A NOTE ON THE STRONG ERDŐS–HAJNAL PROPERTY FOR GRAPHS WITH BOUNDED VC-MINIMAL COMPLEXITY

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Abstract. Inspired by Adler's idea on VC minimal theories [1], we introduce VC-minimal complexity. We show that for any $N \in \mathbb{N}^{>0}$, there is $k_N > 0$ such that for any finite bipartite graph $(X, Y; E)$ with VC-minimal complexity $< N$, there exist $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq k_N|X|$, $|Y'| \geq k_N|Y|$ such that $X' \times Y' \subseteq E$ or $X' \times Y' \cap E = \emptyset$.

§1. Introduction. *Erdős–Hajnal conjecture* [8] says for any graph H there is $\varepsilon > 0$ such that if a graph G does not contain any induced subgraph isomorphic to H then G has a clique or an anti-clique of size $\geq |G|^{\varepsilon}$. More generally, we say a family of finite graphs has the *Erdős–Hajnal property* if there is $\varepsilon > 0$ such that for any graph G in the family, G has a clique or an anti-clique of size $\geq |G|^{\varepsilon}$. A family of finite graphs has the *strong Erdős–Hajnal property* if there is $\varepsilon > 0$ such that for any graph $G = (V, E)$ in the family, there exist $X, Y \subseteq V$ such that $X \cap Y = \emptyset$, $|X| \geq \varepsilon|V|$, $|Y| \geq \varepsilon|V|$, and $X \times Y \subseteq E$ or $X \times Y \subseteq \neg E$. The strong Erdős–Hajnal property implies the Erdős-Hajnal property (see [2, Theorem 1.2.]). Malliaris and Shelah proved in [12] that the family of stable graphs has the Erdős–Hajnal property. Chernikov and Starchenko gave another proof for stable graphs in [3] and in [4] they proved that the family of distal graphs has the strong Erdős–Hajnal property. In general, we are interested in whether the family of bounded VC-dimension (i.e., NIP [14]) graphs, which contains both stable graphs and distal graphs, has the Erdős–Hajnal property. Motivation for studying this problem was given in [10], which also gave a lower bound $e^{(\log n)^{1-o(1)}}$ for largest clique or anti-clique in a graph with bounded VC-dimension. In this paper, we consider graphs of bounded VC-minimal complexity, a special case of NIP graphs. Roughly speaking, we say a bipartite graph $(X, Y; E)$ has VC-minimal complexity $< N$ if for all $a \in X$, the set $\{y \in Y : (a, y) \in E\}$ is a finite union of Swiss Cheeses such that the sum of the number of holes and the number of Swiss Cheeses is $< N$. We will show that the strong Erdős–Hajnal property holds for the family of finite bipartite graphs $(X, Y; E)$ of bounded VC-minimal complexity. One example is definable relations $E(x, y)$ with $|x| = 1, |y| = 1$ in $ACVF$ (algebraically closed valued field). Since $ACVF$ allows Swiss Cheese decomposition [11], given any $\mathcal{M} \models ACVF$ and any definable relation $E \subseteq M \times M$, the family $\{(X, Y; E_{\restriction X \times Y}) : X, Y$

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finite subsets of $M\}$ has bounded VC-minimal complexity, and thus the strong Erdős–Hajnal property holds. This partially generalizes [4, Example 4.11(2)].

Recently, in [13], Nguyen, Scott, and Seymour proved that any family of finite graphs with bounded VC-dimension has the Erdős–Hajnal property, which gave another proof that any family of finite graphs with bounded VC-minimal complexity has the Erdős–Hajnal property.

In addition to its relation with the Erdős–Hajnal property, the strong Erdős–Hajnal property is of independent interest. On one hand, in model theory, partitioned formulas and hence bipartite graphs are often used (e.g., definition of stable formulas and definition of NIP formulas). On the other hand, in [9, Theorem 1], Erdős, Hajnal, and Pach proved the following fact:

FACT 1.1. *Let H be a fixed graph with k vertices. Any H-free graph with n vertices or its complement has a complete bi-partite subgraph with $\lfloor (n/k)^{1/(k-1)} \rfloor$ vertices in its classes.*

This fact roughly says that given a fixed H, for any H-free graph, there exists in it or in its complement a complete bipartite graph of polynomial size. So it's natural to ask when we can find in a graph or its complement a complete bipartite graph of linear size. A random graph argument in [5] together with results in [6] characterized families of graphs that are defined by omitting a finite set of graphs and satisfy the strong Erdős–Hajnal property: If a family of graphs is defined by omitting a finite set of graphs, then it has the strong Erdős–Hajnal property iff it omits some forest together with its complement. We will show that the family of all forests has bounded VC-minimal complexity and thus, the case of bounded VC-minimal complexity is not covered in [6]. It is also an example where a family of graphs has bounded VC-minimal complexity and is not defined by omitting a finite set of graphs.

We will prove the following main theorem:

THEOREM 1.2. *For $N > 0$, let $k_N = \dfrac{1}{2^{N+4}}$. If a finite bipartite graph $(X, Y; E)$ has VC-minimal complexity $< N$ then there exist $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq k_N|X|$, $|Y'| \geq k_N|Y|$ such that $X' \times Y' \subseteq E$ or $X' \times Y' \cap E = \emptyset$.*

§2. Preliminaries. The following Definitions 2.1, 2.2, 2.5 are based on notions in [1].

DEFINITION 2.1. Given a set U, a family of subsets $\Psi = \{B_i : i \in I\} \subseteq \mathcal{P}(U)$, where I is some index set, is called a *directed family* if for any $B_i, B_j \in \Psi$, $B_i \subseteq B_j$ or $B_j \subseteq B_i$ or $B_i \cap B_j = \emptyset$.

DEFINITION 2.2. Given a directed family Ψ of subsets of U, a set $B \in \Psi$ is a called a Ψ-*ball*. A set $S \subseteq U$ is a Ψ-*Swiss cheese* if $S = B \setminus (B_0 \cup \cdots \cup B_n)$, where each of $B, B_0, \ldots, B_n$ is a Ψ-ball. We will call B an *outer ball* of S, and each B_i is called a *hole* of S.

DEFINITION 2.3. A *graph* G is a pair (V, E) where V is a finite set of vertices and $E \subseteq V \times V$ is a binary symmetric anti-reflexive relation.

DEFINITION 2.4. A *bipartite graph* is a triple $(X, Y; E)$ where X, Y are finite sets and $E \subseteq X \times Y$.

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NOTATION. *Given a bipartite graph* $(X, Y; E)$, $a \in X$, $S \subseteq Y$, *we define* $E(a, S)$ *as the set* $\{b \in S : (a, b) \in E\}$.

DEFINITION 2.5. Given a finite bipartite graph $(X, Y; E)$, we say it has *VC-minimal complexity* $< N$ if there is a directed family Ψ of subsets of Y such that for each $a \in X$, $E(a, Y)$ is a finite disjoint union of Ψ-Swiss cheeses and the number of outer balls + the number of holes $< N$. That is, $E(a, Y) = (B_{11} \setminus (B_{12} \cup \cdots \cup B_{1d(1)})) \dot\cup \cdots \dot\cup (B_{s1} \setminus (B_{s2} \cup \cdots \cup B_{sd(s)}))$ where $d(1) + \cdots + d(s) < N$.

§3. Proof.

THEOREM 3.1. *For* $N > 0$, *let* $k_N = \dfrac{1}{2^{N+4}}$. *If a finite bipartite graph* $(X, Y; E)$ *has VC-minimal complexity* $< N$ *then there exist* $X' \subseteq X$, $Y' \subseteq Y$ *with* $|X'| \geq k_N|X|$, $|Y'| \geq k_N|Y|$ *such that* $X' \times Y' \subseteq E$ *or* $X' \times Y' \cap E = \emptyset$.

PROOF. We prove by induction on N. If $N = 1$ then for all $a \in X$, $E(a, Y) = \emptyset$. So $X \times Y \subseteq \neg E$.

Suppose true for N and we show for $N + 1$.

Let $(X, Y; E)$ be a finite bipartite graph with VC-minimal complexity $< N + 1$. Then there is a directed family Ψ such that for each $a \in X$,

$$E(a, Y) = (B^a_{11} \setminus (B^a_{12} \cup \cdots \cup B^a_{1d(1)}))\dot\cup \cdots \dot\cup(B^a_{s_a 1} \setminus (B^a_{s_a 2} \cup \cdots \cup B^a_{s_a d(s_a)})),$$

where the B^a_{kl}'s are Ψ-balls and $d(1) + \cdots + d(s_a) < N + 1$. Consider the finite family

$$\mathcal{F} := \{B^a_{kl} : a \in X, k, l \in \mathbb{N}\} \cup \{Y\}.$$

Since $\mathcal{F}$ is finite and $|Y| \geq \frac{1}{8}|Y|$, there is a minimal $Z \in \mathcal{F}$ such that $|Z| \geq \frac{1}{8}|Y|$ (minimal with respect to the partial order $\subseteq$). Let

$$\mathcal{F}' := \{B^a_{kl} : a \in X, k, l \in \mathbb{N}, B^a_{kl} \subsetneq Z\}.$$

Let $C_1, \dots, C_m$ be maximal elements in $\mathcal{F}'$. Then $\forall a \in X, \forall k, l \in \mathbb{N}, \forall t \in \{1, \dots, m\}$, if $B^a_{kl} \subsetneq Z$ then $B^a_{kl} \cap C_t = \emptyset$ or $B^a_{kl} \subseteq C_t$. Let $R = Z \setminus (C_1 \cup \cdots \cup C_m)$.

CLAIM 3.2. $\forall a \in X$, $E(a, R) = R$ *or* $E(a, R) = \emptyset$.

PROOF. $E(a, Y) = (B^a_{11} \setminus (B^a_{12} \cup \cdots \cup B^a_{1d(1)})) \dot\cup \dots \dot\cup (B^a_{s_a 1} \setminus (B^a_{s_a 2} \cup \cdots \cup B^a_{s_a d(s_a)}))$. Suppose $E(a, R) \neq \emptyset$. Then for some $k \in \{1, \dots, s_a\}$,

$$(B^a_{k1} \setminus (B^a_{k2} \cup \cdots \cup B^a_{kd(k)})) \cap R \neq \emptyset.$$

May assume $(B^a_{11} \setminus (B^a_{12} \cup \cdots \cup B^a_{1d(1)})) \cap R \neq \emptyset$. So $B^a_{11} \cap Z \neq \emptyset$. Since Z is Y or a Ψ-ball, $B^a_{11} \subsetneq Z$ or $B^a_{11} \supseteq Z$. If $B^a_{11} \subsetneq Z$ then $B^a_{11} \subseteq C_1 \cup \cdots \cup C_m$ and $B^a_{11} \cap R = \emptyset$, a contradiction. Hence $B^a_{11} \supseteq Z$. Similarly, for any hole $K \in \{B^a_{12}, \dots, B^a_{1d(1)}\}$, if $K \cap R \neq \emptyset$, then $K \subsetneq Z$ or $K \supseteq Z$. If $K \subsetneq Z$ then $K \subseteq C_1 \cup \cdots \cup C_m$ and $K \cap R = \emptyset$, a contradiction. If $K \supseteq Z$, then

$$(B^a_{11} \setminus (B^a_{12} \cup \cdots \cup B^a_{1d(1)})) \cap R \subseteq (B^a_{11} \setminus (B^a_{12} \cup \cdots \cup B^a_{1d(1)})) \cap Z = \emptyset,$$

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a contradiction. Hence we must have $Z \subseteq B^a_{11}$ and $K \cap R = \emptyset$ for all $K \in \{B^a_{12}, \dots, B^a_{1d(1)}\}$. So $R \subseteq B^a_{11} \setminus (B^a_{12} \cup \dots \cup B^a_{1d(1)}) \subseteq E(a; Y)$. $\dashv$

Since $R \cup C_1 \cup \dots \cup C_m = Z$ and $|Z| \geq \frac{1}{8}|Y|$, by Claim 3.2, we may assume that $|C_1 \cup \dots \cup C_m| \geq \frac{1}{16}|Y|$.

Let t_0 be smallest such that $|C_1 \cup \dots \cup C_{t_0}| \geq \frac{1}{32}|Y|$. Because $|C_1 \cup \dots \cup C_{t_0-1}| < \frac{1}{32}|Y|$ and $|C_{t_0}| < \frac{1}{8}|Y|$ (by minimality of Z),

$$\frac{1}{32}|Y| \leq |C_1 \cup \dots \cup C_{t_0}| \leq (\frac{1}{32} + \frac{1}{8})|Y|.$$

Let $C := C_1 \cup \dots \cup C_{t_0}$.

Consider

$$A_1 := \{a \in X : \exists k, l \in \mathbb{N}, B^a_{kl} \subseteq C\},$$

$$A_2 := \{a \in X : \forall k, l \in \mathbb{N}, B^a_{kl} \nsubseteq C\}.$$

Since $A_1 \cup A_2 = X$, we have $|A_1| \geq \frac{1}{2}|X|$ or $|A_2| \geq \frac{1}{2}|X|$.

Suppose $|A_1| \geq \frac{1}{2}|X|$. For $a \in A_1$,

$$\begin{aligned} E(a, Y \setminus C) = {} & ((B^a_{11} \setminus B^a_{12} \cup \dots \cup B^a_{1d(1)}) \cap (Y \setminus C))\dot{\cup} \dots \\ & \dot{\cup}((B^a_{s_a 1} \setminus B^a_{s_a 2} \cup \dots \cup B^a_{s_a d(s_a)}) \cap (Y \setminus C)) \\ = {} & (((B^a_{11} \cap (Y \setminus C)) \setminus (B^a_{12} \cap (Y \setminus C)) \cup \dots \cup (B^a_{1d(1)} \cap (Y \setminus C)))\dot{\cup} \dots \\ & \dot{\cup}(B^a_{s_a 1} \cap (Y \setminus C)) \setminus (((B^a_{s_a 2} \cap (Y \setminus C)) \cup \dots \cup (B^a_{s_a d(s_a)} \cap (Y \setminus C)))). \end{aligned}$$

Since $a \in A_1$, for some B^a_{kl}, $B^a_{kl} \subseteq C$. If B^a_{kl} is an outer ball, say $B^a_{kl} = B^a_{11}$, then

$$\begin{aligned} E(a, Y \setminus C) = {} & ((B^a_{21} \cap (Y \setminus C)) \setminus (B^a_{22} \cap (Y \setminus C)) \cup \dots \cup (B^a_{2d(2)} \cap (Y \setminus C)))\dot{\cup} \dots \\ & \dot{\cup}(B^a_{s_a 1} \cap (Y \setminus C)) \setminus (((B^a_{s_a 2} \cap (Y \setminus C)) \cup \dots \cup (B^a_{s_a d(s_a)} \cap (Y \setminus C)))). \end{aligned}$$

If B^a_{kl} is a hole, say $B^a_{kl} = B^a_{12}$, then

$$\begin{aligned} E(a, Y \setminus C) = {} & ((B^a_{11} \cap (Y \setminus C)) \setminus (B^a_{13} \cap (Y \setminus C)) \cup \dots \cup (B^a_{1d(1)} \cap (Y \setminus C)))\dot{\cup} \dots \\ & \dot{\cup}(B^a_{s_a 1} \cap (Y \setminus C)) \setminus (((B^a_{s_a 2} \cap (Y \setminus C)) \cup \dots \cup (B^a_{s_a d(s_a)} \cap (Y \setminus C)))). \end{aligned}$$

Hence $(A_1, Y \setminus C; E)$ is a bipartite graph of VC-minimal complexity $< N$ such that for any $a \in A_1$, $E(a, Y \setminus C)$ is a disjoint union of Ψ'-Swiss cheeses, where

$$\Psi' := \{D \cap (Y \setminus C) : D \in \Psi\}.$$

By inductive hypothesis, there exist $F \subseteq A_1$, $G \subseteq Y \setminus C$ with

$$|F| \geq k_N |A_1| \geq \frac{1}{2} k_N |X|,$$

$$|G| \geq k_N |Y \setminus C| \geq (1 - \frac{1}{32} - \frac{1}{8}) k_N |Y|,$$

such that $F \times G \subseteq E$ or $F \times G \subseteq \neg E$. So the conclusion holds for $N + 1$.

Suppose $|A_2| \geq \frac{1}{2}|X|$. Then $\forall a \in A_2, \forall k, l \in \mathbb{N}, B^a_{kl} \nsubseteq C$.

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Claim 3.3. *$\forall a \in A_2$, $E(a, C) = \emptyset$ or $E(a, C) = C$.*

Proof. We first show that for all $a \in A_2$, for all $k, l \in \mathbb{N}$, if $B^a_{kl} \cap C \neq \emptyset$ then $C \subseteq B^a_{kl}$.

Fix $a \in A_2$ and $k, l \in \mathbb{N}$. If $B^a_{kl} \cap C \neq \emptyset$, then $B^a_{kl} \cap Z \neq \emptyset$. So $B^a_{kl} \subsetneq Z$ or $B^a_{kl} \supseteq Z$. If $B^a_{kl} \subsetneq Z$, then $B^a_{kl} \subseteq C_t$ for some $t \in \{1, \dots, m\}$. But since $B^a_{kl} \nsubseteq C$, $B^a_{kl} \cap C = \emptyset$, a contradiction. Hence we must have $B^a_{kl} \supseteq Z$ when $B^a_{kl} \cap C \neq \emptyset$. Thus $\forall a \in A_2$, $\forall k, l \in \mathbb{N}$, $B^a_{kl} \cap C = \emptyset$ or $B^a_{kl} \supseteq C$.

For $a \in A_2$, if $E(a, C) \neq \emptyset$, may assume $C \cap (B^a_{11} \setminus (B^a_{12} \cup \dots \cup B^a_{1d(1)})) \neq \emptyset$. So $C \cap B^a_{11} \neq \emptyset$ and $C \subseteq B^a_{11}$. For any hole $K \in \{B^a_{12}, \dots, B^a_{1d(1)}\}$, if $C \cap K \neq \emptyset$, then $C \subseteq K$ and $C \cap B^a_{11} \setminus (B^a_{12} \cup \dots \cup B^a_{1d(1)})) = \emptyset$, a contradiction. So for any hole K, $C \cap K = \emptyset$. Thus $C \subseteq (B^a_{11} \setminus (B^a_{12} \cup \dots \cup B^a_{1d(1)}))$. Hence $\forall a \in A_2$, $E(a, C) = \emptyset$ or $E(a, C) = C$. ⊣

So the conclusion holds for $N + 1$ (because $|A_2| \geq \frac{1}{2}|X|$ and $|C| \geq \frac{1}{32}|Y|$). ⊣

§4. Corollary. We can apply Theorem 3.1 to VC-minimal theories (ACVF in particular). The following notions and fact about VC-minimal theories come from [1]. We rephrase them as in [7] for notational convenience.

Definition 4.1 [1, Definition 5][7, Definition 2.1(1)]. A set of formulae $\Psi = \{\psi_i(x, \bar{y_i}) : i \in I\}$ is called a *directed family* if for any $\psi_0(x, \bar{y_0}), \psi_1(x, \bar{y_1}) \in \Psi$ and any parameters $\bar{a}_0, \bar{a}_1$ taken from any model of T, one of the following is true:
(i): $\psi_0(x, \bar{a_0}) \subseteq \psi_1(x, \bar{a_1})$;
(ii): $\psi_1(x, \bar{a_1}) \subseteq \psi_0(x, \bar{a_0})$;
(iii): $\psi_0(x, \bar{a_0}) \cap \psi_1(x, \bar{a_1}) = \emptyset$.

Definition 4.2 [1, Definition 3][7, Definition 2.1(2)]. A theory T is *VC-minimal* if there is a directed family Ψ such that for any formula $\varphi(x, \bar{y})$ and any parameters $\bar{c}$ taken from any model of T, $\varphi(x, \bar{c})$ is equivalent to a finite boolean combination of formulae $\psi_i(x, \bar{b_i})$, where each $\psi_i \in \Psi$.

Fact 4.1 [1, Proposition 7][7, Theorem 2.6]. *Fix T a VC-minimal theory and a directed family of formulae Ψ for T. For every formula $\tau(x, \bar{y})$, there are a finite set $\Psi_0 \subseteq \Psi$ and natural numbers n_1 and n_2 such that for every parameter tuple $\bar{a}$, $\tau(x, \bar{a})$ can be decomposed as the union of at most n_1 disjoint Swiss cheeses, each of them having at most n_2 holes, such that all balls appearing in the decomposition are instances of formulae in Ψ_0.*

By Fact 4.1, for any VC-minimal theory T, any model $\mathcal{M} \models T$ and any definable relation $E(x, \bar{y}) \subseteq M \times M^{|\bar{y}|}$, there is $N \in \mathbb{N}^{>0}$ such that for any finite disjoint $X, Y \subseteq M$, the bipartite graph $(X, Y; E)$ has VC-minimal complexity $< N$. Thus we have the following:

Corollary 4.1.1. *Given a VC-minimal theory T, a model $\mathcal{M} \models T$ and an $\mathcal{L}$-formula $\varphi(x, \bar{y}, \bar{z})$, let $N \in \mathbb{N}^{>0}$ satisfy: for any $\bar{b} \in M^{|\bar{y}|}$, $\bar{c} \in M^{|\bar{z}|}$, $\varphi(x, \bar{b}, \bar{c})$ can be decomposed as the union of at most n_1 disjoint Swiss cheeses, each of them having at most n_2 holes, with $n_1 n_2 < N$. Then for any fixed $\bar{c} \in M^{|\bar{z}|}$, any pair of finite*

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sets $X \subseteq M$, $Y \subseteq M^{|\bar{y}|}$, *there exist* $X' \subseteq X$, $Y' \subseteq Y$ *such that* $|X'| \geq \frac{1}{2^{N+4}}|X|$, $Y' \geq \frac{1}{2^{N+4}}|Y|$, *and* $\forall x \in X'\ \forall \bar{y} \in Y'\ \varphi(x, \bar{y}, \bar{c})$ *or* $\forall x \in X'\ \forall \bar{y} \in Y'\ \neg\varphi(x, \bar{y}, \bar{c})$.

Remark ([4, Example 4.11(2)] shows). Let $\mathcal{M} \models ACVF_{0,0}$ and let a formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ be given. Then there is some $\delta = \delta(\varphi) > 0$ such that for any definable relation $E(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y}, \bar{c})$ for some $\bar{c} \in M^{|\bar{z}|}$ and finite disjoint $X \subseteq M^{|\bar{x}|}$, $Y \subseteq M^{|\bar{y}|}$, there are some $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \delta|X|$, $|Y'| \geq \delta|Y|$ and $X' \times Y' \subseteq E$ or $X' \times Y' \subseteq \neg E$. By [11], $ACVF$ has Swiss Cheese decomposition and thus is a VC-minimal theory. So by Corollary 4.1.1, the same conclusion also holds in $ACVF_{p,q}$ for nonzero p, q when $|\bar{x}| = 1$.

Note: [4, Example 4.11(2)] allows $|\bar{x}| > 1$ and $|\bar{y}| > 1$ for definable relations $E(\bar{x}, \bar{y})$ in $ACVF_{0,0}$. But for positive characteristics, when $|\bar{x}| > 1$, $|\bar{y}| > 1$, there is a counter example given in [4, Proposition 6.2]:

Fact 4.2 [4, Proposition 6.2]. *Let* $p > 0$, *and let* $\mathbb{F} = \mathbb{F}_p^{alg}$. *For a set of points* $P \subseteq \mathbb{F}^2$ *and a set of lines* L *in* $\mathbb{F}^2$ *we denote by* $I(P, L) \subseteq P \times L$ *the incidence relation, i.e.,* $I(P, L) = \{(p, l) \in P \times L : p \in l\}$. *Then for all constants* $\delta > 0$, *there exist a finite (sufficiently large) set of points* $P \subseteq \mathbb{F}^2$ *and a finite (sufficiently large) set of lines* L *in* $\mathbb{F}^2$ *such that for all* $P_0 \subseteq P$ *and* $L_0 \subseteq L$ *with* $|P_0| \geq \delta|P|$, $|L_0| \geq \delta|L|$, $I(P_0, L_0) \neq \emptyset$.

Thus the strong Erdős–Hajnal property fails for I *in* $\mathbb{F}$ *and we cannot generalize Corollary* 4.1.1 *to the case where both* $|\bar{x}| > 1$ *and* $|\bar{y}| > 1$.

Remark. In [6], Chudnovsky, Scott, Seymour, and Spirkl proved the following fact:

Fact 4.3 [6, 1.2]. *For every forest* H, *there exists* $\varepsilon > 0$ *such that for every graph* G *with* $|G| > 1$ *that is both* H*-free and* $\overline{H}$*-free, there is a pair of disjoint subsets* (A, B) *with* $|A|, |B| \geq \varepsilon|G|$ *such that* $A \times B \subseteq E$ *or* $A \times B \cap E = \emptyset$.

The family of forests can be shown to have VC-minimal complexity ≤ 2 and thus the VC-minimal case is not covered in [6].

For a forest H, $v \in V(H)$, let $B_{v,\lhd}$ denote the set of the predecessor of v and $B_{v,\rhd}$ denote the set of successors of v. Consider the family $\mathcal{F}_H := \{B_{v,\lhd}, B_{v,\rhd} : v \in H\}$. $\mathcal{F}_H$ is directed: Let $v, w \in V(H)$. If $B_{v,\lhd} \cap B_{w,\rhd} \neq \emptyset$, then since $B_{v,\lhd}$ is a singleton, $B_{v,\lhd} \subseteq B_{w,\rhd}$. Similarly, if $B_{v,\lhd} \cap B_{w,\lhd} \neq \emptyset$, then $B_{v,\lhd} \subseteq B_{w,\lhd}$. If $B_{v,\rhd} \cap B_{w,\rhd} \neq \emptyset$, then $v = w$ and $B_{v,\rhd} = B_{w,\rhd}$.

For any $v \in V(H)$, $E_v = B_{v,\lhd} \sqcup B_{v,\rhd}$. So given any forest $H = (V(H), E)$ and disjoint $X, Y \subseteq V(H)$, the bipartite graph $(X, Y; E)$ has VC-minimal complexity ≤ 2.

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