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LETTERS TO THE EDITOR

NUMBER OF SUCCESSES IN MARKOV TRIALS

U. NARAYAN BHAT AND RAM LAL, *Southern Methodist University*

Abstract

Markov trials are a sequence of dependent trials with two outcomes, success and failure, which are the states of a Markov chain. The distribution of the number of successes in n Markov trials and the first-passage time for a specified number of successes are obtained using an augmented Markov chain model.

MARKOV CHAIN; FIRST-PASSAGE TIME

Let $\{Y_n, n = 0, 1, 2 \cdots\}$ be a two-state Markov chain with states 0 (failure) and 1 (success) and transition probabilities given by

$$p_{ij} = P(Y_{n+1} = j \mid Y_n = i), \qquad n = 0, 1, 2, \cdots, \qquad i, j = 0, 1.$$

(1)
$$\begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \quad (0 \leqq a, b \leqq 1).$$

When $|1 - a - b| < 1$, the limiting probabilities exist independent of the initial state, and are given by

(2)
$$(\pi_0, \pi_1) = \left(\frac{b}{a+b}, \frac{a}{a+b}\right) = (1-p, p), \quad \text{say}.$$

Consider a sequence of Markov trials where the outcomes of repeated trials are governed by the Markov chain defined above. As in Bernoulli trials $X_n = \sum_1^n Y_r$ gives the number of successes in n trials. Using combinatorial arguments, Gabriel (1959) has given the distribution and moments of X_n. Here we provide an alternative and simpler method which is much more suitable for numerical calculations.

The key feature of this approach is an augmented Markov chain (X_n, Y_n) with the state space $\{00, 11, 10, 21, 20, 31, \cdots\}$. Define

(3)
$$P_{ij,kl} = P(X_n = k, Y_n = l \mid X_0 = i, Y_0 = j)$$

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Postal address for both authors: Department of Statistical Science, Southern Methodist University Dallas, TX 75275, USA.

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and the matrix $[p_{ij,kl}]$

$$
(4)\qquad P=\begin{array}{c}
 \\ 00\\ 11\\ 10\\ 21\\ 31\\ \cdot\\ \cdot\\ \cdot
\end{array}
\begin{array}{c|c|c|c}
\begin{array}{cc} 00 & 11 \end{array} & \begin{array}{cc} 10 & 21 \end{array} & \begin{array}{cc} 20 & 31 \end{array} & \cdots \\
\begin{array}{cc} 1-a & a \\ 0 & 0 \end{array} & \begin{array}{cc} 0 & 0 \\ b & 1-b \end{array} & 0 & \\ \hline
0 & \begin{array}{cc} 1-a & a \\ 0 & 0 \end{array} & \begin{array}{cc} 0 & 0 \\ b & 1-b \end{array} & \\ \hline
 & & \begin{array}{cc} 0 & 0 \end{array} & \\ \hline
 & & & \ddots
\end{array}
=\begin{bmatrix} A & B & & \\ & A & B & \\ & & A & B \\ & & & \ddots \end{bmatrix}.
$$

The first two rows of the matrix P^n (nth power of P) give the distribution of the number of defectives in n trials (after the initial trial). Let $p^{(n)}_{ik,kl}$ be the (ij, kl) element of P^n. Assuming that the process is in equilibrium we may write

$$P(X_0=0,\ Y_0=0)=1-p; \qquad P(X_0=1,\ Y_0=1)=p.$$

Then

$$
(5)\qquad P(X_n=k)=(1-p)[p^{(n)}_{00,k1}+p^{(n)}_{00,k0}]+p[p^{(n)}_{11,k1}+p^{(n)}_{11,k0}]
$$
$$k=0,1,2,\cdots,n.$$

A simple way of obtaining $p^{(n)}_{ij,kl}$ is by matrix multiplication after truncating the state space of the countably-infinite-state Markov chain appropriately. If explicit expressions for $p^{(n)}_{ij,kl}$ or the distribution (5) are needed, they can be determined by expanding the nth power of the triangular matrix P using recursive relations. Let $D^{(n)}_1$ be the ith submatrix of the first row of the resulting triangular matrix. The following theorem gives explicit expressions for the elements of $D^{(n)}_i$.

Theorem 1. For $n\geqq i$, let

$$
(6)\qquad D^{(n)}_i=\begin{bmatrix} d^{(n)}_i(11) & d^{(n)}_i(12)\\ d^{(n)}_i(21) & d^{(n)}_i(22)\end{bmatrix}
$$

and define

$$\delta^{**}_{gh}=1 \quad\text{if}\quad g<h, \quad\text{and}\quad =0 \quad\text{if}\quad g\geqq h$$
$$\delta^{**}_{0in}=\delta^{**}_{0i}\delta^{**}_{0n}(1-\delta^{**}_{ni}).$$

We have

$$
(7)\qquad
\begin{aligned}
d^{(n)}_i(11)&=\sum_{j=\min(1,i)}^{i}\delta^{**}_{i+j-1,n}\binom{i-1}{j-1}\binom{n-i}{j}(1-a)^{n-i-j}(1-b)^{i-j}a^jb^j\\
d^{(n)}_i(12)&=\sum_{j=1}^{i+1}\delta^{**}_{i+j-1,n}\binom{i}{j-1}\binom{n-i-1}{j-1}(1-a)^{n-i-j}(1-b)^{i+1-j}a^jb^{j-1}\\
d^{(n)}_i(21)&=\delta^{**}_{0in}(1-a)^{n-i}(1-b)^{i-1}b\\
&\quad+\sum_{j=1}^{i-1}\delta^{**}_{i+j-1,n}\binom{i-1}{j}\binom{n-i}{j}(1-a)^{n-i-j}(1-b)^{i-1-j}a^jb^{j+1}\\
d^{(n)}_i(22)&=\delta_{in}(1-b)^i+\sum_{j=1}^{i}\delta^{**}_{i+j-1,n}\binom{i}{j}\binom{n-i-1}{j-1}(1-a)^{n-i-j}(1-b)^{i-j}a^jb^j.
\end{aligned}
$$

Using the above expression in (5) the distribution of the number of successes in $(n+1)$

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trials (including the initial trial) follows:

$$P(X_n=k)=(1-p)\sum_{j=\min(1,k)}^{k}\binom{k-1}{j-1}(1-a)^{n-k-j}(1-b)^{k-j}a^jb^{j-1}$$
$$\left(b\delta^{**}_{k+j-1,n}\binom{n-k}{j}+(1-a)\delta^{**}_{0k}\,\delta^{**}_{k+j-2,n}\binom{n-k}{j-1}\right)$$
(8)
$$+p\,\delta^{**}_{0k}(1-b)^{k-1}(\delta_{k-1,n}+\delta^{**}_{0kn}b(1-a)^{n-k})$$
$$+p\sum_{j=1}^{k-1}\binom{k-1}{j}(1-a)^{n-k-j}(1-b)^{k-1-j}a^jb^j\left(b\,\delta^{**}_{k+j-1,n}\binom{n-k}{j}\right.$$
$$\left.+(1-a)\delta^{**}_{k+j-2,n}\binom{n-k}{j-1}\right),\qquad k=0,1,2,\cdots,n+1.$$

When the initial state distribution is given by (2), the mean and variance of X_n are obtained as

(9)
$$E(X_n)=np$$
$$\mathrm{Var}\,(X_n)=np(1-p)+2p(1-p)\frac{\rho}{1-\rho}\left(n-\frac{1-\rho^n}{1-\rho}\right),$$

when $\rho=1-(a+b)$ is the lag-1 serial correlation of the process Y_n.

To determine distribution characteristics of the first-passage time to get more than c successes convert state $(c+1,1)$ into an absorbing state. Now the state space is given by $\{00, 11, 20, 21, \cdots, c0, (c+1,1)\}$. Let R be the corresponding transition probability matrix and let T_{ij} be the number of transitions required to visit state $(c+1,1)$. Because of the absorbing state $(c+1,1)$, for the unconditional first-passage time T, we get

(10)
$$P(T\leqq n)=(1-p)R^{(n)}_{00,(c+1,1)}+pR^{(n)}_{11,(c+1,1)}$$

where $R^{(n)}_{ij,lm}$ are the elements of R^n and we have assumed that the process starts with an equilibrium state.

Expanding the triangular matrix R recursively, R^n is found to have $D_i^{(n)}$ as elements except for the last two columns corresponding to states $\{c0, (c+1,1)\}$. Let $E_i^{(n)}$ be the element submatrices in the last column. After simplifications we get the following result.

Theorem 2. For $n\geqq K-i$, $i=1,2,\cdots,K-1$, let

(11)
$$E_i^{(n)}=\begin{bmatrix}e_i^{(n)}(11) & e_i^{(n)}(12)\\ e_i^{(n)}(21) & e_i^{(n)}(22)\end{bmatrix}$$

and define δ functions as in Theorem 1. We have

$$e_i^{(n)}(11)=\sum_{j=1}^{K-i}\delta^{**}_{K-1-i+j,n}\binom{n-K+i}{j}\binom{K-1-i}{j-1}(1-a)^{n-K+i-j}(1-b)^{K-i-j}a^jb^j$$
$$e_i^{(n)}(12)=\sum_{j=1}^{K-i}\delta^{**}_{K-1-i+j,n}\left(\sum_{g=0}^{n-K+i-j}\binom{j-1+g}{g}(1-a)^g\right)\binom{K-1-i}{j-1}(1-b)^{K-i-j}a^jb^{j-1}$$
$$-e_i^{(n)}(11)$$
$$e_i^{(n)}(21)=\delta^{**}_{K-1-i,n}(1-a)^{n-K+i}b(1-b)^{K-1-i}$$
$$+\sum_{j=1}^{K-1-i}\delta^{**}_{K-1-i+j,n}\binom{n-K+i}{j}\binom{K-1-i}{j}(1-a)^{n-K+i-j}(1-b)^{K-1-i-j}a^jb^{j+1}$$
$$e_i^{(n)}(22)=\delta^{**}_{K-1-i,n}(1-b)^{K-1-i}$$
$$+\sum_{j=1}^{K-1-i}\delta^{**}_{K-1-i+j,n}\left(\sum_{g=0}^{n-K+i-j}\binom{j-1+g}{g}(1-a)^g\right)\binom{K-1-i}{j}$$
$$(1-b)^{K-1-i-j}a^jb^j-e_i^{(n)}(21).$$

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Let $T_{ij,kl}$ be the number of visits of the process to state (kl) before it eventually gets absorbed in state $(c+1,1)$, having originally started from state (ij). Mean and variance of $T_{ij,kl}$ are obtained in terms of the elements of the fundamental matrix of R (Kemeny and Snell (1960), Chapter 3). Noting that, for the first-passage transition $(00) \to (c+1, 1)$, $T_{00} = \sum_{k,l} T_{00,kl}$, we get $E[T_{00}] = (c(a+b)+1)/a$ and $V[T_{00}] = [1-a+(2-a-b)bc]/a^2$.

The results obtained above are useful in developing techniques for the quality control of production processes in which the quality characteristics (whether defective or not) of successive items are Markov dependent. For a sequential sampling inspection plan, the elements of the matrix R^n give the probability of acceptance when n items are inspected. The exact distribution of the number of defectives in a sample of size n given in Equation (8) directly leads to a control chart with appropriate limits. These procedures are investigated in more detail in Bhat et al. (1987) and Bhat and Lal (1988).

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