < \prod_{\chi \neq \chi_0, \chi_1} L(1, \chi) < \exp(\omega(k)).\]

This, combined with (1), (3) and Lemma 8, gives the following

**Theorem 1.** \(c(\varepsilon)k^{-\varepsilon} < \prod_{\chi \neq \chi_0} L(1, \chi) < \exp(c(\log \log k + \omega(k))).\)

In a similar way, we have

**Theorem 2.** \(c(\varepsilon)k^{-\varepsilon} < | \prod_{\chi \neq \chi_{-1}} L(1, \chi) | < \exp(c(\log \log k + \omega(k))).\)

5. For the cyclotomic field \(\Omega = F(\zeta), \Omega_0 = F(\zeta + \zeta^{-1})\), where \(\zeta\) is a \(l\)-th root of unity, \(l\) being an odd prime, it is well known that
\[ \prod_{\chi \neq \chi_0} L(1, \chi) = \frac{(2\pi)^m R_h}{2 \sqrt{|d|}} \]

\[ \prod_{\chi \neq \chi_0, \chi \neq \chi_0} L(1, \chi) = \frac{2^m R_\Theta h_0}{2 \sqrt{|d_0|}} \]

where \( h, h_0 \), are respectively the class numbers of \( \Omega \) and \( \Omega_0 \) and \( R, R_\Theta \) the regulators of them, and further \( m = \frac{l - 1}{2}, |d| = l^{l-2}, |d_0| = l^{m-1} \) and \( R = R_\Theta 2^{m-1} \).

Combined (13) with Theorem 1, we can infer the following

**Theorem 3.** For any positive \( \varepsilon > 0 \),

\[ 2 \left( \frac{1}{\sqrt{2 \pi}} \right)^{l-1} \frac{c(\varepsilon)}{l^2} < Rh < 2 \left( \frac{1}{\sqrt{2 \pi}} \right)^{l-1} (\log l)^\varepsilon. \]

In this special case, the result is sharper than the one obtained by R. Brauer (see [2]) for the general finite algebraic extension.

Let \( h_1 \) be the so-called first factor of the class number of \( \Omega \). Then

\[ h_1 = h/h_0 = (l\sqrt{l\pi l\pi})^{m-1} \prod_{\chi \neq \chi_0} L(1, \chi). \]

Combined this with Theorem 2, we can infer the following

**Theorem 4.** For any positive \( \varepsilon > 0 \),

\[ 2 l \left( \frac{\sqrt{l}}{2 \pi} \right)^{l-1} \frac{c(\varepsilon)}{l^2} < h_1 < 2 l \left( \frac{\sqrt{l}}{2 \pi} \right)^{l-1} (\log l)^\varepsilon. \]

The second inequality is better than the result obtained by Ankeny and Chowla (see [1]) on the extended Riemann hypothesis.

**References**


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