A NOTE ON THE ABSOLUTE NÖRLUND SUMMABILITY OF CONJUGATE FOURIER SERIES

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Introduction

Let $\sum a_n$ be an infinite series, with sequence of partial sums $\{u_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and write

$$P_n = p_0 + p_1 + \cdots + p_n.$$

The sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} \frac{p_{n-k}}{P_n} u_k, \qquad P_n \neq 0,$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{u_n\}$, generated by the sequence $\{p_n\}$. The series $\sum a_n$ is said to be summable (N, p_n) to sum s, if $\lim_{n\to\infty} t_n = s$. It is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\{t_n\} \in BV$.

Necessary and sufficient conditions in order that the method (N, p_n) be regular are

$$p_n = o(|P_n|), \qquad n \to \infty,$$

and

$$\sum_{k=0}^{n} |p_k| = O(|P_n|), \qquad n \to \infty.$$

For absolute regularity of the method, necessary and sufficient conditions are $p_n = o(|P_n|)$ and

$$\sum_{n=k}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n+1-k}}{P_{n+1}} \right| < K,$$

K being independent of k, $k = 0, 1, 2, \cdots$ (cf. Mears [5] and Knopp and Lorentz [4]).

The object of this note is to establish a theorem on the summability $|N, p_n|$ of the conjugate series $\sum (b_n \cos nt - a_n \sin nt)$ of a Lebesgue integrable, 2π-periodic function $f(t)$. Before stating this theorem, we introduce the following notation:

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}$$

$$P_n^* = \sum_0^n |p_k|$$

$$R_n = \frac{(n+1)p_n}{P_n}$$

$$S_n = \frac{1}{P_n}\sum_0^n \frac{P_k}{(k+1)}$$

$$\sigma_n = |P_n|\sum_n^\infty \frac{1}{(k+1)|P_k|}$$

$$\Delta q_n = q_n - q_{n+1}$$

$$T_n = \frac{1}{|P_n|}\sum_1^n k|p_{k-1}-p_k|.$$

We now state the main result of this paper.

THEOREM. *Let* $\{p_n\}$ *be a sequence of numbers such that* $P_n^* = O(|P_n|)$, $\{S_n\} \in B$ *and* $\{R_n\} \in BV$. *If* $\psi(t) \in BV(0, \pi)$ *and* $\int_0^\pi |\psi(t)|\, dt/t$ *exists, then the conjugate Fourier series of* $f(t)$ *is summable* $|N, p_n|$ *at* $t = x$.

For some of the existing results on the $|N, p_n|$ summability, the reader is referred to Bosanquet and Hyslop [1, Theorem 4], Pati [6] and [7, Theorem 2], and Wang [8, Theorem 2]. Results in [1] pertain to the $|C, \alpha|$ summability to which the $|N, p_n|$ method reduces in the case where $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$. These results are all special cases of the theorem proved here.

Preliminary lemmas

LEMMA 1. *If* $\{p_n\}$ *defines a regular method of summation* (N, p_n) *then* $\{S_n\} \in B$ *if and only if* $\{\sigma_n\} \in B$.

This has been proved elsewhere [2, Lemma 1].

LEMMA 2. *If* $\{p_n\}$ *defines a regular method of summation* (N, p_n), *and if* $\{R_n\} \in BV$, *then* $\{S_n\} \in B$ *is sufficient for* $\{S_n\} \in BV$.

PROOF. Noting that

$$S_{n+1} = \frac{1}{P_{n+1}}\sum_{k=0}^{n+1}\frac{P_k}{k+1} = \frac{1}{(n+2)} + \frac{P_n}{P_{n+1}} S_n,$$

we see that

$$\begin{aligned}
S_{n+1}-S_n &= \frac{1}{(n+2)} - \frac{p_{n+1}}{P_{n+1}} S_n \\
&= \frac{1}{(n+2)P_n}\left\{P_n - R_{n+1}\sum_{k=0}^{n} \frac{P_k}{k+1}\right\} \\
&= \frac{1}{(n+2)P_n}\sum_{k=0}^{n} \frac{(k+1)p_k - R_{n+1}P_k}{(k+1)} \\
&= \frac{1}{(n+2)P_n}\sum_{k=0}^{n} \left(\frac{P_k R_k - R_{n+1}P_k}{(k+1)}\right) \\
&= \frac{1}{(n+2)P_n}\sum_{k=0}^{n} \frac{P_k}{k+1}\sum_{v=k}^{n}(\Delta R_v) \\
&= \frac{1}{(n+2)P_n}\sum_{v=0}^{n}(\Delta R_v)\sum_{k=0}^{v} \frac{P_k}{k+1} \\
&= \frac{1}{(n+2)P_n}\sum_{k=0}^{n} S_k P_k(\Delta R_k).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=0}^{\infty}|\Delta S_n| &\leqq \sum_{n=0}^{\infty} \frac{1}{(n+1)|P_n|}\sum_{k=0}^{n}|S_k P_k(\Delta R_k)| \\
&= \sum_{k=0}^{\infty}|(\Delta R_k)S_k P_k| \sum_{n=k}^{\infty}\frac{1}{(n+1)|P_n|} \\
&= \sum_{k=0}^{\infty}|(\Delta R_k)|\,|S_k|\,|\sigma_k| \\
&\leqq K, \qquad \text{by Lemma 1.}
\end{aligned}$$

Lemma 3. [8, Theorem 2]. *Let $\{p_n\}$ be any sequence of numbers such that $P_n^* = O(|P_n|)$, $\{T_n\} \in B$, $\{R_n\} \in BV$ and $\{S_n\} \in BV$. If $\psi(t) \in BV(0, \pi)$ and*

$$\int_0^{\pi} |\psi(t)| \frac{dt}{t}$$

exists, then the conjugate Fourier series of $f(t)$ is summable $|N, p_n|$ at $t = x$.

Lemma 4. *If $P_n^* = O(|P_n|)$ and $\{R_n\} \in BV$, then $\{T_n\} \in B$.*

Proof. We have

$$\begin{aligned}
\Delta R_k &= \frac{(k+1)\Delta p_k}{P_k} + p_{k+1}\left(\frac{k+1}{P_k} - \frac{k+2}{P_{k+1}}\right) \\
&= \frac{(k+1)\Delta p_k}{P_k} + \frac{(k+2)p_{k+1}^2}{P_k P_{k+1}} - \frac{p_{k+1}}{P_k} \\
&= \frac{1}{P_k}\{(k+1)\Delta p_k + p_{k+1}R_{k+1} - p_{k+1}\}.
\end{aligned}$$

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Hence

$$\begin{aligned} T_n &= \frac{1}{|P_n|}\sum_{k=0}^{n-1}(k+1)|\Delta p_k| \\ &\leqq \frac{1}{|P_n|}\sum_{k=0}^{n-1}|P_k(\Delta R_k)| + \frac{1}{|P_n|}\sum_{k=0}^{n-1}|(1-R_{k+1})p_{k+1}| \\ &\leqq K\sum_{k=0}^{n-1}|\Delta R_k| + K \\ &\leqq K. \end{aligned}$$

Proof of the theorem

$\{R_n\} = (n+1)p_n/P_n \in BV$ implies that $p_n = o(|P_n|)$, so that the sequence $\{p_n\}$ of the theorem satisfies the hypotheses of Lemma 2. We now use Lemmas 2, 3 and 4 to complete the proof.

Remarks

The method of proof used here furnishes an alternate proof for the theorem of [2].

One observes that in the light of Lemma 4, the condition $\{T_n\} \in B$ in Lemma 3 can be omitted. A similar remark applies to the condition

$$n|p_n| < K|P_n|$$

assumed in some of the results of Hille and Tamarkin [3] on Fourier effectiveness of regular (N, p_n) methods. This condition may be dropped since it is assumed that $\{T_n\} \in B$.

We have

$$\{T_n\} = \left\{\frac{1}{P_n}\sum_{k=1}^{n} k|p_{k-1} - p_k|\right\} \in B$$

and this implies that

$$\{Z_n\} = \left\{\frac{1}{P_n}\sum_{k=1}^{n} k(p_{k-1} - p_k)\right\} \in B.$$

Furthermore

$$Z_n = \frac{P_{n-1}}{P_n} - \frac{np_n}{P_n}$$

and therefore, fcr a regular method of summation (N, p_n), $\{Z_n\} \in B$ if and only if $n|p_n| < K|P_n|$ holds.

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