ON INTEGER MATRICES AND INCIDENCE MATRICES OF CERTAIN COMBINATORIAL CONFIGURATIONS, II: RECTANGULAR MATRICES

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Introduction. In this paper we establish a connection between rectangular integer matrices and incidence matrices of resolvable balanced incomplete block designs. The definition of these terms has been given in paper I of this series.

Our theorem can be stated as follows:

THEOREM 2. *Let A be a $v \times b$ matrix with integer elements such that*

$$A'A = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1r} \\ B_{21} & B_{22} & \dots & B_{2r} \\ \dots & \dots & \dots & \dots \\ B_{r1} & B_{r2} & \dots & B_{rr} \end{bmatrix}. \tag{2.1}$$

Here the B_{ij} are $n_1 \times n_j$ matrices $(i, j = 1, 2, \dots r)$ $\sum_{i=1}^{r} n_i = b$ and

(i) *trace of $B_{ii} \leqslant v$ $(i = 1, 2, \dots, r)$,*

(ii) *sum of the elements of $B_{ij} = \tilde{v}$ $(i, j = 1, 2, \dots, r)$,*

(iii) *the square of the length of any row vector of A is odd and the scalar product of any two row vectors of A is $\lambda \neq 0$,*

(iv) *$r(vr - b) \geqslant b\lambda(v - 1)$.*

Then A or $-A$ is the incidence matrix for a resolvable b.i.b. design.

Proof. Let $A = (a_{ij}) \qquad (i = 1, 2, \dots, v;\ j = 1, 2, \dots, b)$,

$$N_i = n_1 + n_2 + \dots + n_i \qquad (i = 1, 2, \dots, r),$$

$N_0 = 0$. The submatrix of A consisting of its $N_{i-1} + 1, N_{i-1} + 2, \dots, N_i$th column is denoted by A_i $(i = 1, 2, \dots, r)$. Let $s_{\nu i}$ be the sum of the elements of the νth row of A_i $(i = 1, 2, \dots, r;\ \nu = 1, 2, \dots, v)$.

As $B_{ij} = A_i' A_j$ $(i, j) = 1, 2, \dots, r)$, we have

$$[1 \quad 1 \quad \dots \quad 1]\, B_{ij} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [1 \quad 1 \quad \dots \quad 1]\, A_i A_j \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{2.2}$$

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and so by (ii) of the hypothesis

$$\sum_{\nu=1}^{v} s_{\nu i} s_{\nu j} = \tilde{v} \qquad (i, j = 1, 2, \ldots, r). \tag{2.3}$$

Thus

$$\sum_{\nu=1}^{v} (s_{\nu i} - s_{\nu j})^2 = \tilde{v} + \tilde{v} - 2\tilde{v} = 0 \tag{2.4}$$

and hence

$$s_{\nu i} = s_{\nu}, \text{ say} \qquad (i = 1, 2, \ldots, r; \nu = 1, 2, \ldots, v). \tag{2.5}$$

Also by (i) of the hypothesis,

$$\sum_{j=N_{i-1}+1}^{N_i} a_{ij}^2 + \sum_{j=N_{i-1}+1}^{N_i} a_{2j}^2 + \ldots + \sum_{j=N_{i-1}+1}^{N_i} a_{vj}^2 \leqslant v \qquad (i = 1, 2, \ldots, r). \tag{2.6}$$

None of the v sums on the left can be zero. For, if, say, the first sum vanishes, then

$$0 = \sum_{j=N_{i-1}+1}^{N_i} a_{1j}^2 = \sum_{j=N_{i-1}+1}^{N_i} a_{1j} = s_1.$$

Now by (1.5) the sum of the elements of the first row of A is zero. But

$$0 = \sum_{i=1}^{b} a_{1i} = \sum_{i=1}^{b} a_{1i}^2 \equiv 1 \pmod 2,$$

by (iii) of the hypothesis, and this is a contradiction. Hence (2.6) holds with the equality sign, and thus each of the v sums on the left equals 1. Hence there is precisely one non-zero element in each row of A_i $(i = 1, 2, \ldots, r)$. This non-zero element can be ± 1. If the element is 1 [is -1], then, by (2.5), all the non-zero elements in that row of A are equal to 1 [to -1]. Suppose now that A contains both 1 and -1. Consider three rows of A, two of these having their non-zero elements with the same sign and the other having its non-zero elements with the opposite sign. Since the scalar product of any two of these is λ by (iii) of the hypothesis and $\lambda \neq 0$, we arrive at a contradiction. Hence all the non-zero elements of A have the same sign. Thus either A or $-A$ have all their non-zero elements equal to 1. We may assume the former. Then the row sums of A are clearly equal to r. Thus AA' is a matrix with r along its main diagonal, by (iii) of the hypothesis, and λ elsewhere. If k_i denotes the sum of the elements of the ith column of A $(i = 1, 2, \ldots, b)$, then

$$\sum_{i=1}^{b} k_i^2 = (r + \lambda(v - 1))v \tag{2.7}$$

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and

$$\sum_{i=1}^{b} k_i = rv. \tag{2.8}$$

Hence

$$\sum_{i=1}^{b} (k_i - k)^2 = (r + \lambda(v - 1))v - krv = (b\lambda(v - 1) - r(vr - b))vb^{-1}, \tag{2.9}$$

where

$$k = rvb^{-1}. \tag{2.10}$$

Using (iv) of the hypothesis, we infer from (2.9) that all the column sums of A are equal to k. It now follows that $v/k = n_i = n$, say $(i = 1, 2, \ldots, r)$. Hence the 0–1 matrix A is an incidence matrix of a resolvable b.i.b. design. This completes the proof.

Reference

1. R. C. Bose, *A note on the resolvability of balanced incomplete block designs*, Sankhyā, *6* (1942), 105–110.

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