A NOTE ON THE ABSOLUTE NÖRLUND SUMMABILITY
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Introduction

Let \( \sum a_n \) be an infinite series, with sequence of partial sums \( \{u_n\} \). Let \( \{p_n\} \) be a sequence of constants, real or complex, and write

\[ P_n = p_0 + p_1 + \cdots + p_n. \]

The sequence-to-sequence transformation

\[ t_n = \sum_{k=0}^{n} \frac{p_{n-k}}{P_n} u_k, \quad P_n \neq 0, \]

defines the sequence \( \{t_n\} \) of Nörlund means of the sequence \( \{u_n\} \), generated by the sequence \( \{p_n\} \). The series \( \sum a_n \) is said to be summable \((N, p_n)\) to sum \( s \), if

\[ \lim_{n \to \infty} t_n = s. \]

It is said to be absolutely summable \((N, p_n)\), or summable \(|N, p_n|\), if \( \{t_n\} \in BV \).

Necessary and sufficient conditions in order that the method \((N, p_n)\) be regular are

\[ p_n = o(|P_n|), \quad n \to \infty, \]

and

\[ \sum_{k=0}^{n} |p_k| = O(|P_n|), \quad n \to \infty. \]

For absolute regularity of the method, necessary and sufficient conditions are \( p_n = o(|P_n|) \) and

\[ \sum_{n=k}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n+1-k}}{P_{n+1}} \right| < K, \]

\( K \) being independent of \( k, k = 0, 1, 2, \cdots \) (cf. Mears [5] and Knopp and Lorentz [4]).

The object of this note is to establish a theorem on the summability \(|N, p_n|\) of the conjugate series \( \sum (b_n \cos nt - a_n \sin nt) \) of a Lebesgue integrable, \( 2\pi \)-periodic function \( f(t) \). Before stating this theorem, we introduce the following notation:
We now state the main result of this paper.

**Theorem.** Let \( \{p_n\} \) be a sequence of numbers such that \( P_n^* = O(|p_n|) \), \( \{S_n\} \in B \) and \( \{R_n\} \in BV \). If \( \psi(t) \in BV(0, \pi) \) and \( \int_0^\pi |\psi(t)| \, dt/t \) exists, then the conjugate Fourier series of \( f(t) \) is summable \( N, p_n \) at \( t = x \).

For some of the existing results on the \( N, p_n \) summability, the reader is referred to Bosanquet and Hyslop [1, Theorem 4], Pati [6] and [7, Theorem 2], and Wang [8, Theorem 2]. Results in [1] pertain to the \( C, \alpha \) summability to which the \( N, p_n \) method reduces in the case where \( p_n = \left( \frac{n+\alpha-1}{\alpha-1} \right) \), \( \alpha > 0 \). These results are all special cases of the theorem proved here.

**Preliminary lemmas**

**Lemma 1.** If \( \{p_n\} \) defines a regular method of summation \( (N, p_n) \) then \( \{S_n\} \in B \) if and only if \( \{\sigma_n\} \in B \).

This has been proved elsewhere [2, Lemma 1].

**Lemma 2.** If \( \{p_n\} \) defines a regular method of summation \( (N, p_n) \), and if \( \{R_n\} \in BV \), then \( \{S_n\} \in B \) is sufficient for \( \{S_n\} \in BV \).

**Proof.** Noting that

\[
S_{n+1} = \frac{1}{P_{n+1}} \sum_{k=0}^{n+1} \frac{P_k}{k+1} \]

\[
= \frac{1}{n+2} + \frac{P_n}{P_{n+1}} S_n,
\]

we see that...
\[ S_{n+1} - S_n = \frac{1}{(n+2)} - \frac{p_{n+1}}{P_{n+1}} S_n \]

\[ = \frac{1}{(n+2)P_n} \left( P_n - R_{n+1} \sum_{k=0}^{n} \frac{P_k}{k+1} \right) \]

\[ = \frac{1}{(n+2)P_n} \sum_{k=0}^{n} \frac{(k+1)p_k - R_{n+1} P_k}{(k+1)} \]

\[ = \frac{1}{(n+2)P_n} \sum_{k=0}^{n} \frac{P_k R_k - R_{n+1} P_k}{(k+1)} \]

\[ = \frac{1}{(n+2)P_n} \sum_{k=0}^{n} \frac{P_k}{k+1} \sum_{v=k}^{n} (\Delta R_v) \]

\[ = \frac{1}{(n+2)P_n} \sum_{k=0}^{n} S_k P_k(\Delta R_k) . \]

Hence

\[ \sum_{n=0}^{\infty} |\Delta S_n| \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)|P_n|} \sum_{k=0}^{n} |S_k P_k(\Delta R_k)| \]

\[ = \sum_{k=0}^{\infty} |(\Delta R_k)S_k P_k| \sum_{n=k}^{\infty} \frac{1}{(n+1)|P_n|} \]

\[ = \sum_{k=0}^{\infty} |(\Delta R_k)| |S_k| |\sigma_k| \]

\[ \leq K, \quad \text{by Lemma 1.} \]

**Lemma 3.** [8, Theorem 2]. Let \( \{p_n\} \) be any sequence of numbers such that

\[ P_n^* = O(|P_n|), \{T_n\} \in B, \{R_n\} \in BV \text{ and } \{S_n\} \in BV. \]

If \( \psi(t) \in BV(0, \pi) \) and

\[ \int_0^{\pi} |\psi(t)| \frac{dt}{t} \]

exists, then the conjugate Fourier series of \( f(t) \) is summable \(|N, p_n| \) at \( t = \pi \).

**Lemma 4.** If \( P_n^* = O(|P_n|) \) and \( \{R_n\} \in BV \), then \( \{T_n\} \in B. \)

**Proof.** We have

\[ \Delta R_k = \frac{(k+1)\Delta p_k}{P_k} + \frac{(k+1)\Delta p_{k+1}}{P_{k+1}} \]

\[ = \frac{(k+1)\Delta p_k}{P_k} + \frac{(k+2)p_k^2}{P_k P_{k+1}} - \frac{p_{k+1}}{P_k} \]

\[ = \frac{1}{P_k} \{(k+1)\Delta p_k + p_{k+1} R_{k+1} - p_{k+1}\}. \]
Hence

\[
T_n = \frac{1}{|P_n|} \sum_{k=0}^{n-1} (k+1)|\Delta p_k|
\]

\[
\leq \frac{1}{|P_n|} \sum_{k=0}^{n-1} |P_k(\Delta R_k)| + \frac{1}{|P_n|} \sum_{k=0}^{n-1} |(1-R_{k+1})p_{k+1}|
\]

\[
\leq K \sum_{k=0}^{n-1} |\Delta R_k| + K
\]

Proof of the theorem

\( \{R_n\} = (n+1)p_n/P_n \in BV \) implies that \( p_n = o(|P_n|) \), so that the sequence \( \{p_n\} \) of the theorem satisfies the hypotheses of Lemma 2. We now use Lemmas 2, 3 and 4 to complete the proof.

Remarks

The method of proof used here furnishes an alternate proof for the theorem of [2].

One observes that in the light of Lemma 4, the condition \( \{T_n\} \in B \) in Lemma 3 can be omitted. A similar remark applies to the condition

\[ n|p_n| < K|P_n| \]

assumed in some of the results of Hille and Tamarkin [3] on Fourier effectiveness of regular \((N, p_n)\) methods. This condition may be dropped since it is assumed that \( \{T_n\} \in B \).

We have

\[ \{T_n\} = \left\{ \frac{1}{P_n} \sum_{k=1}^{n} k|p_{k-1} - p_k| \right\} \in B \]

and this implies that

\[ \{Z_n\} = \left\{ \frac{1}{P_n} \sum_{k=1}^{n} k(p_{k-1} - p_k) \right\} \in B. \]

Furthermore

\[ Z_n = \frac{P_{n-1} - np_n}{P_n} \]

and therefore, for a regular method of summation \((N, p_n)\), \( \{Z_n\} \in B \) if and only if \( n|p_n| < K|P_n| \) holds.
References


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