Preregular maps between Banach lattices

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A continuous linear map from a Banach lattice $E$ into a Banach lattice $F$ is preregular if it is the difference of positive continuous linear maps from $E$ into the bidual $F''$ of $F$. This paper characterizes Banach lattices $B$ with either of the following properties:

(1) for any Banach lattice $E$, each map in $L(E, B)$ is preregular;

(2) for any Banach lattice $F$, each map in $L(B, F)$ is preregular.

It is shown that $B$ satisfies (1) (respectively (2)) if and only if $B'$ satisfies (2) (respectively (1)). Several order properties of a Banach lattice satisfying (2) are discussed and it is shown that if $B$ satisfies (2) and if $B$ is also an atomic vector lattice then $B$ is isomorphic as a Banach lattice to $l^1(\Gamma)$ for some index set $\Gamma$.

1. Introduction

The following natural question arises in the theory of Banach lattices: Given Banach lattices $E$ and $F$, is each map in the space $L(E, F)$ of continuous linear maps from $E$ into $F$ the difference of positive (continuous) linear maps? It is known that if $F$ is a $C(X)$ for $X$ an extremally disconnected, compact Hausdorff space $X$ or if $E$ is an $\ell_1$-space and $F$ has the monotone convergence property then the answer to
this question is affirmative ([10], Chapter 4, (3.7) and (3.8)). On the
other hand, \( L(L^2, L^2) \) contains maps that are not the difference of
positive linear maps, ([10], Chapter 4, (3.3)). Schlotterbeck [16] has
shown that if \( F \) is an AM-space or if \( E \) is an AL-space then each map
in \( L(E, F) \) is the difference of positive linear maps into the bidual \( F'' \)
of \( F \).

The main goal of this paper is to characterize (in §3) Banach lattices
\( B \) with either of the following properties.

PROPERTY I. For any Banach lattice \( E \), each map in \( L(E, B) \) is
the difference of positive linear maps of \( E \) into \( B'' \).

PROPERTY II. For any Banach lattice \( F \), each map in \( L(B, F) \) is
the difference of positive linear maps of \( B \) into \( F'' \).

The characterizations that we obtain indicate that such spaces are
similar to AM- and AL-spaces. In particular, we show that a Banach
lattice has Property I (respectively Property II) if and only if \( B' \) has
Property II (respectively Property I).

In §4, we study some order properties of a Banach lattice with
Property II. We also show that \( L^p[0, 1] \), \( 1 < p < \infty \), possesses neither
Property I nor Property II. Finally in §5 we show that with the additional
assumption that a Banach lattice \( G \) is an atomic lattice, \( G \) is
isomorphic as a Banach lattice to \( l^1(\Gamma) \) for some index set \( \Gamma \) whenever
\( G \) has Property II.

2. Preliminary material

For general terminology and notation concerning functional analysis we
refer the reader to [15] while our reference for ordered locally convex
spaces will be [10].

By a map between Banach spaces we will always mean a continuous linear
map. A sequence in a Banach space is summable (respectively absolutely
summable) if it is unconditionally convergent (respectively absolutely convergent).

A map from a Banach lattice \( E \) into a Banach space \( F \) is order
summable if it maps positive summable sequences into absolutely summable
sequences. \( S_+(E, F) \) denotes the space of order summable maps from \( E \) into \( F \). A map from a Banach space into a Banach lattice is majorizing if its adjoint is order summable. Majorizing maps can also be defined as maps that take null sequences into order bounded sets; (see [16], Chapter 1, and [4]).

A map from a Banach space \( E \) into a Banach space \( F \) is absolutely summable if it maps summable sequences into absolutely summable sequences. The space of absolutely summable maps of \( E \) into \( F \) is denoted by \( S(E, F) \). A map between Banach spaces is absolutely majorizing (hyper-majorizing in [16]) if its adjoint is absolutely summable. The following results characterize these types of maps. For proofs see [16], (3.5), (3.6), and (3.7), or [6], (6.6), (6.7), and (6.8).

**Proposition 2.1.** If \( E \) and \( F \) are Banach spaces and if \( T \in L(E, F) \), then the following statements are equivalent:

1. \( T \) is absolutely summable;
2. \( T' \) is absolutely majorizing;
3. for every Banach lattice \( H \) and \( S \in L(H, E) \), \( T \circ S \) is order summable;
4. for each \( S \in L(c_0, E) \), \( T \circ S \) is order summable.

**Proposition 2.2.** If \( E \) and \( F \) are Banach spaces and if \( T \in L(E, F) \), then the following statements are equivalent:

1. \( T \) is absolutely majorizing;
2. \( T' \) is absolutely summable;
3. for every Banach lattice \( H \) and \( S \in L(F, H) \), \( S \circ T \) is majorizing;
4. for every \( S \in L(F, l^1) \), \( S \circ T \) is majorizing.

Absolutely summable and absolutely majorizing maps can be factored through Hilbert spaces ([16], (3.8)) and so are weakly compact.

We need the following four topologies on the tensor product \( E \otimes_F F \) of two Banach spaces \( E \) and \( F \).

1. \( E \otimes_F F \) is the completion of \( E \otimes F \) for the norm
(ii) $E \bar{\otimes} F$ is the completion of $E \otimes F$ for the norm

$$
\|u\| = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.
$$

where $u = \sum_{i=1}^{n} x_i \otimes y_i$ is any representation of $u \in E \otimes F$.

(iii) $E \hat{\otimes} F$ is the completion of $E \otimes F$ for the norm

$$
\|u\| = \inf \left\{ \sup_{|\sigma| \leq 1} \|\sum_{i=1}^{n} x_i \sigma_i y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.
$$

(iv) If $E$ is also a Banach lattice with cone $K$ then $E \check{\otimes} F$ is the completion of $E \otimes F$ for the norm

$$
\|u\| = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \text{ and } x_i \in K \right\}.
$$

Jacobs [4] has shown that $\tau_\varepsilon \leq \tau_\sigma \leq \tau_{|\sigma|} \leq \tau_n$, that

$(E \hat{\otimes} F)' = S(E, F'),$ and that $(E \check{\otimes} F)' = S^+(E, F')$. Moreover, for maps $T \in S^+(E, F)$, the norm of $T$ in $(E \check{\otimes} F)'$ is given by

$$
|\sigma|'(T) = \inf \left\{ M : \sum_{i=1}^{n} \|Tx_i\| \leq M \sup_{i=1}^{n} |\langle x_i, x' \rangle| \right\}
$$

for all finite sets $\{x_1, \ldots, x_n\} \subset K$.

Let $E$ and $F$ be Banach lattices with cones $K$ and $H$ respectively. In $E \otimes F$ define the projective cone $K_p$ by

$$
K_p = \left\{ \sum_{i=1}^{n} x_i \otimes y_i : x_i \in K, y_i \in H \right\}.
$$

Then it is easy to see that for the dual system $(E \check{\otimes} F, S^+(E, F'))$, $K_p'$ equals the cone of positive maps.
in $S_+(E, F')$. It follows from the definition of the dual norm $|a|'$ that this latter cone is normal in $S_+(E, F')$ for the $|a|'$-topology.

Since this cone is also generating in $S_+(E, F')$ ([16], (2.2)), it follows from [10], Chapter 2, (1.22), that $\overline{K}_p$ is normal and generating in $E \otimes \|0\| F$.

If $E$ and $F$ are Banach lattices and if $T \in \mathcal{L}(E, F)$, then $T$ is regular (respectively preregular) if $T$ is the difference of positive linear maps of $E$ into $F$ (respectively $E$ into $F''$).

If $E$ and $F$ are Banach spaces and $T \in \mathcal{L}(E, F)$ then $T$ is integral if the bilinear form $b_T$ defined on $E \times F'$ by

$$b_T(x, y) = (Tx, y)$$

is an element of $(E \otimes F')'$. Integral maps are both absolutely summable and absolutely majorizing. For more information see [16], Chapter 3, or [6], Chapters 5 and 6.

A Banach lattice $E$ is an AM-space if $x, y \in E$, $x, y \geq 0$, imply that $\|xy\| = \|x\| \vee \|y\|$. A Banach lattice $E$ is an AL-space if $x, y \in E$, $x, y > 0$ imply that $\|x+y\| = \|x\| + \|y\|$. If $F$ is an AL-space then $E \otimes F = E \otimes F$ for all Banach spaces $E$ (for a proof of this result see [6], (6.3)).

The following characterizations of AM- and AL-spaces (see [16], Chapters 1 and 4) are included so that we may compare them with our results, Theorems 3.3 and 3.7.

**Proposition 2.3.** The following statements about a Banach lattice $E$ are equivalent:

1. $E$ is isomorphic as a Banach lattice to an AM-space;
2. every null sequence in $E$ is majorized;
3. every order summable map from $E$ into a Banach space is integral.

**Proposition 2.4.** The following statements about a Banach lattice $E$ are equivalent:
(1) $E$ is isomorphic as a Banach lattice to an $AL$-space;

(2) every positive summable sequence in $E$ is absolutely summable;

(3) every majorizing map from a Banach space into $E$ is integral.

If $E$ is a Banach lattice and $0 \leq x \in E$ we denote by $E_x$ the linear hull of the order interval $[-x, x]$ with $[-x, x]$ as unit ball.

If we order $E_x$ by restricting the order on $E$ then $E_x$ is an $AM$-space.

3. Preregular maps

In this section we characterize Banach lattices with Property I or Property II and show that Property I and Property II are dual to each other.

DEFINITION. If $E$ is a Banach space, then $l^1[E]$ will denote the space of summable sequences in $E$ with the norm

$$
\varepsilon([x_n]) = \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |\langle x_n, x' \rangle|.
$$

Pietsch [14] has shown that $l^1[E]$ is isomorphic to $l^1 \otimes \varepsilon E$. If, in addition, $E$ is a Banach lattice, then this isomorphism is an order isomorphism where we consider the cone $C$ of positive summable sequences in $l^1[E]$ and the closure $\overline{K}_p$ of the projective cone in $l^1 \otimes \varepsilon E$. In this case the norm

$$
\overline{\varepsilon}([x_n]) = \sup_{\|x\| \leq 1} \sum_{n=1}^{\infty} |\langle x_n, x' \rangle|_{x' \geq 0}
$$

is equivalent to the $\varepsilon$ norm.

LEMMA 3.1. If $E$ is a Banach lattice then $(l^1[E], C)$ is a vector lattice if and only if every summable sequence in $E$ is the difference of positive summable sequences. In this case, $(l^1[E], C)$ is a Banach lattice.
Proof. Necessity is clear, so suppose that for every \( \{x_n\} \in \ell^1[E] \), \( \{x_n\} = \{y_n\} - \{z_n\} \) where \( \{y_n\} \) and \( \{z_n\} \) are positive summable sequences. Since \( 0 \leq x_n^+ \leq y_n \), it easily follows that \( \{x_n^+\} \) is summable. Therefore, \( \{x_n^+\} = \{x_n\}^+ \) and \( \ell^1[E] \) is a vector lattice.

When \( \ell^1[E] \) is a vector lattice, the \( \ell \) norm is clearly monotone on the cone \( C \) and so \( \ell^1[E] \) is a Banach lattice by [8], (8.4).

EXAMPLES 3.2. \( \ell^1 \hat{\otimes}_E \ell^1 \) is not a vector lattice. For if it were, then every summable sequence in \( \ell^1 \) would be the difference of positive summable sequences. But then every summable sequence in \( \ell^1 \) would be absolutely summable (by Proposition 2.4) which would contradict the Dvoretzky-Rogers Theorem.

If \( X \) is a compact Hausdorff space, then \( \ell^1[C(X)] \) is a vector lattice. This is contained in Remarks 3.4, but can easily be shown directly by noting that \( \ell^1 \hat{\otimes}_E C(X) \) is order and topologically isomorphic to \( C(X, \ell^1) \), the space of continuous functions from \( X \) into \( \ell^1 \), and that the cone of positive functions in \( C(X, \ell^1) \) is a lattice cone.

We now characterize Banach lattices with Property I.

THEOREM 3.3. If \( H \) is a Banach lattice with cone \( K \) then the following assertions about \( H \) are equivalent:

1. every summable sequence in \( H \) is the difference of positive summable sequences;

2. \( \left( \ell^1 \hat{\otimes}_E H, K_p \right) \) is a vector lattice;

3. \( H \) has Property I, that is, every map from a Banach lattice into \( H \) is preregular;

4. each map in \( L(c_0, H) \) is preregular;

5. every order summable map from \( H \) into a Banach space is absolutely summable.
Proof. That (1) is equivalent to (2) follows from Lemma 3.1 and that (3) implies (4) is clear.

(2) implies (3). Let $E$ be a Banach lattice and $T \in L(E, H)$. Since $H''$ is a Banach lattice all positive linear maps from $E$ into $H''$ are continuous ([10], Chapter 2, (2.16)), and so it suffices to show that $T^+ : E \rightarrow H''$ exists. To do this we must show that for each $x \in E$, $x \geq 0$, the set

$$B_x = \left\{ \sum_{n=1}^{k} (Tx_n)^+ : x = \sum_{n=1}^{k} x_n, x_n \in K \text{ for all } n \right\}$$

has a supremum in $H''$; (see [10], Chapter 2, Section 2, equation (9)).

By Lemma 3.1, $\ell^1 \otimes_E H$ is a Banach lattice and hence the map $\phi : \ell^1 \otimes_E H \rightarrow \ell^1 \otimes_E H$ defined by $\phi([y_n]) = [y_n^+]$ is continuous. Let

$\psi : \ell^1 \otimes_E H \rightarrow H$ be defined by $\psi([y_n]) = \sum_{n=1}^{\infty} y_n$. Then $\psi$ is continuous, since

$$\|\psi([y_n])\| = \sup_{\|y'\| \leq 1} \left| \sum_{n=1}^{\infty} \langle y_n, y' \rangle \right|$$

$$\leq \sup_{\|y'\| \leq 1} \sum_{n=1}^{\infty} |\langle y_n, y' \rangle|$$

$$= \phi([y_n]).$$

For each $0 \leq x \in E$, define

$$A_x = \left\{ [x_n] \in \ell^1[E] : \text{for some } k, \ x_n = 0 \text{ for } n \geq k+1, \right.$$ 

$$x_n \geq 0 \text{ for all } n \text{ and } \sum_{n=1}^{k} x_n = x \right\}.$$

Then for $[x_n] \in A_x$,

$$\overline{c}[x_n] = \sup_{\|x'\| \leq 1, x' \geq 0} \sum_{n=1}^{k} \langle x_n, x' \rangle = \|x\|.$$
Therefore, the set $A_x$ is bounded in $\ell_1^* \overset{E}{\otimes} E$ and hence the set

$$[\psi \varnothing(1 \otimes T)](A_x) = \{\sum_{n=1}^k (T x_n)^+ : x_n \in A_x\} = B_x$$

is topologically bounded in $H$. A standard argument using the decomposition lemma shows that $B_x$ is also directed ($\preceq$). By [10], Chapter 4, (1.8), $H''$ is boundedly order complete (that is, every topologically bounded, directed ($\preceq$) subset has a supremum) and so $B_x$ has a supremum in $H''$.

(4) implies (5). Suppose that $F$ is a Banach space, that $T : H \to F$ is order summable and that $S \in L(\sigma_0, H)$. Then $S = S_1 - S_2$ where $0 \preceq S_1, S_2 \in L(\sigma_0, H'')$. Therefore, $T'' \circ S = T'' \circ S_1 - T'' \circ S_2$ is order summable and hence $T''$ is absolutely summable by Proposition 2.1. It follows from Propositions 2.1 and 2.2 that $T$ is absolutely summable.

(5) implies (2). If $F$ is a Banach space, then $[H \otimes F]' = S(H, F) = S_+(H, F) = [H \otimes |\sigma| \otimes F]'$. Since $\sigma$ and $|\sigma|$ are both norm topologies it follows that $\sigma = |\sigma|$. Hence, in particular,

$$H \otimes |\sigma| \ell_1^* = H \otimes \ell_1 = H \overset{E}{\otimes} \ell_1 .$$

Therefore, by the discussion in §2, the cone $K_F$ is generating in $\ell_1^* \overset{E}{\otimes} H$ and so the latter space is a vector lattice.

This completes the proof of Theorem 3.3.

REMARKS 3.4. (1) It follows from the proof of Theorem 3.3 that if $H$ is boundedly order complete (for example, if $H$ is a dual Banach lattice and satisfies any of the conditions of Theorem 3.3) then each map in $L(E, H)$ is regular and so $L(E, H)$ is a vector lattice for every Banach lattice $E$.

(2) If $H$ is an AM-space then $H$ has Property II. This follows from [10], Chapter 4, (3.7), and the fact that $H$ is isomorphic as a Banach lattice to a $C(X)$ for a stonian space $X$. 


(3) In Theorem 3.3, (2) is equivalent to (3) in more general circumstances. In particular, if $H$ is a Fréchet lattice such that $H'$ is barrelled or such that $H$ is boundedly order complete then this equivalence holds. If $\{H_n\}$ is a sequence of such spaces with Property I then one can show that $\bigcap_{n=1}^{\infty} H_n$ is also a Fréchet lattice with Property I. Therefore, $\bigcap_{n=1}^{\infty} C(X_n)$, where each $X_n$ is a compact, Hausdorff space, is an example of a Fréchet lattice with Property I.

(4) If $H$ is a nuclear Fréchet lattice then $l^1 \hat{\otimes}_e H = l^1 \hat{\otimes}_H H$ which equals the space of absolutely summable sequences in $H$ ([14]). The latter space is a vector lattice and so $l^1 \hat{\otimes}_e H$ is a vector lattice. Since $H'$ is barrelled, it follows that $H$ has Property I.

(5) The characterization in Proposition 2.3 of an AM-space indicates that a Banach lattice with Property I is similar to an AM-space.

We now show that Properties I and II are dual to each other.

**PROPOSITION 3.5.** A Banach lattice $B$ has Property II (respectively Property I) if and only if its dual $B'$ has Property I (respectively Property II).

Proof. $B$ has Property II implies $B'$ has Property I. Suppose that $E$ is a Banach lattice and that $T \in L(E, B')$. Then $T' : B'' \to E'$ and the map $S = T'|_B : B \to E'$ is regular, since $B$ has Property II.

Therefore, $S' : E'' \to B'$ is regular. If $x \in E$ and $y \in B$ then $$\langle S'x, y \rangle = \langle x, Sy \rangle = \langle x, T'y \rangle = \langle Tx, y \rangle.$$ Therefore, $S'|_E = T$ and hence $T$ is regular.

$B'$ has Property I implies $B$ has Property II. Suppose that $E$ is a Banach lattice and that $T \in L(B, E)$. Then $T' : E' \to B'$ is regular and so $T'' : B'' \to E''$ is regular. Therefore, $T = T''|_B$ is preregular.

$B$ has Property I implies $B'$ has Property II. We first show that every majorizing map into $B'$ is absolutely majorizing. Let $E$ be a
Banach space and $S \in L(E, B')$ be majorizing. Then $S' : B'' \to E'$ is order summable and so $T = S'|_B : B \to E'$ is order summable and hence absolutely summable since $B$ has Property I. Therefore, $T' : E'' \to B'$ is absolutely majorizing and so, by Proposition 2.2, $R \circ T'$ is majorizing for all $R \in L(B', l^1)$ . Hence, $(R \circ T')|_E = R \circ (T'|_E)$ is majorizing.

Since this is true for all $R \in L(B', l^1)$ , Proposition 2.2 implies that $T'|_E = S$ is absolutely majorizing.

We now show that $B''$ has Property I. This will imply that $B'$ has Property II by an earlier part of this theorem. By Theorem 3.3 it is enough to show that $L(c_0, B'')$ is a vector lattice and to do this it suffices to show that the projective cone in $c_0 \mathcal{F} B'$ is normal. Since $c_0 \mathcal{F} B' = B' \mathcal{F} c_0$, this is equivalent to showing that $L(B', l^1)$ is a vector lattice. So, let $T \in L(B', l^1)$ . To see that $T^+$ exists in $L(B', l^1)$ let $x \in B'$ , $x \geq 0$ , and let $B'_{\infty}$ be the linear hull of $[-x, x]$ with unit ball $[-x, x]$ . Let $I : B'_{\infty} \to B'$ be the inclusion map. Since $B'_{\infty}$ is an AM-space and $I$ is a positive map, $I$ is majorizing and hence absolutely majorizing by what we have just proved about $B'$ . Proposition 2.2 thus implies that $T \circ I$ is majorizing. It follows that $T \circ I([-x, x]) = T([-x, x])$ is bounded above in $l^1$ (see [16], (1.5)). Therefore, $T^+$ exists and is necessarily continuous. Since $T$ was chosen arbitrarily in $L(B', l^1)$, it follows that $L(B', l^1)$ is a vector lattice.

$B'$ has Property II implies $B$ has Property I. Suppose that $E$ is a Banach lattice and $T \in L(E, B)$ . Then $T' : B' \to E'$ is regular and so $T'' : E'' \to B''$ is regular. Therefore, $T''|_E = T$ is preregular.

**COROLLARY 3.6.** If $H$ is a Banach lattice with Property I then $L(E, H'')$ is an order complete vector lattice for any Banach lattice $E$ .

**Proof.** This follows from Proposition 3.5, (3.4) of Chapter 4 in [10], and the fact that there is a positive continuous projection from $H''$ into $H''$ .
THEOREM 3.7. If $G$ is a Banach lattice then the following assertions about $G$ are equivalent:

(1) for any Banach lattice $F$, the closure of the projective cone $K_{p}$ is normal in $G \otimes F$;

(2) $K_{p}$ is normal in $G \otimes c_{0}$;

(3) $G$ has Property II, that is, every map from $G$ into a Banach lattice is preregular;

(4) $L(G, l^{1})$ is a vector lattice;

(5) every majorizing map from a Banach space into $G$ is absolutely majorizing.

Proof. That (1) implies (2), and that (2) is equivalent to (4) is clear. If (2) holds then $L(c_{0}, G')$ is a vector lattice and so $G'$ has Property I by Theorem 3.3. Therefore, $G$ has Property II by Proposition 3.5. (3) easily implies (1) by [10], Chapter 2, (1.22), and so (1), (2), (3), and (4) are equivalent.

Suppose that (3) holds, that $E$ is a Banach space, and that $T : E \rightarrow G$ is majorizing. Then $T' : E' \rightarrow G'$ is order summable and hence absolutely summable since $G'$ has Property I. Therefore, $T$ is absolutely majorizing and hence (5) holds. Finally, if (5) holds then one can show that $L(G, l^{1})$ is a vector lattice in a manner similar to the latter part of the proof of Proposition 3.5. This completes the proof.

We remark that Proposition 2.4 and Theorem 3.7 show that a Banach lattice with Property II has a characterization similar to that of an $AL$-space.

4. Order properties of a Banach lattice with Property II

PROPOSITION 4.1. A Banach lattice $G$ with Property II has $\sigma(G, G')$ compact order intervals.

Proof. Let $x \geq 0$ in $G$. Since $G_{x}$ is an $AM$-space, the inclusion map $I : G_{x} \rightarrow G$ is majorizing, hence absolutely majorizing by Property II. Therefore $I([-x, x]) = [-x, x]$ is $\sigma(G, G')$ compact in $G$. 

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DEFINITIONS. A sequence \( \{x_n\} \) in a Banach space \( E \) is called weakly summable if \( \sum_{n=1}^{\infty} |\langle x_n, x' \rangle| < \infty \) for all \( x' \in E' \).

It follows from the fact that weakly bounded sets in \( E \) are norm bounded that if \( \{x_n\} \) is a weakly summable sequence then

\[
e(\{x_n\}) = \sup_{\|x'\| \leq 1} \sum_{n=1}^{\infty} |\langle x_n, x' \rangle|\]

is finite. Note that \( \{x_n\} \) is summable in \( E \) if it is weakly summable and the net \( \{\{x_n\}_{n \in \sigma} : \sigma \text{ is a finite subset of } N\} \), where \( N \) is the set of natural numbers, converges to \( \{x_n\}_{n=1}^{\infty} \) for the \( \varepsilon \)-topology on \( l^1[E] \).

For more details see [12], Chapter 1.

A Banach lattice has the monotone convergence property if the filter of sections of every directed (\( \leq \) ) topologically bounded subset converges to its supremum.

PROPOSITION 4.2. For a Banach lattice \( E \) the following statements are equivalent:

1. \( E \) has the monotone convergence property;
2. every positive weakly summable sequence is summable;
3. every increasing, topologically bounded sequence converges to its supremum.

Proof. (1) implies (2). Let \( \{x_n\} \) be a positive weakly summable sequence in \( E \). For each finite subset \( \sigma \) of \( N \),

\[
\| \sum_{n \in \sigma} x_n \| \leq \sup_{\|x'\| \leq 1} \left( \sum_{n \in \sigma} \langle x_n, x' \rangle \right) \\
\leq \sup_{\|x'\| \leq 1} \sum_{n=1}^{\infty} \langle x_n, x' \rangle \leq e(\{x_n\}) < \infty.
\]

Therefore \( \left\{ \sum_{n \in \sigma} x_n : \sigma \text{ is a finite subset of } N \right\} \) is a topologically bounded, directed (\( \leq \)) subset of \( E \) and so converges by the monotone
convergence property. Hence \( \{x_n\} \) is summable.

(2) implies (3). Let \( \{x_n\} \) be an increasing, topologically bounded sequence in \( E \). Without loss of generality we can assume that \( x_n \geq 0 \) for all \( n \). Define \( y_n = x_n - x_{n-1} \) (where \( x_0 = 0 \)). Then

\[
\sum_{n=1}^{k} y_n = \sum_{n=1}^{k} (x_n - x_{n-1}) = x_k
\]

and so for \( 0 \leq x' \in E' \) we have that

\[
\sum_{n=1}^{k} (y_n, x') = (x_k, x') \leq \|x_k\|\|x'\|
\]

\[
\leq \left( \sup_n \|x_n\| \right) \|x'\| < \infty.
\]

This is true for all \( k \) so that the sequence \( \{y_n\} \) is weakly summable and hence summable by (2). Since \( E \) is complete, \( \sum_{n=1}^{\infty} y_n \) exists, that is, the sequence \( \{x_n\} \) converges to some \( x \in E \), and \( x = \sup\{x_n\} \) since the cone in \( E \) is closed.

(3) implies (1). Assume that \( E \) does not have the monotone convergence property. Then there exists a topologically bounded, directed (\( \leq \)) net \( \{x_\alpha : \alpha \in A\} \) such that \( \alpha \geq \beta \) if and only if \( x_\alpha \geq x_\beta \) and such that \( \{x_\alpha\} \) does not converge. Hence there is a \( \delta > 0 \) such that there is no \( \alpha_0 \in A \) with the property that \( \|x_\alpha - x_\beta\| < \delta \) for all \( \alpha, \beta \geq \alpha_0 \). Let \( \alpha_1 \in A \) and choose \( \alpha_2 > \alpha_1 \) such that \( \|x_{\alpha_2} - x_{\alpha_1}\| \geq \frac{\delta}{2} \). Now choose \( \alpha_3 > \alpha_2 \) such that \( \|x_{\alpha_3} - x_{\alpha_2}\| \geq \frac{\delta}{2} \). Continuing in this way we get a monotone increasing, topologically bounded sequence \( \{x_\alpha\} \) that does not converge.

**PROPOSITION 4.3.** A Banach lattice \( G \) with Property II has the monotone convergence property.

**Proof.** Let \( \{x_n\} \) be a weakly summable positive sequence in \( G \). By (1.3.5) in [12], \( \{\lambda_n x_n\} \) is summable for each \( \{\lambda_n\} \in c_0 \) and so we can
define a map \( T : \mathcal{A}_0 \to G \) by \( T(\{\lambda_n\}) = \sum_{n=1}^{\infty} \lambda_n x_n \). \( T \) is continuous since

\[
\sup_{\{\lambda_n\} \leq 1} \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \sup_{\{\lambda_n\} \leq 1} \sup_{\left\| x' \right\| \leq 1} \left| \left\langle \sum_{n=1}^{\infty} \lambda_n x_n, x' \right\rangle \right|
\]

\[
\leq \sup_{\left\| x' \right\| \leq 1} \sum_{n=1}^{\infty} \left| \left\langle x_n, x' \right\rangle \right|
\]

\[
= \varepsilon(\{x_n\}) < \infty.
\]

\( T \) is positive since \( \{x_n\} \) is positive, and hence \( T \) is majorizing since \( \mathcal{A}_0 \) is an AM-space. Therefore, by Property II, \( T \) is absolutely majorizing and hence weakly compact.

Let \( U \) be the unit ball in \( \mathcal{A}_0 \). Then \( T(U) \) is weakly relatively compact and contains the net \( A = \left\{ \sum_{n \in I} x_n : I \text{ is a finite subset of } N \right\} \).

Hence every subnet of \( A \) contains a convergent subnet which necessarily must converge to \( \sup A \) ([10], Chapter 2, (3.1)). By a standard property of net convergence this implies that \( A \) converges to \( \sup A \) for \( \mathcal{A}(G, G') \) and hence for the norm topology of \( G \) by [10], Chapter 2, (3.4). It follows that \( \{x_n\} \) is summable and so \( G \) has the monotone convergence property by Proposition 4.2.

In an AL-space every summable sequence is absolutely summable. A Banach lattice with Property II has a weaker property. In order to describe this property we need the following definitions.

DEFINITIONS (see [13]). A sequence \( \{x_n\} \) in a normed space \( E \) is called (weakly) \( p \)-summable \( (p \geq 1) \) if \( \sum_{n=1}^{\infty} \left| \langle x_n, x' \rangle \right|^p < \infty \) for all \( x' \in E' \). \( \{x_n\} \) is called absolutely \( p \)-summable if \( \sum_{n=1}^{\infty} \|x_n\|^p < \infty \). A continuous linear map \( T \) from a normed space \( E \) into a normed space \( F \) is called absolutely \( p \)-summable if \( T \) maps \( p \)-summable sequences into absolutely \( p \)-summable sequences.

PROPOSITION 4.4. Every positive summable sequence in a Banach
lattice $G$ with Property II is absolutely 2-summable.

Proof. Let $\{x_n\}$ be a positive summable sequence in $G$, let

$$x = \sum_{n=1}^{\infty} x_n,$$

and consider the $AM$-space $G_x$. The sequence $\{x_n\}$ is weakly summable in $G_x$. For, let $a \in G_x^\prime$, $a \geq 0$. Then

$$\sum_{n=1}^{k} |\langle x_n, a \rangle| = \sum_{n=1}^{k} \langle x_n, a \rangle = \langle \sum_{n=1}^{k} x_n, a \rangle \leq \langle x, a \rangle.$$ 

Since $k$ is arbitrary it follows that $\sum_{n=1}^{\infty} |\langle x_n, a \rangle| < \infty$ and so $\{x_n\}$ is weakly summable in $G_x$. Therefore $\{x_n\}$ is 2-summable in $G_x$.

Consider the inclusion map $I : G_x \to G$. As we have seen before in Proposition 4.2, $I$ is absolutely majorizing and hence $I$ can be factored through a Hilbert space $H$, that is, there exist continuous linear maps $I_1 : G_x \to H$ and $I_2 : H \to G$ such that the following diagram commutes:

$$\begin{array}{c}
G_x \\
\downarrow \quad I_1 \\
H \\
\uparrow \quad I_2 \\
G \\
\end{array}$$

By [5], (4.3), $I_1$ is absolutely 2-summable. Therefore $I$ is absolutely 2-summable and so $\{I(x_n)\} = \{x_n\}$ is absolutely 2-summable in $G$.

**EXAMPLE 4.5.** $L^p[0,1]$, $1 < p \leq \infty$, does not have Property II. For $p = \infty$ this follows immediately since $L^\infty[0,1]$ does not have the monotone convergence property. For $p < \infty$ we consider two cases. First assume that $p > 2$. Let $\{E_n\}$ be a sequence of disjoint sets of positive measure. Let $\phi_n(x)$ be that positive multiple of the characteristic function of $E_n$ such that $\int_0^1 |\phi_n(x)|^p = 1$. Choose a sequence $\{a_n\}$
such that \( a_n \geq 0 \), \( \sum_{n=1}^{\infty} a_n^2 = \infty \), and \( \sum_{n=1}^{\infty} a_n^p < \infty \). Then the series
\[
\sum_{n=1}^{\infty} a_n \phi_n(x)
\]
converges unconditionally in \( L^p[0, 1] \) but
\[
\sum_{n=1}^{\infty} \|a_n \phi_n(x)\|^2 = \sum_{n=1}^{\infty} \left( \int_0^1 |a_n \phi_n(x)|^p \right)^{2/p} = \sum_{n=1}^{\infty} a_n^2 = \infty.
\]
Hence \( L^p[0, 1] \) cannot have Property II by Proposition 4.4.

Now assume that \( 1 < p \leq 2 \) and let \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \).

Suppose on the contrary that \( L^p[0, 1] \) does have Property II. Let 
\( T : l^2 \rightarrow l^2 \), be a nonregular map (see [10], pp. 171-172). We note that for 
\( r > 1 \), \( l^2 \) is isomorphic to the complemented subspace of \( L^r[0, 1] \) 
generated by the Rademacher functions and, moreover, this isomorphism sends 
the \( n \)th unit vector in \( l^2 \) into the \( n \)th Rademacher function in \( L^r[0, 1] \) 
(see [9]). So, let \( S : l^2 \rightarrow L^q[0, 1] \) be this isomorphism and let 
\( R : L^p[0, 1] \rightarrow l^2 \) be a continuous projection. Then the map \( T \) can be 
factored as
\[
\begin{array}{c}
S \\
\uparrow \\
L^q[0, 1] \rightarrow L^p[0, 1] \rightarrow l^2
\end{array}
\]
where \( I \) is the continuous, positive injection of \( L^q[0, 1] \) into 
\( L^p[0, 1] \) which exists since \( q \geq p \). Since we are assuming that \( L^p[0, 1] \) 
has Property II, \( T \circ R \) is a regular map and \( S \) is a regular map by 
Proposition 3.5. Hence \( T = T \circ R \circ I \circ S \) is regular, a contradiction.

5. The atomic case

DEFINITIONS. A positive element \( x \) in a vector lattice is called an 
atom if \( 0 \leq y \leq x \) implies that \( y = ax \) for some \( a \in [0, 1] \). An order
A complete vector lattice $E$ is called atomic if the band generated by the atoms is equal to $E$.

The $l^p$ spaces, $1 \leq p < \infty$, are examples of atomic lattices.

In this section we prove that if $G$ is a Banach lattice with Property II, and, in addition, $G$ is atomic, then $G$ is isomorphic as a Banach lattice to $L^1(\Gamma)$ for some index set $\Gamma$.

First, suppose that $G$ is any Banach lattice with Property II and that $\{x_n\}$ is a positive summable sequence in $G$ that is disjoint; that is, $x_n \wedge x_m = 0$ for $n \neq m$. Define the functional $x'_m$ on $L.H.\{x_n\}$ (where $L.H.$ denotes linear hull) by

$$\langle x'_m, \sum \alpha_n x_n \rangle = \alpha_m \|x_m\|,$$

where, of course, $\alpha_n = 0$ for all but a finite number of $n$. Then $x'_m$ is continuous on $L.H.\{x_n\}$ since

$$\|\langle x'_m, \sum \alpha_n x_n \rangle\| = \|\alpha_m \|x_m\|\|
= \|\alpha_m \|x_m\| \leq \sum \|\alpha_m \|x_n\|
= \|\sum \alpha_n x_n\| = \|\sum \alpha_n x_n\|\|,$$

the next to last equality resulting from the fact that the $x_n$'s are disjoint. It follows that each $x'_m$ can be extended to a continuous linear functional $x'_m$ of norm 1 on $X = L.H.\{x_n\}$.

If we define $z_n = x_n/\|x_n\|$, then $\{z_n, x'_m\}$ is a biorthogonal system (that is, $\langle z_n, x'_m \rangle = \delta_{nm}$) in $X$. Moreover, $\{z_n\}$ is an unconditional basis for $X$ such that if $\sum_{n=1}^{\infty} \alpha_n z_n$ is convergent in $X$ then

$$\sum_{n=1}^{\infty} \alpha_n^2 < \infty.$$ To see this, define $U_m$ on $X$ by
\[ U_m(x) = \sum_{n=1}^{m} \langle x', x \rangle z_n, \quad x \in X. \]

\( U_m \) is continuous since \( x_n' \) is continuous for each \( n \) and if
\[
\sum \alpha_n z_n \in \text{L.H.}\{z_n\} = \text{L.H.}\{x_n\} \quad \text{then}
\]
\[
\left\| U_m \left( \sum \alpha_n z_n \right) \right\| = \left\| \sum_{n=1}^{m} \alpha_n z_n \right\|
\]
\[
= \left\| \sum_{n=1}^{m} \left| \alpha_n \right| z_n \right\| \leq \left\| \sum \alpha_n z_n \right\|.
\]

Since \( U_m(z_n) = z_m \) it follows that \( \left\| U_m \right\| = 1 \). It now follows from [7], Corollary 3, p. 31, that \( \{z_n\} \) is a basis for \( X \).

If \( x \in X \) and \( x = \sum_{n=1}^{\infty} \alpha_n z_n \) then \( \alpha_n = \langle x, x_n' \rangle \) for all \( n \) and so
\[
x = \sum_{n=1}^{\infty} \langle x, x_n' \rangle z_n.
\]
Then the disjointness of the \( z_n' \)'s and continuity of the lattice operations imply that
\[
|x| = \sum_{n=1}^{\infty} |\langle x, x_n' \rangle| z_n.
\]
Hence, given \( \delta > 0 \) there exists an \( n_0 \) such that
\[
\left\| \sum_{n=n_0}^{\infty} |\langle x, x_n' \rangle| z_n \right\| < \delta.
\]

If \( \sigma \) is any finite subset of \( N \) such that \( \sigma \) is disjoint from \( \{1, 2, \ldots, n_0\} \) then
\[
\left\| \sum_{n \in \sigma} \langle x, x_n' \rangle z_n \right\| = \left\| \sum_{n \in \sigma} |\langle x, x_n' \rangle| z_n \right\|
\]
\[
\leq \left\| \sum_{n=n_0}^{\infty} |\langle x, x_n' \rangle| z_n \right\|
\]
\[
< \delta.
\]
Hence, $\sum_{n=1}^{\infty} (x, x'_n)z_n$ converges unconditionally to $x$, $\sum_{n=1}^{\infty} |(x, x'_n)|z_n$ converges unconditionally to $|x|$, and it follows from Proposition 4.4 that

$$\sum_{n=1}^{\infty} ||(x, x'_n)z_n||^2 = \sum_{n=1}^{\infty} ||\alpha_n z_n||^2 \leq \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty.$$ 

We make use of these results in the following proposition.

**PROPOSITION 5.1.** Suppose that $G$ is a Banach lattice with Property II and that $\{x_n\}$ is a positive, disjoint, summable sequence in $G$. If $X = \overline{\text{L.H.}}\{x_n\}$ is complemented in $G$ then $\{x_n\}$ is absolutely summable.

**Proof.** Let $P$ be a continuous projection of $G$ onto $X$. By the considerations preceding the statement of this proposition, we can define a map $T : X \to l^2$ by

$$T\left(\sum_{n=1}^{\infty} (x, x'_n)z_n\right) = \{(x, x'_n)\}.$$

A simple application of the Closed Graph Theorem shows that $T$ is continuous (since convergence implies coordinate convergence in $X$ and in $l^2$). Let $R : l^2 \to L^1[0, 1]$ be the isomorphism that sends $e_n$ into the $n$th Rademacher function $\phi_n$. If $S = R \circ T \circ P$, then $S : G \to L^1[0, 1]$ and $S(x_n) = \|x_n\|\phi_n$. Moreover, since $G$ has Property II, $S$ is a regular map and hence $S$ is order summable. Therefore, the positive sequence $\{x_n\}$ gets mapped into the absolutely summable sequence $\{\|x_n\|\phi_n\}$. Since $\|\phi_n\|_{L^1} = 1$, it follows that

$$\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \|x_n\|\phi_n\| < \infty$$

and so $\{x_n\}$ is absolutely summable.

**COROLLARY 5.2.** Suppose that $G$ is a Banach lattice with Property II
and that \( \{x_n\} \) is a positive, disjoint, summable sequence of atoms in \( G \).

Then \( \{x_n\} \) is absolutely summable.

Proof. First note that in Banach lattices \( E \) with the monotone convergence property the closure of a lattice ideal is a band. For, let \( L \) be a lattice ideal in \( E \) and let \( x \geq 0 \) be an element of the band generated by \( L \). Then \( x = \sup \{y \in L : 0 \leq y \leq x\} \). The set \( \{y \in L : 0 \leq y \leq x\} \) is directed \((\leq)\), and so by the monotone convergence property its filter of sections converges to \( x \). Therefore \( x \) is in the closure of \( L \).

If we now consider our original sequence \( \{x_n\} \) then \( \text{L.H.}\{x_n\} \) is a lattice ideal in \( G \). For if \( 0 \leq y \leq \sum_{n=1}^{k} \alpha_n x_n \), then \( \alpha_n \geq 0 \) and

\[
y = y \land \sum_{n=1}^{k} \alpha_n x_n
\]

\[
= y \land \left( \alpha_1 x_1 \lor \alpha_2 x_2 \lor \ldots \lor \alpha_k x_k \right)
\]

\[
= \left( y \land \alpha_1 x_1 \right) \lor \left( y \land \alpha_2 x_2 \right) \lor \ldots \lor \left( y \land \alpha_k x_k \right)
\]

\[
= \sum_{n=1}^{k} \beta_n x_n \in \text{L.H.}\{x_n\},
\]

since each \( x_n \) is an atom. Therefore \( X = \text{L.H.}\{x_n\} \) is a band in \( G \) and so is complemented in \( G \) by \([10]\), Chapter 2, (4.9). The result now follows from Proposition 5.1.

**THEOREM 5.3.** If \( G \) is an atomic Banach lattice with Property II then \( G \) is order and topologically isomorphic to the Banach lattice \( L^1(\Gamma) \) for some index set \( \Gamma \).

Proof. By Zorn's Lemma there exists a maximal, disjoint collection \( \{z_\alpha : \alpha \in \Lambda\} \) of atoms of norm one. \( G \) is equal to the band generated by the \( \{z_\alpha\} \) and, since \( \text{L.H.}\{z_\alpha\} \) is a lattice ideal, if \( x \geq 0 \) in \( G \) then

\[x = \sup \{y \in \text{L.H.}\{z_\alpha\} : 0 \leq y \leq x\} \,.
\]

By the monotone convergence property the filter of sections of \( \{y \in \text{L.H.}\{z_\alpha\} : 0 \leq y \leq x\} \) converges to \( x \) and so, since \( G \) is a Banach
space, there exists a sequence \( \{y_n\} \in L.H.\{z_\alpha\} \) such that \( y_n \to x \). It follows for each \( x \geq 0 \) in \( G \), and hence for each \( x \) in \( G \), that \( x \) is in the closure of the linear hull of a countable number of the \( \{z_\alpha\} \).

Then, by the methods similar to those used in the remarks preceding Proposition 5.1, it is easy to see that \( \{z_\alpha : \alpha \in A\} \) is an unconditional basis for \( G \) and for each \( x \in G \), \( x = \sum_{\alpha \in A} a_\alpha z_\alpha \) where all but a countable number of the \( a_\alpha \)'s are zero. Moreover, it follows from Corollary 5.2 that \( \sum_{\alpha \in A} a_\alpha z_\alpha \) is absolutely summable and so \( \sum_{\alpha \in A} |a_\alpha| < \infty \).

Hence we may define a map \( T : G \to l^1(A) \) by \( T(z_\alpha) = e_\alpha \), where \( e_\alpha \) is the \( \alpha \)th unit vector in \( l^1(T) \). \( T \) is clearly a positive, one-to-one, onto map. The continuity of \( T \) follows from the Closed Graph Theorem and the fact that in \( G \) and in \( l^1(A) \) convergence implies coordinate convergence. Hence \( T \) is an isomorphism by the Open Mapping Theorem.

**COROLLARY 5.4.** Suppose that \( G \) is a Banach lattice with Property II. If \( G \) has atoms then \( G \) has a complemented subspace that is order isomorphic to an \( l^1(I) \) for some index set \( I \).

Proof. Let \( G_1 \) be the band in \( G \) generated by the atoms and let \( P : G \to G_1 \) be the canonical, positive, continuous band projection. Let \( F \) be a Banach lattice and \( T \in L(G_1, F) \). Then \( T \circ P \) is preregular and so \( T = T \circ P \circ I \) is preregular where \( I : G_1 \to G \) is the inclusion map. Therefore, \( G_1 \) has Property II. The result now follows from Theorem 5.3.

**COROLLARY 5.5.** \( l^p \), \( 1 < p < \infty \) has neither Property I nor Property II.

We conjecture that any Banach lattice with Property II is isomorphic as a Banach lattice to an \( AL \)-space.
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