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An Inductive Limit Model for the K-Theory of the Generator-Interchanging Antiautomorphism of an Irrational Rotation Algebra

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Abstract. Let A_θ be the universal C^*-algebra generated by two unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$ and let Φ be the antiautomorphism of A_θ interchanging U and V. The K-theory of $R_\theta = \{a \in A_\theta : \Phi(a) = a^*\}$ is computed. When θ is irrational, an inductive limit of algebras of the form $M_q(C(\mathbb{T})) \oplus M_{q'}(\mathbb{R}) \oplus M_q(\mathbb{R})$ is constructed which has complexification A_θ and the same K-theory as R_θ.

1 Introduction

It was shown in [6] and later, with a simplified proof, in [7] that the irrational rotation algebra A_θ, generated by unitaries U, V with $VU = e^{2\pi i\theta}UV$, can be written as an inductive limit of algebras of the form $M_q\big(C(\mathbb{T})\big) \oplus M_{q'}\big(C(\mathbb{T})\big)$, where $C(\mathbb{T})$ denotes the algebra of continuous complex-valued functions on the unit circle $\mathbb{T}$ and $M_q\big(C(\mathbb{T})\big)$ denotes the algebra of $q \times q$ matrices with entries in $C(\mathbb{T})$. It was subsequently shown by Walters in [14], with a simplified proof given by Boca in [2], that the algebras $M_q\big(C(\mathbb{T})\big) \oplus M_{q'}\big(C(\mathbb{T})\big)$ can be chosen to be invariant under the flip given by $U \to U^*$, $V \mapsto V^*$. Similar results were obtained in [13] for the antiautomorphisms given by $U \mapsto U$, $V \mapsto V^*$ and $U \mapsto -U$, $V \mapsto V^*$, but it was shown that the other naturally occurring antiautomorphism Φ, given by $\Phi(U) = V$ and $\Phi(V) = U$, does not admit such a decomposition.

A similar situation obtains for the period 4 (Fourier) automorphism given by $U \mapsto V$ and $V \mapsto U^*$. It was shown in [12] that there is no inductive limit decomposition of Elliott-Evans type which is invariant under this automorphism. However in [16] Walters raised the possibility of an invariant inductive limit decomposition using algebras of the form $M_q\big(C(\mathbb{T})\big) \oplus M_q\big(C(\mathbb{T})\big) \oplus M_{q'} \oplus M_q$. He produced an inductive limit decomposition of A_θ using such algebras and an order 4 automorphism σ of A_θ compatible with the decomposition and with the same induced map on $K_1(A_\theta)$ as the Fourier automorphism.

In this paper the construction of [16] is slightly modified to obtain an inductive limit decomposition invariant under an antiautomorphism of period 2 with the same effect on $K_1(A_\theta)$ as Φ. In this setting it is possible to obtain a more detailed agreement between the two antiautomorphisms by showing that the K-theories of the

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associated real algebras are identical. It is straightforward to calculate the K-theory of the inductive limit, but not immediately clear how to compute the K-theory of $R_\theta = \{a \in A_\theta : \Phi(a) = a^*\}$ since it has no (obvious) cross product structure. The calculation, which occupies most of this paper, is achieved by combining a standard exact sequence for real C^*-algebras with the exact sequence for real C^*-algebras produced in [11]. Walters, in [15], has calculated, for a dense G_δ set of real parameters θ, the K-theory of the analogous fixed point algebra of the Fourier automorphism, but his methods are different from (and more difficult than) those employed here.

2 Computing the K-Theory of R_θ

As a first step in the calculation of $K_0(R_\theta)$, it will be shown that Boca's construction from [3] produces a projection p in R_θ with trace θ. The features of this construction which are required to show this will now be described.

For each $r \in \mathbb{R}$ let $e(r) = e^{2\pi i r}$ and let β be the Heisenberg cocycle on $\mathbb{R}^2$, defined by $\beta\big((x, y), (x', y')\big) = e(xy')$. Let D be the lattice $\{\sqrt{\theta}(n_1, n_2) : n_1, n_2 \in \mathbb{Z}\}$ and let $D^\perp = \{\frac{1}{\sqrt{\theta}}(m_1, m_2) : m_1, m_2 \in \mathbb{Z}\}$ (defined so that $D^\perp = \{z \in \mathbb{R}^2 : \beta(z, w) = \beta(w, z)$ for all $w \in D\}$). In accordance with page 278 of [9], choose the Haar measures on $D, D^\perp$ to assign each point the masses $\sqrt{\theta}, 1$ respectively. Then define the twisted group algebras $C^*(D, \beta)$ and $C^*(D^\perp, \bar\beta)$ as the C^*-completions of $L_1(D, \beta)$ and $L_1(D^\perp, \bar\beta)$ with the multiplications

$$(fg)(w) = \int_D f(w')g(w - w')\beta(w', w - w')\, dw' \quad \text{for } w \in D$$

$$(fg)(z) = \int_{D^\perp} f(z')g(z - z')\overline{\beta(z', z - z')}\, dz' \quad \text{for } z \in D^\perp$$

and the involutions $f^*(w) = \beta(w, w)\overline{f(-w)}$ for $w \in D$ and $f^*(z) = \overline{\beta(z, z) f(-z)}$ for $z \in D^\perp$.

The Schwartz space $S(\mathbb{R})$ is a $C^*(D, \beta) - C^*(D^\perp, \bar\beta)$ bimodule under the actions defined, for $a \in S(D)$, $b \in S(D^\perp)$ and $h \in S(\mathbb{R})$, by

$$(ah)(s) = \sqrt{\theta} \sum_{(x,y)\in D} a(x, y)h(s + x)e(sy)$$

$$(hb)(s) = \sum_{(x,y)\in D^\perp} b(x, y)h(s - x)e\big(y(x - s)\big).$$

Furthermore it becomes a $C^*(D, \beta) - C^*(D^\perp, \bar\beta)$ equivalence bimodule under the $C^*(D, \beta)$ and $C^*(D^\perp, \bar\beta)$ valued inner products $\langle\ ,\ \rangle_D$ and $\langle\ ,\ \rangle_{D^\perp}$ defined for $f, g \in S(\mathbb{R})$ by

$$\langle f, g\rangle_D(x, y) = \int_{\mathbb{R}} f(s)\overline{g(s + x)}e(-sy)\, ds$$

$$\langle f, g\rangle_{D^\perp}(x, y) = \int_{\mathbb{R}} \overline{f(s)}g(s + x)e(sy)\, ds.$$

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If $f \in S(\mathbb{R})$ is defined by $f(s) = e^{-\pi s^2}$ and $0 < \theta < 0.948$, then $\langle f, f \rangle_{D^\perp}$ is invertible and

$$p = \left\langle f\langle f, f\rangle_{D^\perp}^{-1/2}, f\langle f, f\rangle_{D^\perp}^{-1/2} \right\rangle_D$$

defines a projection p in $C^*(D, \beta)$ with $\tau_D(p) = \theta$, where τ_D is the unique normalised trace on $C^*(D, \beta) \cong A_\theta$. Using the isomorphism between A_θ and $A_{1-\theta}$ it follows that for all θ either p or $1 - p$ is a projection in A_θ with trace θ.

Let J, $\mathcal{F}$ be the bounded invertible operators on $L_2(\mathbb{R})$ defined for $f \in S(\mathbb{R})$ by $(Jf)(s) = \overline{f(s)}$ and $(\mathcal{F}f)(s) = \int_{\mathbb{R}} f(x)e(-xs)\,dx$ and let $F = J\mathcal{F}$, so $(Ff)(s) = \int_{\mathbb{R}} \overline{f(x)}e(xs)\,dx$. F is an invertible antilinear operator on $L_2(\mathbb{R})$ and therefore $\Phi(a) = F^{-1}a^*F$ defines an antiautomorphism of $B\big(L_2(\mathbb{R})\big)$.

Lemma 2.1 *Φ restricts to the involutory antiautomorphism of $C^*(D, \beta)$ which interchanges the canonical unitary generators.*

Proof It suffices to show that $\Phi(\chi_{(\sqrt{\theta},0)}) = \chi_{(0,\sqrt{\theta})}$ and $\Phi(\chi_{(0,\sqrt{\theta})}) = \chi_{(\sqrt{\theta},0)}$, where χ_d is the characteristic function of $\{d\}$ for $d \in D$. Let $h \in S(\mathbb{R})$ and $s \in \mathbb{R}$. Then

$$\begin{aligned}\big(F\Phi(\chi_{(\sqrt{\theta},0)})h\big)(s) &= (\chi^*_{(\sqrt{\theta},0)}Fh)(s) = (\chi_{(-\sqrt{\theta},0)}Fh)(s) \\ &= \sqrt{\theta}(Fh)(s - \sqrt{\theta}) = \sqrt{\theta}\int_{\mathbb{R}} \overline{h(x)}e\big(x(s-\sqrt{\theta})\big)\,dx,\end{aligned}$$

whereas

$$\begin{aligned}(F\chi_{(0,\sqrt{\theta})}h)(s) &= \int_{\mathbb{R}} \overline{(\chi_{(0,\sqrt{\theta})}h)(x)}e(xs)\,dx \\ &= \sqrt{\theta}\int_{\mathbb{R}} e(-\sqrt{\theta}x)\overline{h(x)}e(xs)\,dx.\end{aligned}$$

Thus $F\chi_{(0,\sqrt{\theta})} = F\Phi(\chi_{(\sqrt{\theta},0)})$, so $\chi_{(0,\sqrt{\theta})} = \Phi(\chi_{(\sqrt{\theta},0)})$. A similar calculation gives $\chi_{(\sqrt{\theta},0)} = \Phi(\chi_{(0,\sqrt{\theta})})$. ■

Proposition 2.2 *If $0 < \theta < 1$ then R_θ contains a projection p with trace θ.*

Proof By Lemma 2.1 and the preceding remarks it suffices to show that $pF = Fp$ where $p = \left\langle f\langle f, f\rangle_{D^\perp}^{-1/2}, f\langle f, f\rangle_{D^\perp}^{-1/2} \right\rangle_D$ and $f(s) = e^{-\pi s^2}$. It is shown in [3] that $\mathcal{F}p = p\mathcal{F}$, so it suffices to show that $Jp = pJ$.

For $h \in S(\mathbb{R})$ and $s \in \mathbb{R}$,

$$\begin{aligned}(h\langle f, f\rangle_{D^\perp})(s) &= \sum_{(x,y)\in D^\perp} \langle f, f\rangle_{D^\perp}(x, y)h(s-x)e\big(y(x-s)\big) \\ &= \sum_{(x,y)\in D^\perp} \int_{\mathbb{R}} f(t)f(t+x)e(ty)\,dt\, h(s-x)e\big(y(x-s)\big).\end{aligned}$$

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Thus

$$(h\langle f,f\rangle_{D^\perp}J)(s)=\sum_{(x,y)\in D^\perp}\int_{\mathbb{R}} f(t)f(t+x)e(-ty)\,dt\,\overline{h(s-x)}e\big(-y(x-s)\big)$$
$$=(hJ\langle f,f\rangle_{D^\perp})(s).$$

It follows in turn that $J\langle f,f\rangle_{D^\perp}=\langle f,f\rangle_{D^\perp}J$, $J\langle f,f\rangle_{D^\perp}^{-1/2}=\langle f,f\rangle_{D^\perp}^{-1/2}J$ and $f\langle f,f\rangle_{D^\perp}^{-1/2}J=fJ\langle f,f\rangle_{D^\perp}^{-1/2}=f\langle f,f\rangle_{D^\perp}^{-1/2}$. Putting $g=f\langle f,f\rangle_{D^\perp}^{-1/2}$, a calculation for $\langle g,g\rangle_D$ similar to that given above for $\langle f,f\rangle_{D^\perp}$ then shows that $Jp=pJ$, as required. ■

The principal tool used to calculate the K-theory of R_θ will be two exact sequences, which both rely on the K-theoretic maps $\alpha_i\colon K_i(A_\theta)\to K_i(A_\theta)$, where α is the anti-linear automorphism defined by $\alpha(x)=\Phi(x^*)$. The proof of Proposition 2.7 in III of [8] shows that, when $r_i\colon K_i(A_\theta)\to K_i(R_\theta)$ and $c_i\colon K_i(R_\theta)\to K_i(A_\theta)$ arise from the maps $r(x+iy)=\begin{pmatrix} x & -y\\ y & x\end{pmatrix}$ and the inclusion $c(x)=x$, then $r_i\circ c_i=2\,\mathrm{id}$ and $c_i\circ r_i=\mathrm{id}+\alpha_i$.

Although the principal interest of this paper is in the case of irrational θ, the calculation of the K theory of R_θ can be carried out for both rational and irrational θ simultaneously if the complexification map $c_0\colon K_0(R_\theta)\to K_0(A_\theta)$ is shown to be a surjection.

Proposition 2.3 *The complexification map $c_0\colon K_0(R_\theta)\to K_0(A_\theta)$ is a surjection.*

Proof When θ is irrational, then $K_0(A_\theta)$ is generated by $[1]$ and $[p]$ for any projection p in A_θ with trace θ. Thus the result follows from Proposition 2.2.

When $\theta=p/q$ with $(p,q)=1$ then, as shown for example in [4], A_θ is isomorphic to

$$\Big\{\, f\in C([0,1]^2,M_q): f(\lambda,1)=W_1f(\lambda,0)W_1^* \text{ for all } 0\le\lambda\le 1,$$
$$f(1,\mu)=W_2f(0,\mu)W_2^* \text{ for all } 0\le\mu\le 1\,\Big\},$$

where M_q denotes the algebra of $q\times q$ complex matrices (with $q=1$ when $\theta=1$) and W_1 and W_2 are two particular $q\times q$ matrices. Let $e\in R_\theta$ be the Boca projection with trace $\frac{1}{q}$ and note that, by continuity, the usual normalised trace of $e(\lambda,\mu)$ is equal to $\frac{1}{q}$ for each $(\lambda,\mu)\in[0,1]^2$. Thus e is a full projection in R_θ, so that $eR_\theta e$ is stably isomorphic (as a real C^*-algebra) to R_θ. Since $eR_\theta e$ is isomorphic to

$$R_1=\{f\in C([0,1]^2,\mathbb{C}): f(\lambda,1)=f(\lambda,0),\, f(1,\mu)=f(0,\mu),$$
$$f(\lambda,\mu)=\overline{f(\mu,\lambda)} \text{ for all } \lambda,\mu\},$$

it suffices to prove the result when θ has any fixed value, such as $\frac{1}{2}$.

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As observed in [17], the arguments for the irrational case apply also when $\theta = \frac{p}{q}$ to show that $K_0(R_\theta)$ is generated by $[1]$ and $[f]$ where f is a Rieffel projection with trace $\frac{1}{q}$. Thus, if $\theta = \frac{1}{2}$ and e is a Boca projection with trace $\frac{1}{2}$ then $[e] = a[1] + (1-2a)[f]$ for some $a \in \mathbb{Z}$, from which it follows that $c_0\big(K_0(R_\theta)\big) \supseteq \mathbb{Z} \times (1-2a)\mathbb{Z}$ and $c_0 r_0\big(K_0(A_\theta)\big) \supseteq 2\mathbb{Z} \times 2(1-2a)\mathbb{Z}$, so $\det(\mathrm{id} + \alpha_0) = \det(c_0 r_0) \neq 0$. The only possibilities for an order 2 automorphism α_0 of $\mathbb{Z}^2$ are $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ with $a^2 + bc = 1$. The only one of these for which $\det(\mathrm{id} + \alpha_0) \neq 0$ is id. Hence $c_0 r_0 = 2\,\mathrm{id}$ and $c_0\big(K_0(R_\theta)\big) \supseteq 2\mathbb{Z}^2$. When combined with $c_0\big(K_0(R_\theta)\big) \supseteq \mathbb{Z} \times (1-2a)\mathbb{Z}$, this gives $c_0\big(K_0(R_\theta)\big) = \mathbb{Z}^2$, as required. ■

Proposition 2.4 *For any $\theta \le \theta \le 1$, the maps $\alpha_i \colon K_i(A_\theta) \to K_i(A_\theta)$ are periodic of period 4. The matrices defining the corresponding automorphisms of $\mathbb{Z}^2$ are*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{when } i \equiv 0 \pmod 4$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \textit{when } i \equiv 1 \pmod 4$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{when } i \equiv 2 \pmod 4$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{when } i \equiv 3 \pmod 4$$

Proof For any complex C^*-algebra A let $SA = C_0(\mathbb{R}, A)$ and let $\theta_A \colon K_1(A) \to K_0(SA)$ and $\beta_A \colon K_0(A) \to K_1(SA)$ be the isomorphisms defined in Theorem 8.2.2 and Definition 9.1.1 of [1]. The isomorphism θ_A commutes with the maps produced by either a linear or antilinear automorphism of A. When α is an antilinear automorphism, let $\tilde{\alpha}$ be the associated antilinear automorphism of $C\big(S^1, \mathrm{GL}_n(A^+)\big)$ and note that when $f_e \colon z \mapsto ze + (1-e)$ (where e is a projection in A) then $\tilde{\alpha}(f_e) \colon z \mapsto \bar{z}\alpha(e) + \big(1 - \alpha(e)\big)$. Thus $\tilde{\alpha}(f_e) = f_{\alpha(e)}^{-1}$ and so, when τ is the inverse map in $K_1(SA)$, the following diagram commutes.

$$\begin{array}{ccc} K_0(A) & \xrightarrow[\alpha_0]{} & K_0(A) \\ \beta_A \big\downarrow & & \big\downarrow \beta_A \\ K_1(SA) & \xrightarrow[\tilde{\alpha}_1 \circ \tau]{} & K_1(SA) \end{array}$$

It follows that, under the Bott isomorphism $\theta_{SA}\beta_A$ between $K_0(A)$ and $K_2(A)$, the following diagram commutes, where τ is the inverse map.

$$\begin{array}{ccc} K_0(A) & \xrightarrow[\alpha_0]{} & K_0(A) \\ \theta_{SA}\beta_A \big\downarrow & & \big\downarrow \theta_{SA}\beta_A \\ K_2(A) & \xrightarrow[\alpha_2 \circ \tau]{} & K_2(A) \end{array}$$

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It remains to establish the matrices for $\alpha_0 \colon K_0(A_\theta) \to K_0(A_\theta)$ and $\alpha_1 \colon K_1(A_\theta) \to K_1(A_\theta)$. The second is immediate from $\alpha(U) = V^*$ and $\alpha(V) = U^*$, where U, V are the unitary generators of A_θ. In the first case it has already been shown in the rational case that $\alpha_0 = \mathrm{id}$. When θ is irrational, let p be a projection in R_θ given by Proposition 2.3. Then $[1]$ and $[p]$ generate $K_0(A_\theta)$ and $(cr)(1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $(cr)(p) = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, so $\mathrm{id} + \alpha_0 = 2\,\mathrm{id}$ on $K_0(A_\theta)$ and hence $\alpha_0 = \mathrm{id}$. ∎

The first exact sequence used to determine the K-theory of R_θ will be based on the results of [11]. The first step is to compute the K-theory of the real C^*-algebra $C_\theta = A_\theta \times_\alpha \mathbb{Z}$ using the real Pimsner-Voiculescu sequence.

Proposition 2.5 *For any $0 \le \theta \le 1$, let $C_\theta = A_\theta \times_\alpha \mathbb{Z}$ where $\alpha(x) = \Phi(x^*)$ for each $x \in A_\theta$. Then*

$$K_i(C_\theta) \cong \mathbb{Z}^3 \quad \text{when } i \equiv 0,\ 1 \pmod 4,$$
$$K_i(C_\theta) \cong \mathbb{Z} \quad \text{when } i \equiv 3 \pmod 4,$$

and

$$K_i(C_\theta) \cong \mathbb{Z}_2^2 \times \mathbb{Z} \quad \text{when } i \equiv 2 \pmod 4.$$

Proof The real Pimsner-Voiculescu sequence in this case is

$$\cdots \longrightarrow K_0(A_\theta) \xrightarrow[\mathrm{id} - \alpha_0]{} K_0(A_\theta) \longrightarrow K_0(C_\theta) \longrightarrow K_7(A_\theta) \longrightarrow \cdots .$$

From Proposition 2.4, $\mathrm{id} = \alpha_i$ when $i \equiv 0 \pmod 4$ so, starting with $K_0(A_\theta)$ we obtain

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow K_0(C_\theta) \longrightarrow \mathbb{Z}^2 \xrightarrow[\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)]{} \mathbb{Z}^2 \longrightarrow K_7(C_\theta) \longrightarrow \mathbb{Z}^2$$
$$\xrightarrow[2\,\mathrm{id}]{} \mathbb{Z}^2 \longrightarrow K_6(C_\theta) \longrightarrow \mathbb{Z}^2 \xrightarrow[\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)]{} \mathbb{Z}^2 \longrightarrow K_5(C_\theta) \longrightarrow \mathbb{Z}^2 \longrightarrow 0.$$

The initial portion gives $0 \to \mathbb{Z}^2 \to K_0(C_\theta) \to \{(n,n) : n \in \mathbb{Z}\} \to 0$, so $K_0(C_\theta) \cong \mathbb{Z}^3$. The next part gives $0 \to \mathbb{Z}^2/\{(n,-n) : n \in \mathbb{Z}\} \to K_7(C_\theta) \to \ker(2\,\mathrm{id}) \to 0$, yielding $K_7(C_\theta) \cong \mathbb{Z}$.

Finally, the portions $0 \to \mathbb{Z}^2 \xrightarrow[2\,\mathrm{id}]{} \mathbb{Z}^2 \longrightarrow K_6(C_\theta) \longrightarrow \{(n,-n) : n \in \mathbb{Z}\} \to 0$ and $0 \to \mathbb{Z}^2/\{(n,n) : n \in \mathbb{Z}\} \to K_5(C_\theta) \to \mathbb{Z}^2 \to 0$ yield $K_6(C_\theta) \cong \mathbb{Z}_2^2 \times \mathbb{Z}$ and $K_5(C_\theta) \cong \mathbb{Z}^3$. The periodicity of period 4 established in Proposition 2.4 completes the proof. ∎

It follows from Propositions 2.2(ii) and 2.3 of [11] that C_θ is isomorphic to

$$C_\theta = \Big\{ f \in C\big([0,1], M_2(A_\theta)\big) : f(1) = \hat{\alpha}\big(f(0)\big),$$
$$f(t) = (\Psi\hat{\alpha})\big(f(1-t)^*\big) \text{ for each } 0 \le t \le 1 \Big\}$$

where

$$\hat{\alpha}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

and

$$\Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Phi(d) & \Phi(b) \\ \Phi(c) & \Phi(a) \end{pmatrix}.$$

For each $f \in C_\theta$, $f(0) = (\Psi\hat{\alpha})\big(f(1)^*\big) = \Psi\big(f(0)\big)^*$ and $f(\frac{1}{2}) = (\Psi\hat{\alpha})\big(f(\frac{1}{2})\big)^*$. By Proposition 2.4 of [11] it follows that the evaluation map at $\frac{1}{2}$ has image isomorphic to $R_\theta \otimes \mathbb{H}$ and the evaluation map at 0 has image isomorphic to $R_\theta \otimes M_2(\mathbb{R}) \cong M_2(R_\theta)$. Thus, using I to denote $[0,1]$, there is an exact sequence

$$0 \to C_0\big(I, M_2(A_\theta)\big) \to C_\theta \to M_2(R_\theta) \times (R_\theta \otimes \mathbb{H}) \to 0.$$

The associated K-theoretic long exact sequence

$$\cdots \to K_{n+1}(A_\theta) \to K_n(C_\theta) \to K_n(R_\theta) \times K_{n+4}(R_\theta) \to K_n(A_\theta) \to \cdots \tag{2.1}$$

is one of the tools which will be used to calculate the K-theory of R_θ. The other is the sequence, described in Theorem 1.4.7 of [10],

$$\cdots \longrightarrow K_n(R_\theta) \underset{c_n}{\longrightarrow} K_n(A_\theta) \longrightarrow K_{n-2}(A_\theta) \underset{r_{n-2}}{\longrightarrow} K_{n-2}(R_\theta) \longrightarrow K_{n-1}(R_\theta) \longrightarrow \cdots \tag{2.2}$$

in which the middle map from $K_n(A_\theta)$ to $K_{n-2}(A_\theta)$ is the Bott isomorphism.

It follows from (2.1) and Proposition 2.5 that each group $K_n(R_\theta)$ is finitely generated. The following lemma gives some more detailed information.

Lemma 2.6 *For any* $0 \le \theta \le 1$, *there exist* $a_1, \dots, a_7 \in \mathbb{N} \cup \{0\}$ *such that*

$$\begin{aligned} K_0(R_\theta) &\cong \mathbb{Z}^2 \times \mathbb{Z}_2^{a_0}, & K_1(R_\theta) &\cong \mathbb{Z} \times \mathbb{Z}_2^{a_1}, \\ K_2(R_\theta) &\cong \mathbb{Z}_2^{a_2}, & K_3(R_\theta) &\cong \mathbb{Z} \times \mathbb{Z}_2^{a_3}, \\ K_4(R_\theta) &\cong \mathbb{Z}^2 \times \mathbb{Z}_2^{a_4}, & K_5(R_\theta) &\cong \mathbb{Z} \times \mathbb{Z}_2^{a_5}, \\ K_6(R_\theta) &\cong \mathbb{Z}_2^{a_6}, & K_7(R_\theta) &\cong \mathbb{Z} \times \mathbb{Z}_2^{a_7}. \end{aligned}$$

Proof For $i = 2, 6$ then, by Proposition 2.4, $c_i r_i = \mathrm{id} + \alpha_i = 0$. Using $r_i c_i = 2\,\mathrm{id}$ it follows that $2r_i(\mathbb{Z}^2) = r_i c_i r_i(\mathbb{Z}^2) = 0$ and therefore $4K_i(R_\theta) = 2r_i c_i K_i(R_\theta) \subseteq 2r_i(\mathbb{Z}^2) = 0$. Hence $c_i \colon K_i(R_\theta) \to \mathbb{Z}^2$ is the zero map and so $2K_i(R_\theta) = r_i c_i K_i(R_\theta) = 0$, showing that $K_i(R_\theta)$ is a 2-torsion group and, being finitely generated, it is therefore of the required form.

From (2.2) there is an exact sequence

$$\cdots \longrightarrow K_0(R_\theta) \underset{c_0}{\longrightarrow} \mathbb{Z}^2 \underset{r_6}{\longrightarrow} K_6(R_\theta)$$
$$\longrightarrow K_7(R_\theta) \underset{c_7}{\longrightarrow} \mathbb{Z}^2 \underset{r_5}{\longrightarrow} K_5(R_\theta) \longrightarrow K_6(R_\theta) \underset{c_6}{\longrightarrow} \mathbb{Z}^2 \longrightarrow \cdots$$

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Part of this gives $\mathbb{Z}_2^{a_6} \longrightarrow K_7(R_\theta) \underset{c_7}{\longrightarrow} \mathbb{Z}^2 \underset{r_5}{\longrightarrow} K_5(R_\theta) \longrightarrow \mathbb{Z}_2^{a_6} \longrightarrow 0$. From Proposition 2.4 $c_5 r_5 = \mathrm{id} + \alpha_5 = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ so either $\ker(r_5) = \{0\}$ or $\ker(r_5) \cong \mathbb{Z}$. If $\ker(r_5) = \{0\}$ then $\mathrm{Im}(c_7) = 0$ contradicting $c_7 r_7 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Thus $\ker(r_5) = \mathrm{Im}(c_7) \cong \mathbb{Z}$, from which it follows that both $K_5(R_\theta)$ and $K_7(R_\theta)$ are of the form $\mathbb{Z} \times F_i$ for some finite groups F_5, F_7. Then $c_5(F_5) = c_7(F_7) = 0$ and hence $2F_5 = r_5 c_5 F_5 = 0$ and $2F_7 = r_7 c_7 F_7 = 0$, showing that both $K_5(R_\theta)$ and $K_7(R_\theta)$ have the required forms. A similar argument applies to $\mathbb{Z}_2^{a_2} \longrightarrow K_3(R_\theta) \underset{c_3}{\longrightarrow} \mathbb{Z}^2 \underset{r_1}{\longrightarrow} K_1(R_\theta) \longrightarrow \mathbb{Z}_2^{a_2} \longrightarrow 0$, producing the result for $K_1(R_\theta)$ and $K_3(R_\theta)$.

The portion $\mathbb{Z}_2^{a_6} \underset{c_6}{\longrightarrow} \mathbb{Z}^2 \underset{r_4}{\longrightarrow} K_4(R_\theta) \longrightarrow K_5(R_\theta) \underset{c_5}{\longrightarrow} \mathbb{Z}^2 \underset{r_3}{\longrightarrow} K_3(R_\theta)$ has $\mathrm{Im}(c_5) = \ker(r_3) \cong \mathbb{Z}$ since $\ker(r_3) \cong \mathbb{Z}^2$ contradicts $c_3 r_3 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\mathrm{Im}(c_5) = \{0\}$ contradicts $c_5 r_5 = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Thus $0 \to \mathbb{Z}^2 \to K_4(R_\theta) \to \mathbb{Z} \times \mathbb{Z}_2^{a_5} \to \mathbb{Z} \to 0$ and so $0 \to \mathbb{Z}^2 \to K_4(R_\theta) \to \mathbb{Z}_2^{a_5} \to 0$ from which it follows that $K_4(R_\theta)$ has the required form. A similar argument works for $K_0(R_\theta)$. ∎

The exact sequence (2.1) will next be used to limit the size of $a_0, \dots, a_7$.

Lemma 2.7 *Let $a_0, \dots, a_7$ be as defined in Lemma 2.6. Then $a_0 + a_4 \in \{0, 1\}$, $a_1 + a_5 \in \{0, 1, 2\}$, $a_2 + a_6 \in \{1, 2, 3\}$, $a_3 + a_7 = 0$.*

Proof The part of the sequence (2.1) starting at $K_7(C_\theta)$ gives

$$\cdots \longrightarrow \mathbb{Z} \underset{\beta_7}{\longrightarrow} \mathbb{Z}^2 \times \mathbb{Z}_2^{a_3+a_7} \underset{\gamma_7}{\longrightarrow} \mathbb{Z}^2 \underset{\alpha_6}{\longrightarrow} \mathbb{Z}_2^2 \times \mathbb{Z} \underset{\beta_6}{\longrightarrow} \mathbb{Z}_2^{a_2+a_6} \underset{0}{\longrightarrow} \mathbb{Z}^2.$$

If $\mathrm{Im}(\gamma_7) \cong \mathbb{Z}^2$ then $\ker(\beta_6) = \mathrm{Im}(\alpha_6)$ is a torsion group, giving a contradiction to the final part of the sequence. $\mathrm{Im}(\gamma_7) = 0$ is also impossible because α_6 cannot be injective. Hence $\mathrm{Im}(\gamma_7) \cong \mathbb{Z}$ and so $\mathrm{Im}(\beta_7) = \ker(\gamma_7) \cong \mathbb{Z} \times \mathbb{Z}_2^{a_3+a_7}$, which forces $a_3 + a_7 = 0$. The previous part of the sequence (2.1) gives

$$\longrightarrow \mathbb{Z}^3 \underset{\beta_0}{\longrightarrow} \mathbb{Z}^4 \times \mathbb{Z}_2^{a_0+a_4} \underset{\gamma_0}{\longrightarrow} \mathbb{Z}^2 \underset{\alpha_7}{\longrightarrow} \mathbb{Z} \underset{\beta_7}{\longrightarrow} \mathbb{Z}^2$$

and, from $\mathrm{Im}(\alpha_7) = \ker \beta_7 = 0$ it follows that $\mathrm{Im}(\gamma_0) = \mathbb{Z}^2$ and hence $\mathrm{Im}(\beta_0) = \ker(\gamma_0) \cong \mathbb{Z}^2 \times \mathbb{Z}_2^{a_0+a_4}$, from which it follows that $a_0 + a_4 \in \{0, 1\}$. Both possibilities $\mathrm{Im}(\beta_0) \cong \mathbb{Z}^2$ and $\mathrm{Im}(\beta_0) \cong \mathbb{Z}^2 \times \mathbb{Z}_2$ imply that $\ker(\beta_0) \cong \mathbb{Z}$. The part of sequence (2.1) finishing at β_0 is

$$\mathbb{Z}^3 \underset{\beta_1}{\longrightarrow} \mathbb{Z}^2 \times \mathbb{Z}_2^{a_1+a_5} \underset{\gamma_1}{\longrightarrow} \mathbb{Z}^2 \underset{\alpha_0}{\longrightarrow} \mathbb{Z}^3 \underset{\beta_0}{\longrightarrow}$$

and it follows from $\mathrm{Im}(\alpha_0) = \ker(\beta_0) \cong \mathbb{Z}$ that $\mathrm{Im}(\gamma_1) = \ker(\alpha_0) \cong \mathbb{Z}$. Thus $\mathrm{Im}(\beta_1) \cong \mathbb{Z} \times \mathbb{Z}_2^{a_1+a_5}$ from which it follows that $a_1 + a_5 \in \{0, 1, 2\}$. Finally, the part of sequence (2.1) used at the start of the proof has $\ker(\alpha_6) = \mathrm{Im}(\gamma_7) \cong \mathbb{Z}$ so $\mathrm{Im}(\alpha_6) \cong \mathbb{Z} \times \mathbb{Z}_2$ or $\mathrm{Im}(\alpha_6) \cong \mathbb{Z}$. The first possibility leads to $a_2 + a_6 \in \{1, 2\}$ and the second to $a_2 + a_6 \in \{2, 3\}$. ∎

The K-theory of R_θ can now be calculated.

Theorem 2.8 *The K groups of R_θ are given by the following table.*

i	0	1	2	3	4	5	6	7
$K_i(R_\theta)$	$\mathbb{Z}^2$	$\mathbb{Z}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$

Proof From Proposition 2.3 the complexification map $c_0\colon \mathbb{Z}^2\times\mathbb{Z}_2^{a_0}\to\mathbb{Z}^2$ is a surjection. Thus, from $r_0c_0 = 2\,\mathrm{id}$, $r_0(\mathbb{Z}^2) = 2\mathbb{Z}^2$. The exact sequence (2.2) contains the portion

$$\cdots \longrightarrow \mathbb{Z}^2 \xrightarrow[r_0]{} K_0(R_\theta) \longrightarrow K_1(R_\theta) \xrightarrow[c_1]{} \mathbb{Z}^2 \longrightarrow \cdots$$

which is known to be of the form

$$\cdots \longrightarrow \mathbb{Z}^2 \xrightarrow[r_0]{} \mathbb{Z}^2\times\mathbb{Z}_2^{a_0} \xrightarrow[\delta]{} \mathbb{Z}\times\mathbb{Z}_2^{a_1} \xrightarrow[c_1]{} \mathbb{Z}^2 \longrightarrow \cdots$$

From $r_0(\mathbb{Z}^2) = 2\mathbb{Z}^2$ it follows that $\mathrm{Im}(\delta)\cong\mathbb{Z}_2^{2+a_0}$ and thus that $a_1\ge 2+a_0$. However, by Lemma 2.7, $a_1\le 2$ so $a_1 = 2$ and $a_0 = 0$. Then, since $a_1+a_5\le 2$, $a_5 = 0$. Another portion of the sequence (2.2) is

$$\longrightarrow K_0(R_\theta) = \mathbb{Z}^2 \xrightarrow[c_0]{} \mathbb{Z}^2 \xrightarrow[r_6]{} K_6(R_\theta) \longrightarrow K_7(R_\theta) = \mathbb{Z},$$

and, since c_0 is surjective and $K_6(R_\theta) = \mathbb{Z}_2^{a_6}$, $K_6(R_\theta) = 0$.

To calculate $K_4(R_\theta)$ and $K_2(R_\theta)$ note that for $x\in R_\theta$,

$$\begin{aligned}\begin{pmatrix} x & 0\\ 0 & 0\end{pmatrix} &= \frac{1}{2}\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix} + \frac{i}{2}\begin{pmatrix} -ix & 0\\ 0 & ix\end{pmatrix}\\ &= \frac{1}{2}(x\otimes 1_{\mathbb{H}}) + \frac{i}{2}(x\otimes i_{\mathbb{H}})\\ &\in (R_\theta\otimes\mathbb{H}) + i(R_\theta\otimes\mathbb{H}).\end{aligned}$$

Thus $r\left(\begin{smallmatrix} x & 0\\ 0 & 0\end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix} x\otimes 1_{\mathbb{H}} & -x\otimes i_{\mathbb{H}}\\ x\otimes i_{\mathbb{H}} & x\otimes 1_{\mathbb{H}}\end{smallmatrix}\right) = A\left(\begin{smallmatrix} x\otimes 1_{\mathbb{H}} & 0\\ 0 & 0\end{smallmatrix}\right)A^*$ where $\sqrt{2}A = \left(\begin{smallmatrix} 1\otimes 1_{\mathbb{H}} & 1\otimes 1_{\mathbb{H}}\\ 1\otimes i_{\mathbb{H}} & -1\otimes i_{\mathbb{H}}\end{smallmatrix}\right)$, showing that $r([1]) = [1\otimes 1_{\mathbb{H}}]$ and $r([p]) = [p\otimes 1_{\mathbb{H}}]$, whereas $c[1\otimes 1_{\mathbb{H}}] = 2[1]$ and $c[p\otimes 1_{\mathbb{H}}] = 2[p]$. Thus

$$\cdots \longrightarrow \mathbb{Z}^2 \xrightarrow[r_4]{} \mathbb{Z}^2\times\mathbb{Z}_2^{a_4} \xrightarrow[\gamma]{} \mathbb{Z} = K_5(R_\theta) \longrightarrow \cdots$$

with $r_4(\mathbb{Z}^2) = \mathbb{Z}^2$, showing that $a_4 = 0$, and

$$\longrightarrow K_4(R_\theta) = \mathbb{Z}^2 \xrightarrow[c_4]{} \mathbb{Z}^2 \xrightarrow[r_2]{} K_2(R_\theta) \longrightarrow K_3(R_\theta) = \mathbb{Z}$$

with $c_4(\mathbb{Z}^2) = 2\mathbb{Z}^2$, showing that $K_2(R_\theta)\cong\mathbb{Z}_2^2$. ■

Having established the group structure of $K_n(R_\theta)$ it is possible to specify generators explicitly, though this will not be needed in the sequel. This has already been done for $K_0(R_\theta)$. By the identification $K_4(R_\theta)\cong K_0(R_\theta\otimes\mathbb{H})$, the generators for

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$K_4(R_\theta)$ are $[1 \otimes 1_{\mathbb{H}}]$ and $[p \otimes 1_{\mathbb{H}}]$ where $[1]$ and $[p]$ are generators for $K_0(R_\theta)$. The element $[e^{-\pi i\theta}UV^*]$ is a generator of the summand $\mathbb{Z}$ of $K_1(R_\theta)$ and the elements $\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right]$ and $\left[\left(\begin{smallmatrix}1-p&p\\p&1-p\end{smallmatrix}\right)\right]$, where [p] is a generator of $K_0(R_\theta)$, are generators of the two $\mathbb{Z}_2$ summands. These can be obtained in the following way from the generators $[1]$ and $[p]$ of $K_0(A_\theta)$. Note that the relevant portion of the exact sequence (2.1) arising from

$$0 \to C_0\big(I, M_2(A_\theta)\big) \to C_\theta \to M_2(R_\theta) \times (R_\theta \otimes \mathbb{H}) \to 0$$

is

$$\cdots \underset{0}{\longrightarrow} K_1(C_0(I, M_2(A_\theta)) \cong \mathbb{Z}^2 \longrightarrow K_1(C_\theta) \cong \mathbb{Z}^3 \longrightarrow (\mathbb{Z}\times\mathbb{Z}_2^2)\times\mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots,$$

and thus that the two generators of $\mathbb{Z}_2^2$ arise from evaluation at 0 of the elements of C_θ generating the summands containing the image of $K_1\Big(C_0\big(I, M_2(A_\theta)\big)\Big)$.

The two generators $[1]$ and $[p]$ of $K_0(A_\theta)$ give rise to the elements f_1 and f_p of $C_0(I, A_\theta)^+$ defined by $f_1(t) = I + (e^{2\pi it} - 1)1$ and $f_p(t) = I + (e^{2\pi it} - 1)p$, where I is the identity adjoined to $C_0(I, A_\theta)$. The corresponding elements of C_θ are defined by

$$f_p(t) = \begin{cases} \begin{pmatrix} 1 + (e^{4\pi it} - 1)p & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 \le t \le \frac{1}{2} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 + (e^{4\pi it} - 1)p \end{pmatrix} & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

with a corresponding definition of f_1. These formulae arise from using $f(t) = (\Psi\hat{\alpha})\big(f(1-t)^*\big)$ for $\frac{1}{2} \le t \le 1$. Note that $[f_p] = [g_p]$ and $[f_1] = [g_1]$ where

$$g_p(t) = \begin{pmatrix} 1 + (e^{2\pi it} - 1)p & 0 \\ 0 & 1 + (e^{2\pi it} - 1)p \end{pmatrix}$$

for all $0 \le t \le 1$, with a similar formula for g_1. Let $h_p(t) = \left(\begin{smallmatrix} 1-p & pe^{\pi it} \\ pe^{\pi it} & 1-p \end{smallmatrix}\right)$ for $0 \le t \le 1$ with a similar definition of h_1. Then $h_p^2 = g_p$, $h_1^2 = g_1$, $h_p \in C_\theta$ and $h_1 \in C_\theta$. Evaluating at 0 gives the generators $\left[\left(\begin{smallmatrix}1-p&p\\p&1-p\end{smallmatrix}\right)\right]$ and $\left[\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right]$ of the two $\mathbb{Z}_2$ summands of $K_1(R_\theta)$.

Regarding $K_5(R_\theta)$ as $K_1(R_\theta \otimes \mathbb{H})$, a generator is $[e^{-\pi i\theta}UV^* \otimes 1_{\mathbb{H}}]$. The generators of $K_2(R_\theta)$, viewed as $K_1\big(C_0(I, R_\theta)\big)$ are obtained, via the exact sequence (2.2), as the images of the generators of $K_1\big(C_0(I, A_\theta)\big)$ under the realification map. These are given by

$$t \mapsto \begin{pmatrix} 1 + \big(\cos(2\pi t) - 1\big)\, p & p\sin(2\pi t) \\ -p\sin(2\pi t) & 1 + \big(\cos(2\pi t) - 1\big)\, p \end{pmatrix}$$

and

$$t \mapsto \begin{pmatrix} \cos(2\pi t)1 & \sin(2\pi t)1 \\ -\sin(2\pi t)1 & \cos(2\pi t)1 \end{pmatrix}.$$

The latter can also be viewed as the image of the generator of $K_2(\mathbb{R})$ under the map from $K_2(\mathbb{R})$ into $K_2(R_\theta)$ resulting from $\lambda \mapsto \lambda 1$.

The most cumbersome generators to describe are those for $K_3(R_\theta)$ and $K_7(R_\theta)$. To obtain a generator for $K_7(R_\theta)$ note that the exact sequence (2.2) includes the portion

$$\longrightarrow K_1(R_\theta) \underset{c_1}{\longrightarrow} \mathbb{Z}^2 = K_1(A_\theta) \longrightarrow \mathbb{Z}^2 = K_7(A_\theta)$$

$$\underset{r_7}{\longrightarrow} K_7(R_\theta) = \mathbb{Z} \longrightarrow K_0(R_\theta) \underset{c_0}{\longrightarrow} K_0(A_\theta)$$

where c_0 is an isomorphism and the image of c_1 is $[e^{-\pi i\theta}UV^*] = [UV^*]$. It follows that, for either generator $[U]$ or $[V]$ of $K_1(A_\theta)$, the image under r_7 of the corresponding element of $K_7(A_\theta)$ generates $K_7(R_\theta)$. One description of this generator can be obtained by using the results of [5] to identify $K_n(R_\theta)$ with $K_{n+1}(D_\theta)$ where $D_\theta = \bigl\{ f \in C_0(\mathbb{R}, A_\theta) : f(-x) = \Phi\bigl(f(x)^*\bigr) \bigr\}$ $(= C_0^{\mathbb{R}}(i\mathbb{R}) \otimes R_\theta$ in the language of [5]).

The complexification of D_θ is just $C_0(\mathbb{R}, A_\theta)$ and the element of $K_0\bigl(C_0(\mathbb{R}, A_\theta)\bigr)$ corresponding to the element $[U]$ of $K_1(A_\theta)$ is, as described in Theorem 8.2.2 of [1], $[p_U] - \bigl[\bigl(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\bigr)\bigr]$ where $p_U \in C_0\bigl(I, M_2(A_\theta)\bigr)$ is defined by

$$p_U(t) = \begin{pmatrix} 1 + s_t^2 c_t^2 (U + U^* - 2) & c_t s_t (U-1)\bigl(1 + s_t^2(U-1)\bigr) \\ c_t s_t (U^* - 1)\bigl(1 + s_t^2 (U^* - 1)\bigr) & c_t^2 s_t^2 (2 - U^* - U) \end{pmatrix}$$

in which $s_t = \sin(\frac{\pi}{2}t)$ and $c_t = \cos(\frac{\pi}{2}t)$ for $0 \le t \le 1$. The corresponding generator of $K_0(D_\theta)$ is then given by $[P_U] - \bigl[\bigl(\begin{smallmatrix}e&0\\0&e\end{smallmatrix}\bigr)\bigr]$ where

$$P_U = \frac{1}{2}\begin{pmatrix} p_U + \Psi(p_U)^* & -i\Psi(p_U)^* + ip_U \\ i\Psi(p_U)^* - ip_U & p_U + \Psi(p_U)^* \end{pmatrix} \quad \text{and} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

A similar generator can be obtained for $K_3(R_\theta) \cong K_0(D_\theta \otimes \mathbb{H})$ by tensoring with $1_{\mathbb{H}}$.

3 An Inductive Limit Sharing the K-Theory of R_θ

In [16] Walters constructed an inductive limit decomposition of A_θ, when θ is irrational, and a period 4 automorphism of A_θ compatible with the decomposition, producing the same map on $K_1(A_\theta)$ as the Fourier automorphism α given by $\alpha(U) = V$, $\alpha(V) = U^*$. In this section a minor modification of Walters's construction will be used to produce an involutory antiautomorphism Ψ of A_θ compatible with the decomposition and producing the same map on $K_1(A_\theta)$ as the antiautomorphism Φ defined by $\Phi(U) = V$, $\Phi(V) = U$. Furthermore it will be shown that the real inductive limit algebra associated with Ψ has the same K-theory as R_θ, suggesting that R_θ may well be isomorphic to this inductive limit.

Following [16] let θ have continued fraction expansion $[a_0, a_1, \dots]$ where $a_n \ge 1$ for $n \ge 1$ and $a_0 = 0$ and let

$$P_n = \begin{pmatrix} a_{5n} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{5n-1} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{5n-2} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{5n-3} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{5n-4} & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix},$$

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so that $\det(P_n) = -1$. The n-th convergent p_n/q_n of θ is determined by

$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}$$
$$p_0 = 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}$$

and therefore

$$q_{5n+5} = \alpha_{n+1} q_{5n} + \beta_{n+1} q_{5n-1}, \qquad q_{5n+4} = \gamma_{n+1} q_{5n} + \delta_{n+1} q_{5n-1}$$
$$p_{5n+5} = \alpha_{n+1} p_{5n} + \beta_{n+1} p_{5n-1}, \qquad p_{5n+4} = \gamma_{n+1} p_{5n} + \delta_{n+1} p_{5n-1}.$$

As noted in [16], $\alpha_n \geq 5$ and $\gamma_n \geq 5$ for each n, so we can write

$$\alpha_n = 2\alpha_n' + \alpha_n'', \quad \gamma_n = 2\gamma_n' + \gamma_n''$$

where $\alpha_n'', \gamma_n'' \in \{1, 2\}$ and $\alpha_n', \gamma_n' \geq 2$. Then, as in [16], let

$$A_n = M_{q_{5n}}\big(C(\mathbb{T})\big) \oplus M_{q_{5n}}\big(C(\mathbb{T})\big) \oplus M_{q_{5n-1}} \oplus M_{q_{5n}}$$

and equip this with the involutory antiautomorphism Ψ_n defined by

$$\Psi_n(f, g, A, B) = (g^{\mathrm{tr}}, f^{\mathrm{tr}}, A^{\mathrm{tr}}, B^{\mathrm{tr}}),$$

which has the associated real algebra $R_n = \big\{ (f, \bar{f}, A, B) : f \in M_{q_{5n}}\big(C(\mathbb{T})\big), A \in M_{q_{5n-1}}(\mathbb{R}), B \in M_{q_{5n-1}}(\mathbb{R}) \big\}$.

For any $\ell \times \ell$ matrix M, let $I_k \otimes M$ denote the $k\ell \times k\ell$ matrix with K copies of M down the main diagonal and let $M \otimes I_k$ denote the $k\ell \times k\ell$ matrix consisting of $k \times k$ blocks $m_{ij} I_k$ in the obvious way. As in [16] let S_k and $S_k(\mathrm{id})$ be the $k \times k$ matrices with entries in $C(\mathbb{T})$ defined by

$$S_k = \begin{pmatrix} 0 & 1 \\ I_{k-1} & 0 \end{pmatrix} \quad \text{and} \quad S_k(\mathrm{id}) = \begin{pmatrix} 0 & \mathrm{id} \\ I_{k-1} & 0 \end{pmatrix}$$

where id is the identity function on $\mathbb{T} \subseteq \mathbb{C}$. Let $\rho_n \colon A_n \to A_{n+1}$ be defined, for constant $X, Y \in M_{q_{5n}}\big(C(\mathbb{T})\big)$, for $Z \in M_{q_{5n-1}}$ and $Z' \in M_{q_{5n}}$ by

$$\rho_n(\mathrm{id}\, I_{q_{5n}}, 0, 0, 0)$$
$$= \Big([I_{q_{5n}} \otimes S_{\alpha'_{n+1}}(\mathrm{id})]000, [I_{q_{5n}} \otimes S_{\alpha'_{n+1}}]000, [I_{q_{5n}} \otimes S_{\gamma'_{n+1}}]000, [I_{q_{5n}} \otimes S_{\alpha'_{n+1}}]000 \Big),$$
$$\rho_n(0, \mathrm{id}\, I_{q_{5n}}, 0, 0)$$
$$= \Big(0[I_{q_{5n}} \otimes S^{\mathrm{tr}}_{\alpha'_{n+1}}]00, 0[I_{q_{5n}} \otimes S^{\mathrm{tr}}_{\alpha'_{n+1}}(\mathrm{id})]00, 0[I_{q_{5n}} \otimes S^{\mathrm{tr}}_{\gamma'_{n+1}}]00, 0[I_{q_{5n}} \otimes S^{\mathrm{tr}}_{\alpha'_{n+1}}]00 \Big),$$
$$\rho_n(X, Y, Z, Z') = (A, A, B, A),$$

where

$$A = [X \otimes I_{\alpha'_{n+1}}][Y \otimes I_{\alpha'_{n+1}}][Z \otimes I_{\beta_{n+1}}][Z' \otimes I_{\alpha''_{n+1}}]$$

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and

$$B = [X \otimes I_{\gamma'_{n+1}}][Y \otimes I_{\gamma'_{n+1}}][Z \otimes I_{\delta_{n+1}}][Z' \otimes I_{\gamma''_{n+1}}].$$

Here, as in [16], the matrices in square brackets are diagonal blocks in the appropriate matrix of size q_{5n+5} or q_{5n+4}. (The only difference from the map ρ_n defined in [16] is in the third and fourth components of the image of $(0, \mathrm{id}\, I_{q_{5n}}, 0, 0)$, where S^{tr} replaces $\Lambda S \Lambda^*$.)

For each $k \in \mathbb{N}$ let W_{2k} be the $2k \times 2k$ unitary matrix

$$W_{2k} = \frac{1}{\sqrt{2}} \begin{pmatrix} iI_k & -iI_k \\ I_k & I_k \end{pmatrix}$$

and for each $n \in \mathbb{N}$ let V_{n+1} be the matrix in $M_{q_{5n+5}}\big(C(\mathbb{T})\big) \oplus M_{q_{5n+5}}\big(C(\mathbb{T})\big) \oplus M_{q_{5n+4}} \oplus M_{q_{5n+5}}$ defined by

$$V_{n+1} = \big([W_{2q_{5n}\alpha'_{n+1}}]II, [W_{2q_{5n}\alpha'_{n+1}}]II, [W_{2q_{5n}\gamma'_{n+1}}]II, [W_{2q_{5n}\alpha'_{n+1}}]II\big)$$

Then let $\psi_n \colon A_n \to A_{n+1}$ be defined by $\psi_n = (\mathrm{Ad}\, V_{n+1}) \circ \rho_n$.

Lemma 3.1 *For each n,* $\Psi_{n+1}\psi_n = \psi_n \Psi_n$.

Proof Note that for $k \times k$ matrices A, B

$$W_{2k} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} W_{2k}^* = \frac{1}{2} \begin{pmatrix} A+B & i(A-B) \\ i(B-A) & A+B \end{pmatrix} = \left[W_{2k} \begin{pmatrix} B^{\mathrm{tr}} & 0 \\ 0 & A^{\mathrm{tr}} \end{pmatrix} W_{2k}^* \right]^{\mathrm{tr}}.$$

It follows that $\Psi_{n+1}\psi_n(\mathrm{id}\, I_{q_{5n}}, 0, 0, 0) = \psi_n(0, \mathrm{id}\, I_{q_{5n}}, 0, 0) = \psi_n\Psi_n(\mathrm{id}\, I_{q_{5n}}, 0, 0, 0)$, that $\Psi_{n+1}\psi_n(0, \mathrm{id}\, I_{q_{5n}}, 0, 0) = \psi_n(\mathrm{id}\, I_{q_{5n}}, 0, 0, 0) = \psi_n\Psi_n(0, \mathrm{id}\, I_{q_{5n}}, 0, 0)$ and that $\Psi_{n+1}\psi_n(X, Y, Z, Z') = \psi_n(Y^{\mathrm{tr}}, X^{\mathrm{tr}}, Z^{\mathrm{tr}}, Z'^{\,\mathrm{tr}}) = \psi_n\Psi_n(X, Y, Z, Z')$. ■

It follows from Lemma 3.1 that $\psi_n \colon R_n \to R_{n+1}$ where

$$\begin{aligned} R_n &= \{a \in A_n : \Psi_n(a) = a^*\} \\ &= \big\{ (A, \bar{A}, B, C) : A \in M_{q_{5n}}\big(C(\mathbb{T})\big), B \in M_{q_{5n-1}}(\mathbb{R}), C \in M_{q_{5n}}(\mathbb{R}) \big\}. \end{aligned}$$

The elements of R_n will henceforth be identified with triples (A, B, C) where $A \in M_{q_{5n}}\big(C(\mathbb{T})\big)$, $B \in M_{q_{5n-1}}(\mathbb{R})$, $C \in M_{q_{5n}}(\mathbb{R})$. In this context, for constant $X \in M_{q_{5n}}\big(C(\mathbb{T})\big)$, for $Z \in M_{q_{5n-1}}(\mathbb{R})$ and for $Z' \in M_{q_{5n}}(\mathbb{R})$,

$$\psi_n(\mathrm{id}\, I_{q_{5n}}, 0, 0) = \big([T_n]00, [I_{2q_{5n}} \otimes S_{\gamma'_{n+1}}]00, [I_{2q_{5n}} \otimes S_{\alpha'_{n+1}}]00\big),$$

$$\psi_n(X, Z, Z') = (A, B, A),$$

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where

$$A = [r(X \otimes I_{\alpha'_{n+1}})][Z \otimes I_{\beta_{n+1}}][Z' \otimes I_{\alpha''_{n+1}}],$$
$$B = [r(X \otimes I_{\gamma'_{n+1}})][Z \otimes I_{\delta_{n+1}}][Z' \otimes I_{\gamma''_{n+1}}],$$
$$T_n = \operatorname{Ad} W_{2q_{5n}\alpha'_{n+1}}\big([I_{q_{5n}} \otimes S_{\alpha'_{n+1}}(\mathrm{id})][I_{q_{5n}} \otimes S_{\alpha'_{n+1}}]\big)$$
$$= \frac{1}{2}\begin{pmatrix} I_{q_{5n}} \otimes \big(S_{\alpha'_{n+1}} + S_{\alpha'_{n+1}}(\mathrm{id})\big) & iI_{q_{5n}} \otimes \big(S_{\alpha'_{n+1}}(\mathrm{id}) - S_{\alpha'_{n+1}}\big) \\ iI_{q_{5n}} \otimes \big(S_{\alpha'_{n+1}} - S_{\alpha'_{n+1}}(\mathrm{id})\big) & I_{q_{5n}} \otimes \big(S_{\alpha'_{n+1}} + S_{\alpha'_{n+1}}(\mathrm{id})\big)\end{pmatrix},$$
$$r(X \otimes I_{\alpha'_{n+1}}) = \operatorname{Ad} W_{2q_{5n}\alpha'_{n+1}}([X \otimes I_{\alpha'_{n+1}}][\bar{X} \otimes I_{\alpha'_{n+1}}])$$
$$= \begin{pmatrix} \operatorname{Re}(X) \otimes I_{\alpha'_{n+1}} & -\operatorname{Im}(X) \otimes I_{\alpha'_{n+1}} \\ \operatorname{Im}(X) \otimes I_{\alpha'_{n+1}} & \operatorname{Re}(X) \otimes I_{\alpha'_{n+1}}\end{pmatrix}.$$

These formulae enable the K-theory of $R = \lim R_n$ to be computed.

Theorem 3.2 *Let $0 < \theta < 1$ be irrational and let $R = \lim(R_n, \psi_n)$ where $R_n = M_{q_{5n}}\big(C(\mathbb{T})\big) \oplus M_{q_{5n-1}}(\mathbb{R}) \oplus M_{q_{5n}}(\mathbb{R})$ and where ψ_n is defined above. Then the complexification of R is isomorphic to A_θ and the K groups of R are given by the following table.*

i	0	1	2	3	4	5	6	7
$K_i(R)$	$\mathbb{Z}^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$

Proof Recall that the K groups of $\mathbb{R}$ and $C(\mathbb{T})$ are as given in the following table.

i	0	1	2	3	4	5	6	7
$K_i\big(C(\mathbb{T})\big)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$K_i(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$K_i(R_n)$	$\mathbb{Z}^3$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

All cases other than $i = 0, 4$ can be handled by considering separately the effect on the $M_q\big(C(\mathbb{T})\big)$ and $M_{q'}(\mathbb{R}) \oplus M_q(\mathbb{R})$ summands. On the $M_q\big(C(\mathbb{T})\big)$ summands the map ψ_n is specified by

$$\mathrm{id}\, I_{q_{5n}} \mapsto \operatorname{Ad} W_{2q_{5n}\alpha'_{n+1}}\big([I_{q_{5n}} \otimes S_{\alpha'_{n+1}}(\mathrm{id})][I_{q_{5n}} \otimes S_{\alpha'_{n+1}}]\big)\,00$$
$$X \mapsto \operatorname{Ad} W_{2q_{5n}\alpha'_{n+1}}\big([X \otimes I_{\alpha'_{n+1}}][\bar{X} \otimes I_{\alpha'_{n+1}}]\big)\,00.$$

Since the K-theory is not affected by the inner automorphism, ψ_n can be replaced by the sum of a linear and antilinear map specified by

$$\mathrm{id}\, I_{q_{5n}} \mapsto I_{q_{5n}} \otimes S_{\alpha'_{n+1}}(\mathrm{id}), \quad X \mapsto X \otimes I_{\alpha'_{n+1}}$$

and

$$\mathrm{id}\, I_{q_{5n}} \mapsto I_{q_{5n}} \otimes S_{\alpha'_{n+1}}, \quad X \mapsto \bar{X} \otimes I_{\alpha'_{n+1}}.$$

It follows that ψ_n induces the identity map from $K_1\big(C(\mathbb{T})\big) \cong \mathbb{Z}$ to $K_1\big(C(\mathbb{T})\big) \cong \mathbb{Z}$. Furthermore, since only the linear component of the map has a non-zero effect on

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K_1, usual complex Bott periodicity shows that ψ_n also induces the identity map from $K_i\big(C(\mathbb{T})\big) \cong \mathbb{Z}$ to $K_i\big(C(\mathbb{T})\big) \cong \mathbb{Z}$ when $i = 3, 5, 7$. In the cases $i = 3, 5, 7$, for which $K_i(\mathbb{R}) = 0$, ψ_n therefore induces the identity map from $K_i(R_n) \cong \mathbb{Z}$ to $K_i(R_{n+1}) \cong \mathbb{Z}$.

On $K_0\big(C(\mathbb{T})\big) \cong \mathbb{Z}$ both linear and antilinear parts correspond to multiplication by α'_{n+1} on $\mathbb{Z}$. Thus, using the discussion in the proof of Proposition 2.4, the same is true on K_4, but in K_2 and K_6 the antilinear part corresponds to multiplication by $-\alpha'_{n+1}$. Thus, when $i = 2$ or $i = 6$, ψ_n induces the zero map from $K_i\big(C(\mathbb{T})\big)$ to $K_i\big(C(\mathbb{T})\big)$. When $i = 6$, for which $K_i(\mathbb{R}) = 0$, it follows that ψ_n gives the zero map from $K_i(R_n)$ to $K_i(R_{n+1})$.

Turning to the $M_{q'}(\mathbb{R}) \oplus M_q(\mathbb{R})$ summands, ψ_n is given by

$$(Z, Z') \mapsto \big(00[Z \otimes I_{\delta_{n+1}}][Z' \otimes I_{\gamma''_{n+1}}], 00[Z \otimes I_{\beta_{n+1}}][Z' \otimes I_{\alpha''_{n+1}}]\big).$$

It follows that, for any i, the effect on $K_i\big(M_{q_{5n-1}}(\mathbb{R}) \oplus M_{q_{5n}}(\mathbb{R})\big)$ is given by the matrix

$$\begin{pmatrix} \delta_{n+1} & \beta_{n+1} \\ \gamma''_{n+1} & \alpha''_{n+1} \end{pmatrix}.$$

Recall that $\alpha_{n+1}\delta_{n+1} - \beta_{n+1}\gamma_{n+1} = -1$ and that $\alpha''_{n+1} \equiv \alpha_{n+1} \pmod 2$, $\gamma''_{n+1} \equiv \gamma_{n+1} \pmod 2$, so that for $i = 1, 2$, ψ_n induces an isomorphism from $\mathbb{Z}_2^2$ to $\mathbb{Z}_2^2$. Combining this with the earlier results on the $M_q\big(C(\mathbb{T})\big)$ summands, it follows that ψ_n induces an isomorphism from $K_1(R_n) \cong \mathbb{Z} \times \mathbb{Z}_2^2$ onto $K_1(R_{n+1}) \cong \mathbb{Z} \times \mathbb{Z}_2^2$ and a homomorphism with range $\mathbb{Z}_2^2$ from $K_2(R_n) \cong \mathbb{Z} \times \mathbb{Z}_2^2$ onto $\mathbb{Z}_2^2 \subseteq K_2(R_{n+1})$, with ψ_{n+1} then mapping this image isomorphically onto $\mathbb{Z}_2^2 \subseteq K_2(R_{n+2})$.

This leaves K_0 and K_4 to be considered. As in [16] the corresponding map from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ is in each case given by the matrix

$$\begin{pmatrix} \alpha'_{n+1} & \beta_{n+1} & \alpha''_{n+1} \\ \gamma'_{n+1} & \delta_{n+1} & \gamma''_{n+1} \\ \alpha'_{n+1} & \beta_{n+1} & \alpha''_{n+1} \end{pmatrix}$$

(where exactly the same 4×4 matrix as in [16] is obtained after embedding R_n in A_n). The arguments given in the proof of Proposition 2 of [16] show that the limit algebra has $K_i(R)$ isomorphic to $\mathbb{Z}^2$ and that the complexification of R, namely $\lim(A_n, \psi_n)$, is isomorphic to A_θ. ■

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