ON THE FIXED POINTS OF SYLOW SUBGROUPS OF TRANSITIVE PERMUTATION GROUPS: CORRIGENDUM

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(Received 30 August 1977)

Abstract

The proof of Theorem 5 in a paper with the same title is incorrect. In this note weaker versions of that theorem are proved.


In Herzog and Praeger (1976) we stated Theorem 5 which is incorrect for $p > 2$. Theorem 1 and Corollaries 2–4 are unaffected, as well as Lemmas 2.1 and 2.2. It follows from Praeger (1978b) and Theorem 1 that Corollary 7 is true.

Using the results of Praeger (1978b) we shall prove the following weaker version of Theorem 5.

**THEOREM 5'.** Let $G$ be a transitive permutation group on a set $\Omega$ of $n$ points, and let $P$ be a Sylow $p$-subgroup of $G$ for some prime $p$ dividing $|G|$. Suppose that $P$ has $t$ long orbits and $f$ fixed points in $\Omega$, and suppose that $f = tp - 1$. If $P$ has an orbit of length $p$, then $t = 1$, $n = 2p - 1$ and $G \supseteq A_n$.

**Proof.** By Praeger (1978a) it follows that all long orbits of $P$ have the same length, namely $p$. Hence $f = tp - 1 = \frac{1}{2}(n - 1)$, and by Praeger (1978b) $t = 1$, $n = 2p - 1$ and $G \supseteq A_n$.

Finally we shall show that Theorem 5 holds for $p = 2$ and $f > 0$.

**THEOREM 5'**. Let $G$ be a transitive permutation group on a set $\Omega$ of $n$ points, and let $S$ be a nontrivial Sylow 2-subgroup of $G$. Suppose that $S$ has $t$ long orbits and $f$ fixed points in $\Omega$, and suppose that $f = 2t - i_2(n) > 0$. Then $t = f = i_2(n) = 1$ and $G$ is 2-transitive. If the long $S$-orbit has length 2, then $n = 3$ and $G \supseteq S_3$. 383
PROOF. If \( n \leq 3 \), then Theorem 5'' clearly holds. Assume, by induction, that the result is true for transitive groups of degree less than \( n \). By Wielandt (1964) 3.7, \( |N(S) : S| \) is divisible by \( f = 2t - i_d(n) \). Since \( |N(S) : S| \) is odd, \( f \) is odd, and hence \( i_d(n) = 1 \).

Let \( \Sigma = \{B_1, \ldots, B_r\} \) be a set of blocks of imprimitivity for \( G \) in \( \Omega \). Since \( f > 0 \) and \( S \) fixes setwise any block containing a point of \( \text{fix}_\Omega S \), it follows that \( \text{fix}_\Sigma S \) is non-empty. Let \( B \in \text{fix}_\Sigma S \) and set \( f_B = |\text{fix}_B S|, f_\Sigma = |\text{fix}_\Sigma S| \). Denote by \( t_B \) and \( t_\Sigma \) the number of long \( S \)-orbits in \( B \) and \( \Sigma \), respectively. Suppose first that \( S \) acts nontrivially on \( B \). Then by Herzog and Praeger (1976) Theorem 1, \( f_B = 2t_B - d \) for some \( d \geq 1 \). Hence by Herzog and Praeger (1976), Lemma 1.2,

\[
2t - 1 = f = f_\Sigma f_B = 2f_\Sigma t_B - f_\Sigma d \leq 2t - f_\Sigma d
\]
as \( f_\Sigma t_B \) is the number of long \( S \)-orbits in \( U(B | B \in \text{fix}_\Sigma S) \). Therefore \( f_\Sigma = d = 1 \) and \( t_B = f_\Sigma t_B = t \), from which we conclude that \( |\Sigma| = 1 \). On the other hand, if \( S \) acts trivially on \( B \), then by Herzog and Praeger (1976), Lemma 1.2 and Theorem 1,

\[
f = |B|f_\Sigma, \quad t = |B|t_\Sigma \quad \text{and} \quad f_\Sigma = 2t_\Sigma - d
\]

for some \( d \geq 1 \). Hence \( 2t - 1 = f = 2t - |B|d \) and so \( |B| = 1 \). Thus \( G \) is primitive on \( \Omega \).

Let \( \alpha \in \text{fix}_\Omega S \) and let \( \Gamma_1, \ldots, \Gamma_r, r \geq 1 \), be the orbits of \( G_\alpha \) on \( \Omega \setminus \{\alpha\} \). By Wielandt (1964), 18.4, \( S \) acts nontrivially on each \( \Gamma_i \). Let \( S \) have \( f_i \) fixed points and \( t_i \) long orbits in \( \Gamma_i \) for \( 1 \leq i \leq r \). Then by Herzog and Praeger (1976), Theorem 1, \( f_i = 2t_i - d_i \) for some \( d_i \geq 1, 1 \leq i \leq r \), and so

\[
2t - 1 = f = 1 + \sum f_i = 1 + \sum(2t_i - d_i) = 2t + 1 - \sum d_i.
\]

that is, \( \sum d_i = 2 \). If \( r > 1 \), then \( r = 2 \) and \( d_1 = d_2 = 1 \). By induction \( G_\alpha \) is 2-transitive on \( \Gamma_1 \) and \( \Gamma_2 \), a contradiction to Wielandt (1964), 17.7. Hence \( r = 1 \), that is \( G \) is 2-transitive. If \( f > 1 \), then by Wielandt (1964), 3.7 applied to \( G \) and to \( G_\alpha, |N(S) : S| \) is divisible by the even integer \( f(f - 1) \), a contradiction. Hence \( f = 1 \) and so \( t = 1 \). Finally, if \( S \) has an orbit of length 2, then \( n = 3 \) and \( G \cong S_3 \).

References


C. E. Praeger (1978a), 'Sylow subgroups of transitive permutation groups, II' (to appear).

C. E. Praeger (1978b), 'On transitive permutation groups with a subgroup satisfying a certain conjugacy condition' (submitted).