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Conjugacy Classes of Subalgebras of the Real Sedenions

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Abstract. By applying the Cayley–Dickson process to the division algebra of real octonions, one obtains a 16-dimensional real algebra known as (real) sedenions. We denote this algebra by $\mathbf{A}_4$. It is a flexible quadratic algebra (with unit element 1) but not a division algebra.

We classify the subalgebras of $\mathbf{A}_4$ up to conjugacy (*i.e.*, up to the action of the automorphism group G of $\mathbf{A}_4$) with one exception: we leave aside the more complicated case of classifying the quaternion subalgebras. Any nonzero subalgebra contains 1 and we show that there are no proper subalgebras of dimension 5, 7 or $>$ 8. The proper non-division subalgebras have dimensions 3, 6 and 8. We show that in each of these dimensions there is exactly one conjugacy class of such subalgebras. There are infinitely many conjugacy classes of subalgebras in dimensions 2 and 4, but only 4 conjugacy classes in dimension 8.

1 Introduction

The real Cayley–Dickson algebra $\mathbf{A}_n$, $n \geq 0$, is a non-associative algebra of dimension 2^n. It has an involution (*i.e.*, an involutory anti-automorphism) $x \to \bar{x}$. For simplicity, we shall assume that the underlying vector space of $\mathbf{A}_n$ is the coordinate space $\mathbf{R}^{2^n}$. These algebras are defined recursively as follows. The algebra $\mathbf{A}_0$ is just the field $\mathbf{R}$ of real numbers with trivial involution: $\bar{x} = x$, for all $x \in \mathbf{R}$. We can write $x, y \in \mathbf{A}_{n+1}$ as $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbf{A}_n$. Then the product and the involution in $\mathbf{A}_{n+1}$ are defined by

$$xy := (x_1 y_1 - \bar{y}_2 x_2, y_2 x_1 + x_2 \bar{y}_1), \tag{1.1}$$

$$\bar{x} := (\bar{x}_1, -x_2). \tag{1.2}$$

We shall identify $x_1 \in \mathbf{A}_n$ with $(x_1, 0) \in \mathbf{A}_{n+1}$. Then $\mathbf{A}_n$ becomes a subalgebra of $\mathbf{A}_{n+1}$ and the involution of $\mathbf{A}_{n+1}$ extends that of $\mathbf{A}_n$. The identity $1 \in \mathbf{R}$ is also the identity element of each $\mathbf{A}_n$. Thus we obtain an infinite chain of algebras

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \mathbf{A}_2 \subset \mathbf{A}_3 \subset \mathbf{A}_4 \subset \cdots$$

The first four of these algebras are $\mathbf{A}_0 = \mathbf{R}$, $\mathbf{A}_1 = \mathbf{C}$ (the field of complex numbers), $\mathbf{A}_2 = \mathbf{H}$ (the division algebra of Hamilton quaternions), and $\mathbf{A}_3 = \mathbf{O}$ (the non-associative division algebra of Cayley octonions). The next algebra in the above sequence, $\mathbf{A}_4$, is the first one that is not a division algebra, *i.e.*, it has non-zero elements

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x and y such that $xy = 0$. This algebra has attracted the attention of physicists, and there are several papers dedicated to its study (see [1, 9, 10] and the references given there). It has been given the name of (real) *sedenions*, which we have also adopted in this paper.

Let us agree that whenever we write an element $x \in \mathbf{A}_n$ as $x = (x_1, x_2)$, it will be understood that $x_1, x_2 \in \mathbf{A}_{n-1}$ (and that $n > 0$).

For integer i in the range $0 \leq i < 2^n$, let e_i denote the $(i+1)$-st basis vector of $\mathbf{A}_n$. In particular, $e_0 = 1$. We have $\bar{e}_i = -e_i$ for $i \neq 0$ and, of course, $\bar{e}_0 = e_0$. Let $x, y \in \mathbf{A}_n$ be written as $x = \sum \xi_i e_i$ and $y = \sum \eta_i e_i$ where $\xi_i, \eta_i \in \mathbf{R}$.

The *trace* of $x \in \mathbf{A}_n$ is defined by $\operatorname{tr}(x) := x + \bar{x} = 2\xi_0$. The standard inner product $(x \mid y)$ in $\mathbf{A}_n$ can be expressed using the multiplication map as follows:

$$(x \mid y) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \sum \xi_i \eta_i.$$

In particular, the square of the norm of x is given by $\|x\|^2 = x\bar{x} = \sum \xi_i^2$.

We say that $x \in \mathbf{A}_n$ is *pure* if $\operatorname{tr}(x) = 0$. We denote by $\mathbf{A}_n^{\mathrm{pu}}$, the subspace of pure elements of $\mathbf{A}_n$. We say that $x = (x_1, x_2) \in \mathbf{A}_n$ is *doubly pure* in $\mathbf{A}_n$ if x_1 and x_2 are both pure in $\mathbf{A}_{n-1}$. The space of all doubly pure elements of $\mathbf{A}_n$ will be denoted by $\mathbf{A}_n^{\mathrm{pp}}$.

Since $x + \bar{x} = \operatorname{tr}(x) \in \mathbf{R}$, the flexible law $(xy)x = x(yx)$, which is valid in $\mathbf{A}_n$, implies that also $(xy)\bar{x} = x(y\bar{x})$ for all $x, y \in \mathbf{A}_n$. Hence, the map $\mathbf{A}_n \times \mathbf{A}_n \to \mathbf{A}_n$ sending (x, y) to $xy\bar{x}$ is well defined. If x is a unit vector, then $\bar{x} = x^{-1}$ is the inverse of x and the map $\mathbf{A}_n \to \mathbf{A}_n$ sending $y \to xy\bar{x}$ will be called *conjugation* by x.

It is well known that the automorphism group of $\mathbf{A}_n$ is trivial if $n = 0$, it is cyclic of order 2 if $n = 1$, it is isomorphic to $SO(3)$ if $n = 2$, and it is the connected compact simple Lie group of type G_2 if $n = 3$. We shall write G_2 for the automorphism group of $\mathbf{A}_3 = \mathbf{O}$ and G for that of the sedenions, $\mathbf{A}_4$.

Since we will be interested exclusively in the sedenions, we introduce the following alternative notation for some of the standard basis vectors:

$$i = e_1, j = e_2, k = e_3, l = e_4, e = e_8.$$

Then we have

$$ij = k, il = e_5, jl = e_6, kl = e_7, ie = e_9, je = e_{10}, \ldots, (kl)e = e_{15}.$$

Now let

$$\zeta := \frac{-1 + \sqrt{3}e}{2} \in \mathbf{A}_4.$$

Then the conjugation map

$$\mu(x) := \zeta x \bar{\zeta}, \quad x \in \mathbf{A}_4,$$

is an automorphism of $\mathbf{A}_4$ of order 3. One has $\mu(1) = 1$, $\mu(e) = e$, and $\mu(x) = x\zeta$ for $x \in \mathbf{O}^{\mathrm{pu}}$. If $\alpha, \beta \in \mathbf{R}$ and $x, y \in \mathbf{O}^{\mathrm{pu}}$, then

$$\begin{aligned}\mu(\alpha + \beta e + x + ye) &= \alpha + \beta e + x\zeta + (y\zeta)e \\ &= \alpha + \beta e - \frac{1}{2}(x + \sqrt{3}y) + \frac{1}{2}(\sqrt{3}x - y)e.\end{aligned}$$

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There is also an important automorphism τ of $\mathbf{A}_4$ of order 2. It is defined by

$$\tau(x + ye) = x - ye, \quad x, y \in \mathbf{O}.$$

The automorphisms μ and τ generate a subgroup Σ_3 isomorphic to the symmetric group on three letters.

Every automorphism $\sigma \in G_2$ of $\mathbf{O}$ extends to an automorphism $\hat{\sigma}$ of the sedenions by setting

$$\hat{\sigma}(x + ye) := \sigma(x) + \sigma(y)e, \quad x, y \in \mathbf{O}.$$

Hence, by identifying σ and $\hat{\sigma}$, we can consider G_2 as a subgroup of G. In 1967 R. B. Brown [2] proved that

$$G = G_2 \times \Sigma_3.$$

He also made a conjecture about the structure of the automorphism group of $\mathbf{A}_n$ for $n \geq 4$ (using a more general definition of the Cayley–Dickson process). His conjecture was later proved by Eakin and Sathaye [4].

We say that two subalgebras S and S' of $\mathbf{A}_4$ are G-*conjugate* (or simply *conjugate*) if $S' = \sigma(S)$ for some $\sigma \in G$. Our main objective is to determine the conjugacy classes of subalgebras of the sedenions.

Let us highlight some of the main results that we have obtained. Let S denote a proper subalgebra of $\mathbf{A}_4$. Thus $\mathbf{R} \subset S$ and $\dim(S) < 16$.

In Section 4 we show that $\dim(S)$ has to be 1, 2, 3, 4, 6 or 8. Examples of subalgebras in each of these dimensions are easy to provide (see Section 2).

Section 3 contains some results on two special classes of elements: the zero divisors and alternative elements.

If S is not a division algebra, we show that its dimension is 3, 6 or 8 and in each of these cases there is exactly one conjugacy class of subalgebras (see Theorem 8.1). Hence, the classification problem is thereby reduced to the case of division subalgebras.

Any 2-dimensional subalgebra is isomorphic to $\mathbf{C}$ but there are infinitely many conjugacy classes. In Section 5 we study the action of G and G_2 on the unit spheres $S^{13} \subset \mathbf{A}_4^{\mathrm{pp}}$ and $S^{14} \subset \mathbf{A}_4^{\mathrm{pu}}$. We determine the orbit spaces and show that the canonical maps $S^{13} \to S^{13}/G_2$ and $S^{13} \to S^{13}/G$ have cross sections. Then in Section 6 we apply these results to obtain a set of representatives for the G-conjugacy classes of 2-dimensional subalgebras.

There are exactly two isomorphism classes of 4-dimensional subalgebras: the quaternion subalgebras and another division algebra which we call S_4. Two subalgebras of the latter isomorphism type are always conjugate but this is not so in the case of quaternion subalgebras. In fact, there are infinitely many conjugacy classes of quaternion subalgebras.

In Theorem 7.1 we show that any quaternion subalgebra is conjugate to a quaternion subalgebra inside a particular octonion subalgebra, which we call $\mathbf{O}_{i,j,e}$. The first author has now completed the classification of the G-conjugacy classes of quaternion subalgebras (see [3]).

Finally we show that there are exactly three conjugacy classes of 8-dimensional division subalgebras. Two of these classes consist of algebras isomorphic to the octonions.

2 Preliminaries

We say that $x \in \mathbf{A}_n$ is a *zero-divisor* in $\mathbf{A}_n$ if $xy = 0$ for some nonzero element $y \in \mathbf{A}_n$. The *associator* of $x, y, z \in \mathbf{A}_n$ is defined by

$$(x, y, z) := (xy)z - x(yz).$$

It is well known that all algebras $\mathbf{A}_n$ are *flexible, i.e.,* that $(xy)x = x(yx)$ for all $x, y \in \mathbf{A}_n$.

Let us recall the definition of alternative elements. We say that $a \in \mathbf{A}_n$ is *alternative* in $\mathbf{A}_n$ if the associator (a, a, x) vanishes for all $x \in \mathbf{A}_n$. Let us write $a = \alpha + \beta e_{2^{n-1}} + b$ with $\alpha, \beta \in \mathbf{R}$ and $b \in \mathbf{A}_n^{\mathrm{pp}}$. Then it is easy to check that a is alternative iff b is alternative.

For $a \in \mathbf{A}_n$, the left (resp., right) multiplication operator L_a (resp. R_a) is the linear operator on $\mathbf{A}_n$ defined by $L_a(x) := ax$ (resp., $R_a(x) := xa$).

We refer to Moreno's paper [12] and the survey paper [8] by Khalil and Yiu for the proofs of the following facts.

Proposition 2.1

(i) *If $x, y \in \mathbf{A}_n^{\mathrm{pu}}$, then $x \perp y$ iff $xy = -yx$.*
(ii) *$(xy \mid z) = (y \mid \bar{x}z) = (x \mid z\bar{y})$ for all $x, y, z \in \mathbf{A}_n$.*
(iii) *$\|xy\| = \|yx\| = \|\bar{x}y\| = \|x\bar{y}\|$ for all $x, y, z \in \mathbf{A}_n$.*
(iv) *If $x \in \mathbf{A}_n$ is a zero-divisor in $\mathbf{A}_n$, then $x \in \mathbf{A}_n^{\mathrm{pp}}$.*
(v) *A sedenion $x = (x_1, x_2)$ is a zero-divisor iff it is doubly pure, $\|x_1\| = \|x_2\|$, and $x_1 \perp x_2$.*
(vi) *The linear operators L_a and R_a with $a \in \mathbf{A}_n^{\mathrm{pu}}$ are skew-symmetric (with respect to the standard inner product).*
(vii) *For any subalgebra $S \subset \mathbf{A}_n$, $SS^\perp \subset S^\perp$ and $S^\perp S \subset S^\perp$.*

Note that (vi) and (vii) are immediate consequences of (ii).

We shall need yet another known result (see [8, Theorem 3.2.3]). (Note that [12, Corollary 2.14] is incorrectly stated.)

Proposition 2.2 *The group G_2 acts freely and transitively on*

$$\{(x, y) \in \mathbf{A}_4 \times \mathbf{A}_4 : \|x\| = \|y\| = 1,\ xy = 0\}.$$

We conclude this section with examples of subalgebras S of $\mathbf{A}_4$ of dimensions 2, 3, 4, 6 and 8. These examples will be used throughout the paper.

If S is a 2-dimensional subalgebra of $\mathbf{A}_4$, then S is spanned by 1 and an element $a \in \mathbf{A}_4^{\mathrm{pu}}$ with $a^2 = -1$. This subalgebra, isomorphic to $\mathbf{C}$, will be denoted by $\mathbf{C}_a$.

In dimension 3 we give only one example: the subalgebra $S_3 = \langle 1, i + le, j + (kl)e \rangle$ (spanned by the indicated elements) and point out that $(i + le)((j + (kl)e) = 0$. We shall see later (Proposition 8.1) that any 3-dimensional subalgebra is conjugate to S_3.

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In dimension 4 we give three examples:

$$\mathbf{H} = \mathbf{A}_2 = \langle 1, i, j, k \rangle,$$

$$\mathbf{H}_{i,e} := \langle 1, i, e, ie \rangle,$$

$$S_4 := \langle 1, i + je, l + (kl)e, il - (jl)e \rangle.$$

It is easy to write down the multiplication table of S_4. If we set $a := i + je$, $b := l + (kl)e$ and $c := il - (jl)e$, then $ab = 2c$, $bc = 2a$ and $ca = 2b$.

Clearly $\mathbf{H}_{i,e}$ is isomorphic to $\mathbf{H}$. Since the 2-element set $\{\pm e\}$ is G-invariant, these two subalgebras are not conjugate. As the subalgebra S_4 is not associative, it is not isomorphic to $\mathbf{H}$. We shall see later (Corollary 3.5) that any 4-dimensional subalgebra $S \subset \mathbf{A}_4$ is a division algebra and (Theorem 7.1) that S is either isomorphic to $\mathbf{H}$ or conjugate to S_4.

If $a, b \in \mathbf{O}^{\mathrm{pu}}$ are orthogonal unit vectors, then we denote by $\mathbf{H}_{a,b}$ the 4-dimensional subalgebra generated by a and b. There is an isomorphism $\mathbf{H} \to \mathbf{H}_{a,b}$ sending $i \to a$ and $j \to b$. In fact, it follows from Proposition 2.2 that $\mathbf{H}_{a,b}$ is conjugate to $\mathbf{H}$.

In dimension 6 we give only one subalgebra:

$$S_6 := S_3 + S_3 e = \langle 1, i + le, j + (kl)e, e, l - ie, kl - je \rangle.$$

We shall see later (Proposition 8.1) that any 6-dimensional subalgebra is conjugate to S_6.

Finally, in dimension 8 we give 4 examples. The first three are division algebras, while the last one is not:

$$\mathbf{O} = \mathbf{A}_3 = \mathbf{H} + \mathbf{H}l,$$

$$\mathbf{O}_{i,j,e} := \mathbf{H} + \mathbf{H}e,$$

$$R_8 := S_4 + S_4 e,$$

$$S_8 := \mathbf{H} + \mathbf{H}(le).$$

The subalgebra $\mathbf{O}_{i,j,e}$ is isomorphic but not conjugate to $\mathbf{O}$, while R_8 is not isomorphic to $\mathbf{O}$. We shall see later (Theorems 8.1 and 9.3) that any 8-dimensional subalgebra S is conjugate to one of these four. Note that both S_6 and S_8 contain S_3.

3 Alternative Elements and Zero-Divisors of Sedenions

In this section we collect some additional facts about the alternative elements and zero-divisors of sedenions. We start by describing two kinds of subspaces of $\mathbf{A}_4$ which play an important role in this paper.

The first type are the subspaces all of whose elements are alternative elements of $\mathbf{A}_4$. In the case of sedenions there is a simple characterization of doubly pure alternative elements $a = (b, c)$. It says that such a is alternative iff b and c are linearly dependent. Let us say that a subspace is *alternative* if each of its elements is alternative. These subspaces are described in [8, Corollary 4.6.4]. We shall prove the following special case of their result.

Proposition 3.1 *The alternative subspaces $V \subset \mathbf{A}_4^{\mathrm{pp}}$ are of two types:*

(i) *$V = V_0 a$, where V_0 is a subspace of $\mathbf{O}^{\mathrm{pu}}$ and $a \in \mathbf{C}_e$ a unit vector,*
(ii) *$V = x\mathbf{C}_e$ for some $x \in \mathbf{O}^{\mathrm{pu}}$.*

Proof It is clear that the subspaces mentioned in (i) and (ii) are indeed alternative.

Any nonzero alternative element $x \in \mathbf{A}_4^{\mathrm{pp}}$ can be written as $x = va$, where $v \in \mathbf{O}^{\mathrm{pu}}$ and $a \in \mathbf{C}_e$ is a unit vector. Moreover, this decomposition is unique apart from $x = (-v)(-a)$.

Denote by V_0 the orthogonal projection of V to $\mathbf{O}^{\mathrm{pu}}$. Let $x, y \in V$ be nonzero vectors and write them as $x = va$, $y = wb$ with $v, w \in V_0$ and unit vectors $a, b \in \mathbf{C}_e$. Since $x + y \in V$ is alternative, it is easy to see that either v and w or a and b must be linearly dependent. It follows that if $\dim(V_0) > 1$ then V must be of type (i).

If $\dim(V_0) = 1$, then $\dim(V)$ is 1 or 2. In the former case V is of type (i) and in the latter of type (ii). ∎

The second type are the subspaces all of whose elements are zero-divisors in $\mathbf{A}_4$. More generally, if V is a subspace of a subalgebra $S \subset \mathbf{A}_4$ such that every $x \in V$ is a zero-divisor in S, then we say that V is a *ZD-subspace* of S. In particular, we shall prove that the maximum dimension of a *ZD*-subspace of $\mathbf{A}_4$ is 6.

We now give a method for constructing *ZD*-subspaces.

Let V_0 be a subspace of $\mathbf{O}^{\mathrm{pu}}$. If $\varphi \colon V_0 \to \mathbf{O}^{\mathrm{pu}}$ is any linear map, we define the subspace $V_0(\varphi)$ by

$$V_0(\varphi) = \{x + \varphi(x)e : x \in V_0\}.$$

If φ is an isometry, then

$$V_0(\varphi) \perp V_0(-\varphi).$$

A *special isometry* is an isometry $\varphi \colon V_0 \to \mathbf{O}^{\mathrm{pu}}$ such that $x \perp \varphi(x)$ for all $x \in V_0$. It is easy to verify that if φ is a special isometry, then $V_0(\varphi)$ is a *ZD*-subspace of $\mathbf{A}_4$.

Proposition 3.2 *Let V be a ZD-subspace of $\mathbf{A}_4$ and d its dimension. Then there exist a unique d-dimensional subspace $V_0 \subset \mathbf{O}^{\mathrm{pu}}$ and a unique special isometry $\varphi \colon V_0 \to \mathbf{O}^{\mathrm{pu}}$ such that $V = V_0(\varphi)$.*

Proof Recall first that every zero-divisor is doubly pure. Hence, we can define linear maps $\pi_0, \pi_1 \colon V \to \mathbf{O}^{\mathrm{pu}}$ by writing $z \in V$ as $z = (x, y) = x + ye$ and setting $\pi_0(z) = x$ and $\pi_1(z) = y$. It follows from Proposition 2.1(v) that both maps π_0 and π_1 are injective. We denote their respective images by V_0 and V_1. The map $\varphi := \pi_1 \circ \pi_0^{-1} \colon V_0 \to V_1$ is an isomorphism of vector spaces and $V = V_0(\varphi)$. Proposition 2.1(v) also implies that the isometry φ is special.

The uniqueness assertions are obvious. ∎

Next we establish the upper bound for the dimension of *ZD*-subspaces.

Proposition 3.3 *The maximum dimension of a ZD-subspace of $\mathbf{A}_4$ is 6.*

Proof Let V_0 be the orthogonal complement of $\mathbf{C}$ in $\mathbf{O}$. Clearly, $V_0 \subset \mathbf{O}^{\mathrm{pu}}$. Then

$$V := \{x + (ix)e : x \in V_0\}$$

is a ZD-subspace of dimension 6.

In view of the previous proposition, it remains to show that there is no special isometry $\varphi\colon \mathbf{O}^{\mathrm{pu}} \to \mathbf{O}^{\mathrm{pu}}$. This is true because there are no non-vanishing continuous vector fields on even-dimensional spheres [5, Theorem 16.5].

One can also prove this by an elementary argument. Indeed, the condition that $\varphi\colon \mathbf{O}^{\mathrm{pu}} \to \mathbf{O}^{\mathrm{pu}}$ is a special isometry implies that it is a skew-symmetric operator. As $\mathbf{O}^{\mathrm{pu}}$ has dimension 7, such an operator does not exist. ■

We shall need some additional properties of zero-divisors.

Proposition 3.4

(i) *If $z = (a, b) \in \mathbf{A}_4$ is a zero-divisor with $\|a\| = \|b\| = 1$, then*

$$\ker(L_z) = \{(x, a(bx)) : x \in \mathbf{O} \cap \mathbf{H}_{a,b}^{\perp}\}.$$

(ii) *If $(a, b), (c, d) \in \mathbf{A}_4$ satisfy $(a, b) \cdot (c, d) = 0$ and a, b, c, d are unit vectors, then $ab + cd = 0$.*

(iii) *If $x, y, z \in \mathbf{A}_4$ satisfy $xy = yz = zx = 0$, then at least one of x, y, z is 0.*

Proof For (i) see [12, Theorem 1.15] or [8, §3.2].

To prove (ii) we may assume that $a = i$ and $b = j$. By (i) we know that $c = xl$ where $x \in \mathbf{H}$, $\|x\| = 1$, and $d = -kc$. Hence, $cd = -(xl)(k(xl)) = -k$.

(iii) follows from (ii). ■

Corollary 3.5 *Any 4-dimensional subalgebra $S \subset \mathbf{A}_4$ is a division algebra.*

Proof Assume that S is not a division algebra, *i.e.*, that $ab = 0$ for some $a, b \in S$ with $\|a\| = \|b\| = 1$. As $a \perp b$, there exists an orthonormal basis $\{a, b, c\}$ of S^{pu}. By Proposition 2.1(vii), the 1-dimensional subspace $\langle c\rangle$ is L_a and L_b-invariant. As these operators are skew-symmetric, we must have $ac = bc = 0$. We have a contradiction to part (iii) of the proposition. ■

Corollary 3.6 *S_6 is not contained in any 8-dimensional subalgebra $S \subset \mathbf{A}_4$.*

Proof Assume that there is such a subalgebra S and let $P := S \cap S_6^{\perp}$. Choose a nonzero vector $a \in P$. Since $aS_6 \subset P$, the kernel of L_a must be contained in S_6. In fact $\ker(L_a) = S_6^{\mathrm{pp}}$ since zero-divisors are doubly pure. Now the elements $i+le$, $kl-je$ and a contradict part (iii) of the proposition. ■

The following proposition is essentially due to Moreno [12] but it is not stated there in this form.

Proposition 3.7 *Let $a = ri + (si + tj)e$, where $r, s, t \in \mathbf{R}$ and $rt \neq 0$, be a unit vector. Then L_a^2 has eigenvalues -1 and $-1 \pm 2rt$. The -1-eigenspace is the octonion subalgebra $\mathbf{O}_{i,j,e}$. Those with eigenvalues $-1 + 2rt$ and $-1 - 2rt$, respectively, are the ZD-subspaces*

$$\{xl - ((xk)l)e : x \in \mathbf{H}\} \quad \text{and} \quad \{xl + ((xk)l)e : x \in \mathbf{H}\}.$$

Moreover, their sum is equal to $\mathbf{O}_{i,j,e}^{\perp} = \mathbf{H}l + \mathbf{H}(le)$, and its square is contained in $\mathbf{O}_{i,j,e}$.

Proof As $\mathbf{O}_{i,j,e}$ is an alternative algebra, we have $L_a^2(x) = a(ax) = a^2x = -x$ for all $x \in \mathbf{O}_{i,j,e}$. A computation shows that the eigenvalues and eigenspaces are as specified in the lemma. By using Proposition 2.1(ii), it is easy to verify that xl and $(xk)l$ are orthogonal to each other and have the same norm for all $x \in \mathbf{H}$. Hence the eigenspaces belonging to $-1 \pm 2rt$ are *ZD*-spaces. The last assertion is straightforward to verify. ■

4 Non-Existence of Proper Subalgebras of Dimension 5, 7 or > 8

In this section we shall prove that the sedenions have no proper subalgebras of dimension 5, 7 or > 8.

For any subspace $V \subset \mathbf{A}_4$, we set

$$V^{\mathrm{pu}} := V \cap \mathbf{A}_4^{\mathrm{pu}} \quad \text{and} \quad V^{\mathrm{pp}} := V \cap \mathbf{A}_4^{\mathrm{pp}}.$$

Lemma 4.1 *If $S \subset \mathbf{A}_4$ is a subalgebra of odd dimension, then S^{pu} is a ZD-subspace of S (and $\mathbf{A}_4$).*

Proof For $a \in S^{\mathrm{pu}}$, L_a is a skew-symmetric operator, and so is its restriction L_a^S to S. As $\dim(S)$ is odd, L_a^S must be singular, *i.e.,* a is a zero-divisor in S. ■

Proposition 4.2 *$\mathbf{A}_4$ has no proper subalgebras of dimension > 8.*

Proof Assume $S \subset \mathbf{A}_4$ is a proper subalgebra with $d := \dim(S) > 8$. Let $a \in S^{\perp}$, $a \neq 0$. By Proposition 2.1(vii) we have $aS \subset S^{\perp}$ and by Proposition 3.4(i) $\ker(L_a)$ has dimension 0 or 4. It follows that

$$16 - d = \dim(S^{\perp}) \geq \dim(aS) \geq d - 4, \tag{4.1}$$

hence $d \leq 10$.

If $d = 9$, then by Lemma 4.1, S^{pu} is a *ZD*-subspace of dimension 8 which contradicts Proposition 3.3. Thus we must have $\dim(S) = 10$. The inequalities in (4.1) must now be equalities. Hence, $S^{\perp}$ is a *ZD*-subspace. Consequently, $S^{\perp} \subset \mathbf{A}_4^{\mathrm{pp}}$ and so $e \in S$. By Proposition 3.2, there exists a 6-dimensional subspace V_0 of $\mathbf{O}^{\mathrm{pu}}$ and a special isometry $\varphi\colon V_0 \to \mathbf{O}^{\mathrm{pu}}$ such that $S^{\perp} = V_0(\varphi)$. We conclude that the unit vector $a \in \mathbf{O}^{\mathrm{pu}}$ which is orthogonal to V_0 belongs to S. Consequently, the quaternion subalgebra $\mathbf{H}_{a,e}$ spanned by 1, a, e and ae is contained in S. As $S^{\perp} \subset \mathbf{H}_{a,e}^{\perp}$, it follows from Proposition 2.1(vii) that $S^\perp \mathbf{H}_{a,e} \subset S^\perp S \subset S^\perp$. It is not hard to check that the map

$$\mathbf{H}_{a,e} \times S^\perp \to S^\perp, \quad (q,x) \to xq$$

makes $S^\perp$ into a left vector space over $\mathbf{H}_{a,e}$. This is impossible, since $S^\perp$ has dimension 6. ■

Proposition 4.3 $\mathbf{A}_4$ *has no* 7*-dimensional subalgebras.*

Proof Suppose $S \subset \mathbf{A}_4$ is a 7-dimensional subalgebra. By Lemma 4.1, S^{pu} is a ZD-subspace of S. By Proposition 2.2, we may assume that the elements $a := i + je$ and $b := l - (kl)e$ are in S.

By Proposition 3.2, $S^{\mathrm{pu}} = V_0(\varphi)$ for some 6-dimensional subspace $V_0 \subset \mathbf{O}^{\mathrm{pu}}$ and a special isometry $\varphi\colon V_0 \to \mathbf{O}^{\mathrm{pu}}$. Since the subspace $V_0(-\varphi)$ is orthogonal to S, it must be contained in $S^\perp$. In fact we have $S^\perp = \langle e, u, ve\rangle \oplus V_0(-\varphi)$, where u (resp., v) is a unit pure octonion orthogonal to V_0 (resp., $\varphi(V_0)$). It is now easy to verify that $S^\perp \cap \mathbf{O}^{\mathrm{pu}}e = \mathbf{R}ve$.

Since $a, b \in S$, we have $i, l \in V_0$ and $\varphi(i) = j$, $\varphi(l) = -kl$. We conclude that $i - je$ and $l + (kl)e$ belong to $S^\perp$. By Proposition 2.1(vii) we have $SS^\perp \subset S^\perp$, and so the elements

$$\begin{aligned} a(i - je) &= (i + je)(i - je) = 2ke, \\ a(l + (kl)e) &= (i + je)(l + (kl)e) = 2(il - (jl)e), \\ b(i - je) &= (l - (kl)e)(i - je) = -2(il + (jl)e), \end{aligned}$$

belong to $S^\perp$. Hence, ke and $(jl)e$ both belong to $S^\perp \cap \mathbf{O}^{\mathrm{pu}}e$. As this intersection is 1-dimensional, we have a contradiction. ■

Proposition 4.4 $\mathbf{A}_4$ *has no* 5*-dimensional subalgebras.*

Proof Suppose $S \subset \mathbf{A}_4$ is a 5-dimensional subalgebra. By Lemma 4.1, S^{pu} is a ZD-subspace of S. Choose unit vectors $a, b \in S^{\mathrm{pu}}$ such that $ab = 0$. Then necessarily $a \perp b$. Extend them to get an orthonormal basis $\{a, b, c, d\}$ of S^{pu}. By Proposition 2.1(vii), the subspace $\langle c, d\rangle$ is invariant under L_a and L_b. Since their restrictions to this 2-dimensional space are skew-symmetric operators, some nonzero linear combination of a and b will kill both c and d. By choosing a new orthonormal basis of $\langle a, b\rangle$, we may assume that, in addition to $ab = 0$, we also have $ac = ad = 0$.

Since R_b, R_c and R_d kill a, they must leave $P := \langle 1, b, c, d\rangle$ invariant. Thus P is a 4-dimensional subalgebra of S. We may assume that $a = (i + je)/\sqrt{2}$. Since $b, c, bc \in P^{\mathrm{pu}} \subset \ker(L_a)$, Proposition 3.7 implies that $bc = 0$. This contradicts Proposition 3.4(iii). ■

5 The Action of G on the Unit Sphere

In this section we study the action of G and G_2 on the unit spheres $S^{13} \subset \mathbf{A}_4^{\mathrm{pp}}$ and $S^{14} \subset \mathbf{A}_4^{\mathrm{pu}}$. Some of the results that we prove here will be used in the next section to obtain the classification of the G-conjugacy classes of 2-dimensional subalgebras of $\mathbf{A}_4$.

We shall view $\mathbf{A}_4$ as a right $\mathbf{C}_e$-vector space (of dimension 8) via the sedenion multiplication $(x, z) \to xz$ where $x \in \mathbf{A}_4$ and $z \in \mathbf{C}_e$. It is easy to verify that this map indeed makes $\mathbf{A}_4$ into a right $\mathbf{C}_e$-vector space. For elements $z \in \mathbf{C}_e$ we shall write $|z|$ instead of $\|z\|$.

We use the direct decomposition $\mathbf{A}_4 = \mathbf{C}_e \oplus \mathbf{A}_4^{\mathrm{pp}}$ to define a hermitian form on $\mathbf{A}_4$. For arbitrary sedenions x and y there is a unique $\langle x \mid y\rangle \in \mathbf{C}_e$ such that

$$\bar{x}y - \langle x \mid y\rangle \in \mathbf{A}_4^{\mathrm{pp}}.$$

It is easy to verify that $\langle xz_1 \mid yz_2\rangle = \bar{z}_1\langle x \mid y\rangle z_2$ is valid for arbitrary sedenions x, y and arbitrary $z_1, z_2 \in \mathbf{C}_e$, *i.e.*, $\langle x \mid y\rangle$ is a hermitian form. It is closely related to the standard inner product. Indeed we have

$$\langle x \mid y\rangle = (x \mid y) + (xe \mid y)e,$$
$$\langle x \mid x\rangle = (x \mid x) = \|x\|^2.$$

The map $Q\colon \mathbf{A}_4 \to \mathbf{C}_e$ defined by $Q(x) := \langle \tau(x) \mid x\rangle$ is a $\mathbf{C}_e$-quadratic form. If we write $x = x_1 + x_2e$, with $x_1, x_2 \in \mathbf{O}$, then

$$Q(x) = \|x_1\|^2 - \|x_2\|^2 + 2(x_1 \mid x_2)e.$$

It is immediate from this formula that, for $x \in \mathbf{A}_4^{\mathrm{pp}}$, x is a zero divisor in $\mathbf{A}_4$ iff $Q(x) = 0$.

Denote by Δ the closed unit disk in $\mathbf{C}_e$.

Theorem 5.1

(a) *Q maps the unit sphere S^{13} of $\mathbf{A}_4^{\mathrm{pp}}$ onto Δ.*
(b) *An element $x \in S^{13}$ is alternative, resp., zero-divisor iff $|Q(x)| = 1$ resp., $Q(x) = 0$.*
(c) *Two points $x, y \in S^{13}$ belong to the same G_2-orbit iff $Q(x) = Q(y)$. Consequently, Q induces a homeomorphism from the orbit space S^{13}/G_2 onto Δ.*

Proof Let $x \in S^{13}$ be written as $x = x_1 + x_2e$ with $x_1, x_2 \in \mathbf{O}^{\mathrm{pu}}$.

(a) By the Cauchy–Schwarz inequality, we have

$$|Q(x)|^2 = 1 - 4(\|x_1\|^2\|x_2\|^2 - (x_1 \mid x_2)^2) \leq 1.$$

Since Q maps the circle $\{i\cos\theta + ie\sin\theta : 0 \leq \theta \leq 2\pi\}$ onto the boundary of Δ with winding number 2, and S^{13} is simply connected, the image of S^{13} must be the whole disk Δ.

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(b) The assertion about alternative elements follows from the fact that the Cauchy–Schwarz inequality is an equality iff x_1 and x_2 are $\mathbf{R}$-linearly dependent. The case of zero-divisors was already observed above.

(c) Write $y = y_1 + y_2 e$ with $y_1, y_2 \in \mathbf{O}^{\mathrm{pu}}$. Observe that $Q(x) = Q(y)$ means that the ordered vector pairs (x_1, x_2) and (y_1, y_2) are congruent, *i.e.*, have the same Gram matrix. This is exactly what is needed for the G_2-conjugacy of such pairs. ■

Our next objective is to construct a cross section of the map

$$Q\colon S^{13} \to \Delta. \tag{5.1}$$

Proposition 5.2 *Let r and θ be the usual polar coordinates in $\mathbf{C}_e$. For $0 \le r \le 1$ set*

$$\alpha(r) = \sqrt{\frac{1+r}{2}}, \quad \beta(r) = \sqrt{\frac{1-r}{2}}.$$

Then the map $\sigma\colon \Delta \to S^{13}$ defined by

$$\begin{aligned}\sigma(r,\theta) = \frac{1}{2}\bigl(\alpha(r) + \beta(r)\bigr) \cdot (i + je) \\ + \frac{1}{2}\bigl(\alpha(r) - \beta(r)\bigr) \cdot \bigl[(i\cos\theta + j\sin\theta) + (i\sin\theta - j\cos\theta)e\bigr]\end{aligned}$$

is a cross section of the map (5.1).

Proof Clearly, σ is continuous. It remains to verify that $Q \circ \sigma$ is the identity map on Δ, which is straightforward. ■

We remark that the image of σ is a cap of the unit sphere in the 3-dimensional space spanned by the mutually orthogonal vectors $i + je$, $i - je$ and $j + ie$.

Let us explain how the above map σ was actually constructed. Consider the one-parameter subgroup $\{\varphi_\theta\}$ of G_2 which fixes the basic unit l and acts on the quaternion subalgebra $\mathbf{H}$ as conjugation by $\cos(\theta/4) + k\sin(\theta/4)$. Explicitly, we have

$$\varphi_\theta(i) = i\cos(\theta/2) + j\sin(\theta/2), \quad \varphi_\theta(j) = -i\sin(\theta/2) + j\cos(\theta/2).$$

As G_2 acts on $\mathbf{A}_4$ as a group of automorphisms, φ_θ fixes also the basic units $1, k, kl, e, ke, le$ and $(kl)e$. Then one has

$$\sigma(r,\theta) = \varphi_\theta\bigl(\alpha(r)i + \beta(r)je\bigr)\bigl(\cos(\theta/2) + e\sin(\theta/2)\bigr).$$

Note that $\Sigma_3 = \langle \mu, \tau\rangle$ acts on the sphere S^{13} and on the orbit space S^{13}/G_2. Define the action of Σ_3 on the euclidean plane $\mathbf{C}_e$ by letting τ act via its restriction to this plane and by specifyng that μ acts as multiplication by ζ^2. Since $Q(\tau(x)) = \tau Q(x)$ and $Q(\mu(x)) = Q(x\zeta) = Q(x)\zeta^2$, we conclude that the above map $S^{13}/G_2 \to \Delta$ is Σ_3-equivariant. It follows that every G-orbit in S^{13} contains a unique G_2-orbit whose Q-image lies in the closed sector Λ of Δ defined by $0 \le \theta \le \pi/3$ in terms of the polar coordinate θ. In particular, the following corollary is valid.

Corollary 5.3 *Q induces a homeomorphism from the orbit space S^{13}/G onto $\Lambda \subset \Delta$.*

By identifying S^{13}/G with Λ, one can obtain a cross section of the canonical map $S^{13} \to \Lambda$ by taking the restriction of σ. We shall construct a different cross section for this map.

Let $\Gamma_1 \subset S^{13}$ be the closed subset consisting of all unit vectors

$$u = \alpha i + (\beta i + \gamma j)e \tag{5.2}$$

such that $\alpha, \beta, \gamma \geq 0$ and

$$2\alpha\beta \leq \sqrt{3}(2\alpha^2 - 1). \tag{5.3}$$

Proposition 5.4 *The set Γ_1 is a cross section of the canonical map $S^{13} \to S^{13}/G$. In particular, Γ_1 is a set of representatives of the G-orbits in S^{13}.*

Proof It suffices to verify that Q induces a bijection $\Gamma_1 \to \Lambda$. We omit the details of this routine verification. ■

Let Γ_2 be the subset of S^{13} consisting of elements of the form (5.2) such that $\alpha, \beta, \gamma \geq 0$ and

$$2\alpha\beta > \sqrt{3}(1 - 2\alpha^2) \tag{5.4}$$

or $\alpha = \gamma = 1/\sqrt{2}$ and $\beta = 0$.

Let $\tilde{\Gamma} \subset S^{14}$ be the subset consisting of all unit vectors

$$v = e\cos\varphi + u\sin\varphi, \tag{5.5}$$

where $\varphi \in [0, \pi/2)$ and $u \in \Gamma_2$ or $\varphi = \pi/2$ and $u \in \Gamma_1$. One can show that $\tilde{\Gamma}$ is a set of representatives of the G-orbits in S^{14}.

6 Two-Dimensional Subalgebras

Recall that any 2-dimensional subalgebra of $\mathbf{A}_4$ is given by $\mathbf{C}_a = \langle 1, a\rangle$, where $a \in \mathbf{A}_4^{\mathrm{pu}}$ is a unit vector, *i.e.*, $a^2 = -1$. As $\mathbf{C}_{-a} = \mathbf{C}_a$, the problem of classification of 2-dimensional subalgebras up to conjugacy is equivalent to the classification of G-orbits in the real projective space associated with the space of pure sedenions (its points are the 1-dimensional subspaces of $\mathbf{A}_4^{\mathrm{pu}}$). For dimension reasons, there are infinitely many of these conjugacy classes.

We can apply the results of the previous section to obtain a set of representatives of the G-conjugacy classes of 2-dimensional subalgebras.

To any point $(u, \varphi) \in \Gamma_1 \times [0, \pi/2]$ we associate the 2-dimensional subalgebra $S(u, \varphi) = \mathbf{C}_a$ where $a = e\cos\varphi + u\sin\varphi$.

Proposition 6.1 *The subalgebras $S(u, \varphi)$, $(u, \varphi) \in \Gamma_1 \times [0, \pi/2]$, form a set of representatives of the G-conjugacy classes of 2-dimensional subalgebras of $\mathbf{A}_4$.*

Proof Let S be any 2-dimensional subalgebra. We know that $S = \mathbf{C}_a$ for some $a \in S^{14}$. We have $a = e\cos\varphi + u\sin\varphi$ for some $u \in S^{13}$ and $\varphi \in [0,\pi]$. By Proposition 5.4, we may assume that $u \in \Gamma_1$. If $\varphi \in [0,\pi/2]$, then $S = S(u,\varphi)$. Otherwise we replace a by $-a = e\cos(\pi-\varphi) - u\sin(\pi-\varphi)$ which is G_2-conjugate to $e\cos(\pi-\varphi) + u\sin(\pi-\varphi)$. Hence S is G-conjugate to $S(u,\pi-\varphi)$.

Now assume that $S(u,\varphi)$ and $S(u',\varphi')$ are G-conjugate. Then there exists $g \in G$ such that

$$g(e\cos\varphi + u\sin\varphi) = \pm(e\cos\varphi' + u'\sin\varphi').$$

Since $\varphi, \varphi' \in [0,\pi/2]$, we must have $\varphi = \varphi'$. If $\varphi = 0$, then $\varphi' = 0$ and $S(u,\varphi) = S(u',\varphi') = \mathbf{C}_e$. Otherwise $g(u) = \pm u'$ and Proposition 5.4 implies that $u = u'$. ■

Since $\mathbf{C}_a = \mathbf{C}_{-a}$, it is natural to ask: if $a \in S^{14}$, when are a and $-a$ G-conjugate? If $a \in S^{13}$, then a and $-a$ are G_2-conjugate and so we may assume that $a \notin S^{13}$. Since $\tau(e) = -e$, we may also assume that $a \neq \pm e$. The answer is given in the next proposition.

Proposition 6.2 *Let $a \in S^{14}$, $a \notin S^{13}$, $a \neq \pm e$, be written as $a = e\cos\varphi + u\sin\varphi$ where $u \in S^{13}$. Then a and $-a$ are G-conjugate iff $Q(u)^3 \in \mathbf{R}$.*

Proof Assume that a and $-a$ belong to the same G-orbit, *i.e.*,

$$g(a) = -\tau(a) = e\cos\varphi - \tau(u)\sin\varphi$$

for some $g \in G$. As $\cos\varphi \neq 0$, we must have $g \in \langle G_2, \mu\rangle$. Since $\sin\varphi \neq 0$, it follows that $g(u) = -\tau(u)$. By applying Q and using Theorem 5.1, we conclude that $\tau(Q(u)) = \mu^s \cdot Q(u)$ for some $s \in \{0,1,2\}$. Since μ acts on $\mathbf{C}_e$ as multiplication by ζ^2 and τ as complex conjugation, we infer that indeed $Q(u)^3 \in \mathbf{R}$. The converse can be proved similarly. ■

7 Four-Dimensional Subalgebras

In this section we prove that there are only two isomorphism classes of 4-dimensional subalgebras: $\mathbf{H}$ and S_4. The subalgebras isomorphic to S_4 are all conjugate. On the other hand, there are infinitely many conjugacy classes of quaternion subalgebras.

Theorem 7.1 *Any 4-dimensional subalgebra $S \subset \mathbf{A}_4$ is either conjugate to S_4 or to a quaternion subalgebra of $\mathbf{O}_{i,j,e}$.*

Proof Assume first that all elements of S are alternative. Let a, b be an orthonormal pair in S^{pp}. By using the action of G_2, we may assume that $a = ri + sie$ and $b = \alpha i + \beta j + (\gamma i + \delta j)e$ for some real $r, s, \alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \beta\gamma = 0$. It is evident that $S \subset \mathbf{O}_{i,j,e}$ and so S is a quaternion algebra.

Assume now that S contains a non-alternative element. Assume also that S is not conjugate to a subalgebra of $\mathbf{O}_{i,j,e}$. Then Proposition 3.7 implies that S contains a nonzero zero-divisor, say a, of $\mathbf{A}_4$. Without any loss of generality, we may assume that $a = (i + je)/\sqrt{2}$.

The eigenvalues of L_a^2 are $0, -1$ and -2. S is a direct sum of its intersections with the eigenspaces of L_a^2. As S is a division algebra (see Corollary 3.5) and not contained in $\mathbf{O}_{i,j,e}$, we conclude that S contains a nonzero vector, say b, from the -2-eigenspace. This eigenspace is described in Proposition 3.7. By using the action of G_2, we may assume that $b = l + (kl)e$. Hence $S = S_4$. ■

As mentioned in the Introduction, the classification of the conjugacy classes of the quaternion subalgebras has been now completed by the first author [3]. It is more complicated than that for 2-dimensional subalgebras.

8 Non-Division Subalgebras

In this section we determine the conjugacy classes of proper subalgebras $S \subset \mathbf{A}_4$ which are not division algebras. It turns out that there are exactly three such conjugacy classes: one in each of the dimensions 3, 6 and 8.

Recall that the real division algebras have dimensions 1, 2, 4 or 8 (a famous result due to Milnor and Bott [11] and Kervaire [7], independently). Since any subalgebra of dimension 2 or 4 is a division algebra (see Theorem 7.1), the possible dimensions for a proper non-division subalgebra of $\mathbf{A}_4$ are 3, 6 and 8.

The computations needed in the proof of the next theorem were performed by using Maple (a package for symbolic computations).

Theorem 8.1 *If $S \subset \mathbf{A}_4$ is a proper non-division subalgebra, then:*

(i) *S is conjugate to S_3, S_6 or S_8.*
(ii) *If $S \supset S_3$, then S is one of S_3, S_6, S_8 or one of the two particular conjugates of S_8. (One of these conjugates is generated by S_3 and $k\zeta$ and the other by S_3 and $k\zeta^2$.)*

Proof By hypothesis there exist $a, b \in S^{\mathrm{pu}}$ such that $ab = 0$ and $\|a\|^2 = \|b\|^2 = 2$. By Proposition 2.2, we may assume that $a = i + le$ and $b = j + (kl)e$. In particular $S \supset S_3$ and we see that (i) follows from (ii). It remains to prove (ii) for $S \neq S_3$.

Let $P = S \cap S_3^{\perp}$. Since $\dim(P)$ is odd, $aP \subset P$ and L_a is skew-symmetric, there exists a nonzero $c \in P$ such that $ac = 0$. Since $b \perp c$, there exist $\alpha, \beta, \gamma \in \mathbf{R}$ such that

$$\begin{aligned} c &= \alpha(jl + ke) + \beta(k - (jl)e) + \gamma(kl - je) \\ &= (\alpha jl + \beta k + \gamma kl) + (\alpha k - \beta jl - \gamma j)e. \end{aligned}$$

Let M be the 8-by-16 matrix of coefficients of the vectors $a, b, c, bc, a(bc), b(a(bc))$, $c(bc)$ and $c(a(bc))$. All of these vectors belong to S^{pu}. Hence the rank of M is < 8. We use Maple to compute some 8-by-8 minors of M. The one in columns 2, 3, 4, 5, 7, 9, 11 and 16 is equal to $-256\gamma^4(\alpha^2+\beta^2)(\alpha^2+2\beta^2)$. Thus if $\gamma \neq 0$, then $\alpha = \beta = 0$ and S contains the 6-dimensional subalgebra generated by a, b and $kl - je$. By Corollary 3.6, $S = S_6$.

It remains to consider the case $\gamma = 0$. If we perform a similar computation as above, replacing the vector $c(bc)$ with $a(b(a(bc)))$, then the 8-by-8 minor in columns 2, 3, 4, 5, 6, 7, 11 and 12 is equal to $2048\alpha^5(3\beta^2 - \alpha^2)$.

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If $\alpha = 0$, then $k - (jl)e \in S$, and it follows that $S = S_8$. Otherwise $\alpha = \pm\beta\sqrt{3} \neq 0$ and S is one of the two conjugates of S_8 mentioned in the theorem.

For instance, if we take $\beta = 1$ and $\alpha = \sqrt{3}$, then S_3 and c generate the 8-dimensional subalgebra with basis

$$\{1, i + le, j + (kl)e, k + \sqrt{3}ke = -2k\zeta^2, \sqrt{3}(l + ie) + 2le,$$
$$\sqrt{3}il - (il)e, \sqrt{3}jl - (jl)e, \sqrt{3}(kl + je) + 2(kl)e\}.$$

By applying μ to it, we obtain the subalgebra with basis

$$\{1, i + \sqrt{3}l, j + \sqrt{3}kl, k, le - \sqrt{3}ie, (il)e, (jl)e, (kl)e - \sqrt{3}je\}.$$

It is easy to see that this subalgebra is a G_2-conjugate of S_8. ∎

9 Division Subalgebras of Dimension 8

In this section we complete the solution of our problem by classifying the 8-dimensional division subalgebras $S \subset \mathbf{A}_4$. We consider first the case where S consists of alternative elements only.

Proposition 9.1 *If $S \subset \mathbf{A}_4$ is an 8-dimensional division subalgebra such that each element of S is alternative, then $S = \mathbf{O}$, $\mu(\mathbf{O})$ or $\mu^2(\mathbf{O})$.*

Proof The subspace S^{pp} is alternative and has dimension at least 6. By Proposition 3.1, we have $S^{\mathrm{pp}} = V_0 a$ for some subspace $V_0 \subset \mathbf{O}^{\mathrm{pu}}$ and a unit vector $a \in \mathbf{C}_e$. Write $a = \alpha + \beta e$, where $\alpha, \beta \in \mathbf{R}$.

Let $v, w \in V_0$ be orthonormal. Then the element

$$(va)(wa) = 2(vw)(\alpha^2 - \beta^2 - 2\alpha\beta e)$$

belongs to S^{pp}. Hence, we must have

$$\begin{vmatrix} \alpha^2 - \beta^2 & -2\alpha\beta \\ \alpha & \beta \end{vmatrix} = 0,$$

i.e., $a = 1, \zeta$ or ζ^2. The assertion of the proposition now follows. ∎

In the remaining cases, we show S contains a nonzero zero-divisor of $\mathbf{A}_4$.

Lemma 9.2 *Let $S \subset \mathbf{A}_4$ be an 8-dimensional division subalgebra containing a non-alternative element. Then S contains a nonzero zero-divisor of $\mathbf{A}_4$.*

Proof By hypothesis there is an $a \in S$ which is not alternative. We may assume that a is doubly pure and moreover that $a = ri + (si + tj)e$ with $r, s, t \in \mathbf{R}$ and $rt \neq 0$. By Proposition 3.7, the eigenvalues of L_a^2 are -1 and $-1 \pm 2rt$. As S is L_a^2-invariant, it is the direct sum of its intersections with the three eigenspaces of L_a^2. Since the eigenspaces belonging to the eigenvalues $-1 \pm 2rt$ are ZD-spaces, we need consider only the case where S coincides with the -1-eigenspace. Hence, $S = \mathbf{O}_{i,j,e}$ and the assertion obviously holds. ∎

We can now classify the 8-dimensional division subalgebras.

Theorem 9.3 *There are exactly three conjugacy classes of* 8*-dimensional division subalgebras* $S \subset \mathbf{A}_4$*. Their representatives are* $\mathbf{O}$*,* $\mathbf{O}_{i,j,e}$ *and* R_8*. (The first conjugacy class consists just of three algebras:* $\mathbf{O}$*,* $\mu(\mathbf{O})$ *and* $\mu^2(\mathbf{O})$*.)*

Proof Assume that S is not conjugate to $\mathbf{O}$ or $\mathbf{O}_{i,j,e}$. By Proposition 9.1 and Lemma 9.2, S has a nonzero zero-divisor, say a. We may assume that $a = i + je$.

By Proposition 3.7, the eigenspaces of L_a^2 are: $\mathbf{O}_{i,j,e}$ with eigenvalue -1, $\ker(L_a)$ with eigenvalue 0, and $P := \{x + (kx)e : x \in \mathbf{H}l\}$ with eigenvalue -2. Since S is L_a^2-invariant, $a \in S$, and S is a division algebra, it follows that $S \subset \mathbf{O}_{i,j,e} + P$. As $S \neq \mathbf{O}_{i,j,e}$, by a dimension argument we deduce that $Q := S \cap \mathbf{O}_{i,j,e}$ is a quaternion algebra and that $S = Q \oplus P$. As R_8 contains P and a, we conclude that $S = R_8$. ■

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