Proceedings of the Edinburgh Mathematical Society (1985) **28**, 59–66 ©

ON THE (J, p_n, q_n) METHOD OF SUMMATION

by B. KWEE

(Received 31st January 1984)

1. Introduction

In the following discussion we shall assume that $p_n \geqq 0$, $q_n \geqq 0$ for all n and that $q_{n+1} > q_n \to \infty$. The (J, p_n, q_n) method of summation is defined as follows.

The series $\sum_{n=0}^{\infty} a_n$, with the partial sum s_n, is called summable (J, p_n, q_n) to s, and we write $\sum_{n=0}^{\infty} a_n = s(J, p_n, q_n)$ if the series

$$\sum_{n=0}^{\infty} p_n x^{q_n} \tag{1}$$

and $\sum_{n=0}^{\infty} p_n s_n x^{q_n}$ converge to the sum functions $p^*(x)$ and $p^{(s)}(x)$ respectively for $0 < x < 1$ and if $\tau(x) = p^{(s)}(x)/p^*(x) \to s$ as $x \to 1-0$.

It is easy to verify by using Toeplitz's theorem that the (J, p_n, q_n) method is regular when $P_n = p_0 + p_1 + \dots + p_n \to \infty$, namely it sums all convergent series to their natural sums when $P_n \to \infty$.

The special case of (J, p_n, q_n) in which $q_n = n$ for $n \geqq 0$ is the P method defined in Borwein [1].

Assume further that $p_0 > 0$. The series $\sum_{n=0}^{\infty} a_n$ is called summable $(\bar{N}, p_n)$ to s if $\lim_{\to\infty} t_n = s$, where $t_n = 1/P_n \sum_{v=0}^{n} p_v s_v$. It is known that the $(\bar{N}, p_n)$ method is regular when $P_n \to \infty$.

The following four theorems will be proved in this paper. The first and last theorems are concerning the inclusion of two different summability methods. Theorem 3 is the main theorem of this paper and Theorem 2 is a Tauberian theorem.

Theorem 1. *Let $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$, and let the series in* (1) *converge for* $0 < x < 1$. *Then*

$$(\bar{N}, p_n) \subseteq (J, p_n, q_n). \tag{2}$$

The special case of this theorem in which $q_n = n$ is given in Ishiguro [3].

Theorem 2. *Assume that $P_n \to \infty$ and that*

$$\sum_{m=n+1}^{\infty} \frac{p_m p_m^*(x_n)}{P_m} = O(P_n), \tag{3}$$

where x_n *is determined by the equation* $p^*(x_n)=P_n$ *and* $p_m^*(x_n)=\sum_{v=m}^{\infty} p_v x_n^{q_v}$. *If* $\sum_{n=0}^{\infty} a=s(J,p_n,q_n)$ *and if*

$$a_n=o\left[\frac{p_n}{P_n}\right], \tag{4}$$

then $\sum_{n=0}^{\infty} a_n=s$.

There are three theorems in Ishiguro [3] and [4] in all of which it is proved that, under suitable conditions on $\{p_n\}$, (4) is a Tauberian condition for (J,p_n). All the three theorems mentioned are different from Theorem 2 with $q_n=n$ of this paper because the conditions imposed on $\{p_n\}$ are different.

Theorem 3. *Assume that*

$$p_n=o(P_n). \tag{5}$$

and that $p_0>O$, $p_n\geqq O(n\geqq 1)$, $P_n\to\infty$ *as* $n\to\infty$. *Then for any sequence* $\{\alpha_n\}$ *of non-decreasing real numbers such that* $\alpha_n\to\infty$, *however slowly, there is a divergent series with*

$$a_n=O\left[\frac{\alpha_n p_n}{P_n}\right] \tag{6}$$

and $s_n=O(1)$ *which is summable* (J,p_n,q_n).

This theorem asserts that the highest possible order of magnitude of a_n for the converse of the (J,p_n,q_n) method of summation is $O(p_n/P_n)$.

Theorem 4. *Let* $q_0\geqq 1$, $r_n=\log q_n$ *and let* $\sum_{n=0}^{\infty} a_n=s(J,p_n,q_n)$. *If both the series* $\sum_{n=0}^{\infty} p_n x^{r_n}$ *and* $\sum_{n=0}^{\infty} p_n s_n x^{r_n}$ *converge for* $0<x<1$, *then* $\sum_{n=0}^{\infty} a_n=s(J,p_n,r_n)$.

2. Proof of Theorem 1

The proof requires a partial summation, the justification for which, while easy, is not quite obvious. We require the following lemma.

Lemma. *Suppose that, for some fixed* x *with* $O<x<1$, *the series in* (1) *converges. Then*

$$P_n=o(x^{-q_n}) \tag{7}$$

as $n\to\infty$.

Proof. The convergence of the series in (1) for some fixed x with $0<x<1$ implies $p_n^*(x)=\sum_{v=n}^{\infty} p_v x^{q_v}\to 0$ as $n\to\infty$. We can write $p_n=x^{-q_n}(p_n^*(x)-p_{n+1}^*(x))$ so that

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$$P_n=\sum_{v=0}^{n}p_v=\sum_{v=0}^{n}x^{-q_v}(p_v^*(x)-p_{v+1}^*(x))$$

$$=p_0^*(x)x^{-q_0}+\sum_{v=1}^{n}p_v^*(x)(x^{-q_v}-x^{-q_{v-1}})-p_{n+1}^*(x)x^{-q_n} \tag{8}$$

By the hypothesis, $q_n\to\infty$ and $0<x<1$. It follows that

$$\sum_{v=1}^{n}(x^{-q_v}-x^{-q_{v-1}})=x^{-q_n}-x^{q_0}\to\infty$$

as $n\to\infty$. Therefore, since $p_n^*(x)\to 0$ as $n\to\infty$, we have

$$\sum_{v=1}^{n}p_v^*(x)(x^{-q_v}-x^{-q_{v-1}})=o(x^{-q_n}) \tag{9}$$

(7) follows from (8) and (9).

In the above lemma, $\{p_n\}$ can be any sequence subject to the condition that the series in (1) converges for some fixed x with $0<x<1$.

Now assume further that $p_0>0$ and that the series in (1) converges for $0<x<1$. We shall then prove that the inclusion (2) holds.

It is worth noting that for this we do not need to assume that

$$\sum_{n=0}^{\infty}p_n=\infty. \tag{10}$$

If (10) is not satisfied, then (J,p_n,q_n) is not regular. But neither is $(\bar{N},p_n)$, and the inclusion (2) still holds.

Now if $t_n=1/P_n\sum_{v=0}^{n}p_vs_v$ is bounded then it follows from the lemma that, for $0<x<1$,

$$\sum_{n=0}^{\infty}p_ns_nx^{q_n}=p_os_ox^{q_o}+\sum_{n=1}^{\infty}(P_nt_n-P_{n-1}t_{n-1})x^{q_n}$$

$$=\sum_{n=0}^{\infty}P_nt_n(x^{q_n}-x^{q_{n+1}}). \tag{11}$$

Thus we have to verify that the sequence-to-function transformation

$$\tau(x)=\frac{1}{p^*(x)}\sum_{n=0}^{\infty}P_nt_n(x^{q_n}-x^{q_{n+1}}) \tag{12}$$

is regular. The coefficients of t_n in (12) are all positive and, by the special case of (11) in which $s_n=1$ (all n) so that $t_n=1$ (all n), their sum is 1. Thus all that remains to prove is

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that, for fixed n,

$$\frac{P_n(x^{q_n}-x^{q_{n+1}})}{p^*(x)}\to 0$$

as $x\to 1-0$. But the numerator tends to 0, and the denominator is greater than or equal to $p_o x^{q_o}\to p_o$ as $x\to 1-0$. Since $p_o>0$, the result therefore follows.

3. Proof of Theorem 2

Since the assumption that the sequence $\{s_n\}$ is summable (J,p_n,q_n) implies the assumption that the series $\sum_{\nu=0}^{\infty} p_\nu s_\nu x^{q_\nu}$ converges for $0<x<1$, we have

$$\begin{aligned} t_n &= s_n-\frac{p^{(s)}(x_n)}{p^*(x_n)}=\frac{1}{p^*(x_n)}\sum_{\nu=0}^{\infty}(s_n-s_\nu)p_\nu x_n^{q_\nu} \\ &= \frac{1}{p^*(x_n)}\left(\sum_{\nu=0}^{n-1}p_\nu x_n^{q_\nu}\sum_{m=\nu+1}^{n}a_m-\sum_{\nu=n+1}^{\infty}p_\nu x_n^{q_\nu}\sum_{m=n+1}^{\nu}a_m\right) \end{aligned} \tag{13}$$

We first show that the order of the summations on the right hand side of (13) may be changed. The first double sum is a finite sum, and the order of summation may therefore be inverted. The inversion in the order of summation in the second double sum may be justified by absolute convergence. To prove absolute convergence, we do not need the full force of the assumption that $a_n=o(p_n/P_n)$, but only the weaker assumption that $a_n=O(p_n/P_n)$. Thus, for some constant M,

$$\begin{aligned} \sum_{\nu=n+1}^{\infty}p_\nu x^{q_\nu}\sum_{m=n+1}^{\nu}|a_m| &\leqq M\sum_{\nu=n+1}^{\infty}p_\nu x_n^{q_\nu}\sum_{m=n+1}^{\nu}\frac{p_m}{P_n} \\ &= \sum_{m=n+1}^{\infty}\frac{p_m}{P_m}\sum_{\nu=m}^{\infty}p_\nu x_n^{q_\nu} \\ &= \sum_{m=n+1}^{\infty}\frac{p_m}{P_m}p_m^*(x_n)<\infty \end{aligned}$$

by (3), if we take (3) as including the assumption that the sum on the left converges for all n.

We obtain from (13) on interchanging the order of summation

$$t_n=\frac{1}{P_n}\sum_{m=1}^{n}a_m\sum_{\nu=0}^{m-1}p_\nu x_n^{q_\nu}-\frac{1}{P_n}\sum_{m=n+1}^{\infty}a_m p_m^*(x_n).$$

Let $a_m=(p_m/P_m)y_m$ so that $y_m\to 0$ as $m\to\infty$ by (4). We can write

$$t_m=\sum_{m=0}^{\infty}\alpha_{nm}y_m, \tag{14}$$

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where

$$\alpha_{nm}=\begin{cases}\dfrac{p_m}{P_nP_m}\sum_{\nu=0}^{m-1}p_\nu x_n^{q_\nu}, & m\leqq n,\\ -\dfrac{p_m p_m^*(x_n)}{P_nP_m}, & m>n.\end{cases}$$

Necessary and sufficient conditions for (14) to transform every sequence converging to 0 into a sequence converging to 0 are (see Hardy [**2**], Theorem 4)

(i) $\lim_{n\to\infty}\alpha_{nm}=0$

for each m;

(ii) $\sum_{m=0}^{\infty}|\alpha_{nm}|<H$,

where H is independent of n.

Since $0<x_n<1$, we have, for $n\geqq m$,

$$|\alpha_{nm}|\leqq\frac{p_m}{P_nP_m}\cdot P_{m-1}\to 0$$

for each m, when $n\to\infty$. By (3),

$$\sum_{m=0}^{\infty}|\alpha_{nm}|\leqq\frac{1}{P_n}\sum_{m=0}^{n}\frac{p_m}{P_m}\cdot P_{m-1}+\frac{1}{P_n}\sum_{m=n+1}^{\infty}\frac{p_mP_m^*(x_n)}{P_m}<H,$$

where H is independent of n.

Thus $\lim_{n\to\infty}t_n=0$. In view of our hypothesis $\lim_{n\to\infty}[p^{(s)}(x_n)/p^*(x_n)]=s$, we therefore have $\lim_{n\to\infty}s_n=s$. This proves Theorem 2.

4. Proof of Theorem 3

We can assume without loss of generality that

$$p_n\alpha_n=o(P_n). \tag{15}$$

For suppose the theorem proved in this special case. We can then deduce the result in the general case as follows. Since $p_n=o(P_n)$, there is an increasing sequence $\{\alpha_n'\}$ of positive numbers with $\alpha_n'\to\infty$ as $n\to\infty$ such that

$$p_n\alpha_n'=o(P_n).$$

Now define $\alpha_n''=\min(\alpha_n,\alpha_n')$ so that $\alpha_n''\leqq\alpha_n'$ and

$$p_n\alpha_n''=o(P_n).$$

Applying the special case of the theorem in which (15) holds with α_n replaced by α_n'' we see that there is a divergent and bounded sequence $\{s_n\}$ summable (J, p_n, q_n) with

$$a_n = O\left(\frac{\alpha_n'' p_n}{P_n}\right) \tag{16}$$

Since $\alpha_n'' \leqq \alpha_n$, (16) gives us that

$$a_n = O\left(\frac{\alpha_n p_n}{P_n}\right)$$

and the conclusion follows.

By the hypothesis $p_0 > 0$, $p_n \geqq 0$ $(n \geqq 1)$ and $p_n \to \infty$. Hence by a well known theorem the series $\sum_{n=1}^{\infty} (p_n/P_n)$ diverges and this implies that the series $\sum_{n=0}^{\infty} (\alpha_n p_n/P_n)$ of non-negative terms diverges since $\alpha_n \to \infty$.

We will define inductively sequences $\{m_k\}$, $\{n_k\}$ of positive integers with $m_{k+1} > n_k > m_k$. Provided that m_k is sufficiently large, we can choose $n_k > m_k$ so that

$$C_k = \frac{\alpha_{m_k} p_{m_k}}{P_{m_k}} + \frac{\alpha_{m_k+1} p_{m_k+1}}{P_{m_k+1}} + \ldots + \frac{\alpha_{n_k} p_{n_k}}{P_{n_k}} \leqq \pi$$

and

$$C_k + \frac{\alpha_{n_k+1} p_{n_k+1}}{P_{n_k+1}} > \pi.$$

We now define for $m_k \leqq n \leqq n_k$

$$s_n = \sin\left\{\left(\frac{a_{m_k} p_{n_k}}{P_{m_k}} + \frac{\alpha_{m_k+1} p_{m_k+1}}{P_{m_k+1}} + \ldots + \frac{\alpha_n p_n}{P_n}\right)\beta_k\right\}, \tag{17}$$

where β_k is determined by the equation $C_k\ \beta_k = \pi$, and $s_n = 0$ for other values of n. It is clear that the sequence $\{s_n\}$ defined above oscillates between 0 and 1 and so $s_n = 0(1)$ and the series with the partial sum s_n defined in this way diverges.

It follows from (15) that $C_k \to \pi$ as $k \to \infty$ so that the sequence $\{\beta_k\}$ is bounded. Hence

$$a_n = s_n - s_{n-1} = O\left(\frac{\alpha_n p_n}{P_n}\right)$$

for the ranges $m_k \leqq n \leqq n_k$ and $a_n = 0$ outside these ranges so that (6) holds for all n.

Next we show that

$$\frac{P_{n_k}}{P_{m_k}} \to 1 \tag{18}$$

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as $k\to\infty$. Since (5) holds, we have $P_v \leqq 2P_{v-1}$ for all sufficiently large v; thus, if k is sufficiently large, this inequality will hold for all $v \geqq m_k$. Supposing that k is large enough for this to happen, we have, by the definition of n_k,

$$\pi \geqq \sum_{v=m_k}^{n_k} \frac{\alpha_v p_v}{P_v} \geqq \frac{1}{2}\alpha_{m_k} \sum_{v=m_k}^{n_k} \frac{p_v}{P_{v-1}} \geqq \frac{1}{2}\alpha_{m_k} \sum_{v=m_k}^{n_k} \int_{P_{v-1}}^{P_v} \frac{dx}{x}$$

$$= \frac{1}{2}\alpha_{m_k} \log \frac{P_{n_k}}{P_{m_k-1}} \geqq \frac{1}{2}\alpha_{m_k} \log \frac{P_{n_k}}{P_{m_k}} \tag{19}$$

Since $\alpha_{m_k}\to\infty$ as $k\to\infty$, it follows from (19) that $\log(P_{n_k}/P_{m_k})\to 0$ as $k\to\infty$, and this is equivalent to (18).

We shall now show that, when the sequence $\{m_k\}$ is suitably chosen, the series $\sum_{n=0}^{\infty} a_m$, the partial sum of which is defined by (17), is summable $(\bar{N}, p_n)$ to 0. Since $(\bar{N}, p_n) \subseteq (J, p_n, q_n)$, this means what we prove here is actually stronger than Theorem 3.

Let $\{t_n\}$ denote the $(\bar{N}, p_n)$ transform of the sequence $\{s_n\}$ defined by (17) and let $t_n^{(k)}$ denote the contribution to t_n of those s_n with $m_k \leqq n \leqq n_k$; that is to say that

$$t_n^{(k)} = \begin{cases} 0 & (n < m_k), \\ \dfrac{1}{P_n} \displaystyle\sum_{v=m_k}^{n} p_v s_v & (m_k \leqq n \leqq n_k), \\ \dfrac{1}{P_n} \displaystyle\sum_{v=m_k}^{n_k} p_v s_v & (n > n_k). \end{cases}$$

Thus, for $m_k \leqq n < n_{k+1}$, we have

$$t_n = \sum_{r=1}^{k} t_n^{(r)}. \tag{20}$$

Let $\{\eta_k\}$ be a given sequence of positive numbers tending to 0. Suppose that $m_1, m_2, m_3, \ldots m_k$ have been chosen; this fixes $t_n^{(r)}$ for $r = 1, 2, \ldots k$. Now, for given r, $\{t_n^{(r)}\}$ is the $(\bar{N}, p_n)$ transform of a sequence which has all zeros for $n > n_r$. Thus, by the regularity of $(\bar{N}, p_n)$, we have $t_n^{(r)}\to 0$ as $n\to\infty$. Hence we may choose $m_{k+1} > n_k$ so that, for $n \geqq m_{m+1}$,

$$\left| \sum_{r=1}^{k} t_n^{(r)} \right| < \eta_k.$$

Since $\eta_k\to 0$ as $k\to\infty$, it will follow with the aid of (20) that $t_n\to 0$ as $n\to\infty$ if we show that, as $k\to\infty$, $t_n^{(k)}\to 0$ uniformly in the range $m_k \leqq n \leqq m_{k+1}$. Since $0 \leqq s_n \leqq 1$, this follows easily from (18). Thus the series determined by the sequence $\{s_n\}$ we define in (17) is summable $(\bar{N}, p_n)$ to 0 and hence summable (J, p_n, q_n) to 0. This proves Theorem 3.

5. Proof of Theorem 4

We write $r(y)=\sum_{n=0}^{\infty} p_n e^{-r_n y}$, $\phi(w)=\sum_{n=0}^{\infty} p_n e^{-q_n w}$, and $\phi^{(s)}(w)=\sum_{n=0}^{\infty} p_n s_n e^{-q_n w}$. Without loss of generality we may assume that $s=0$, namely that the series is summable (J, p_n, q_n) to 0 so that for any given $\varepsilon>0$ there is a $\delta>0$ such that $|\phi^{(s)}(w)/\phi(w)|<\varepsilon$ for $0<w<\delta$. Thus, by Theorem 30 of Hardy [2],

$$\Psi(y)=\frac{1}{r(y)}\sum_{n=0}^{\infty} p_n s_n e^{-r_n y}=\frac{1}{r(y)\,\Gamma(y)}\int_0^{\infty} w^{y-1}\phi^{(s)}(w)dw$$

$$=\frac{1}{r(y)\,\Gamma(y)}\left(\int_0^{\delta}+\int_{\delta}^{\infty}\right)w^{y-1}\phi^{(s)}(w)dw$$

$$=H_1+H_2.$$

We have

$$|H_1|<\frac{\varepsilon}{r(y)\,\Gamma(y)}\int_0^{\delta} w^{y-1}\phi(w)dw<\frac{\varepsilon}{r(y)\,\Gamma(y)}\int_0^{\infty} w^{y-1}\phi(w)dw=\varepsilon.$$

For fixed $w\geqq\delta$, $e^{-q_n(w-\delta)}$ is a positive non-increasing function of n. Therefore, by Abel's lemma,

$$\left|\sum_{n=0}^{\infty} p_n s_n e^{-q_n w}\right|\leqq e^{-q_0(w-\delta)}\sup_{N\geqq 0}\left|\sum_{n=0}^{N} p_n s_n e^{-q_n\delta}\right|$$

so that $|\phi^{(s)}(w)|\leqq Ke^{-q_0 w}$ for $w\geqq\delta$. From this

$$|H_2|\leqq\frac{K\delta^{y-1}}{r(y)\,\Gamma(y)}\int_{\delta}^{\infty} e^{-q_0 w}dw<\varepsilon$$

for $0<y<\delta_0$. Finally we get $|\Psi(y)|<2\varepsilon$ for $p<y<\delta_0$. This proves Theorem 4.

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Department of Mathematics
University of Malaya
Kuala Lumpur
Malaysia