BULL. AUSTRAL. MATH. SOC. 26A18
VOL. 47 (1993) [41–53]

PERIODS FOR TRIANGULAR MAPS

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We study the sets of periods of triangular maps on a cartesian product of arbitrary spaces. As a consequence we extend Kloeden's Theorem (in a 1979 paper) to a class of triangular maps on cartesian products of intervals and circles. We also show that, in some sense, this is the more general situation in which the Sharkovskiĭ ordering gives the periodic structure of triangular maps.

1. INTRODUCTION

In what follows X will denote the following cartesian product of sets: $\prod_{i=1}^{n} X_i$. A map $f : X \to X$ will be called *triangular* if its i-th component function f^i only depends on the first i variables for $i = 1, 2, \ldots, n$; that is, $f^i(x_1, x_2, \ldots, x_n) = f^i(x_1, x_2, \ldots, x_i)$ for all $(x_1, x_2, \ldots, x_n) \in X$ and all $i = 1, 2, \ldots, n$.

A *fixed point* of a map $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. We say that $x \in X$ is an *m-periodic point* of f if x is a fixed point of f^m but is not a fixed point of f^k for any $1 \leqslant k < m$. The set $\{x, f(x), \ldots, f^{m-1}(x)\}$ will be called an *m-periodic orbit* of f. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f.

This paper deals with the problem of determining the possible sets of periods of triangular maps. By using ideas of Kloeden (see [11]) we shall show that, for these maps, each periodic orbit can be decomposed into a "product" of periodic orbits (see Proposition 2.2). From this fact we shall obtain a characterisation of the possible sets of periods of triangular maps (see Corollary 2.3). It turns out that this characterisation is rather complicated and difficult to use. However, if we restrict our attention to a particular class of triangular maps, a much nicer characterisation of the set of periods can be obtained. To be more precise we have to introduce some notation.

Assume that X_i is a closed interval I of the real line or a circle S^1 for each $i = 1, 2, \ldots, n$. Clearly, if all the X_i are intervals, then X is an n-dimensional rectangle and if all the X_i are circles then it is the n-dimensional torus. Otherwise, X is an n-dimensional generalised cylinder. In what follows, when we do not need to distinguish

Received 4th December 1991.
The authors have been partially supported by the DGICYT grant number PB90-0695.

between these three cases we shall simply say that X is a *cylinder*. We endow X with the product topology. Then, a continuous triangular self-map of a cylinder will be called a *cylinder map*. In the case when a cylinder X has (topological) dimension one then instead of talking about cylinder maps we shall talk about *interval maps* if X is an interval or *circle maps* if it is the circle. If a cylinder X has dimension two then we shall talk about *rectangle maps* if X is a rectangle, *torus maps* if it is a torus and *annulus maps* in the remaining cases.

The structure of the sets of periods of triangular maps on an interval or the circle (that is, of interval and circle maps) is well-known. We shall describe them for completeness.

To describe the structure of the set of periods for interval maps we introduce the *Sharkovskiĭ ordering* ${}_s>$ on the set $\mathbb{N}_s = \mathbb{N} \cup \{2^\infty\}$ as follows (we have to include the symbol 2^∞ in order to ensure the existence of supremum of every subset with respect to the ordering ${}_s>$):

$$\begin{gathered} 3 \ {}_s> 5 \ {}_s> 7 \ {}_s> \dots \ {}_s> 2\cdot 3 \ {}_s> 2\cdot 5 \ {}_s> 2\cdot 7 \ {}_s> \dots \ {}_s> 4\cdot 3 \ {}_s> 4\cdot 5 \ {}_s> 4\cdot 7 \ {}_s> \\ \dots \ {}_s> \dots \ {}_s> 2^n\cdot 3 \ {}_s> 2^n\cdot 5 \ {}_s> 2^n\cdot 7 \ {}_s> \dots \ {}_s> 2^\infty \ {}_s> \\ \dots \ {}_s> 2^n \ {}_s> \dots \ {}_s> 16 \ {}_s> 8 \ {}_s> 4 \ {}_s> 2 \ {}_s> 1. \end{gathered}$$

We shall also use the symbol ${}_s\geqslant$ in the natural way. For $s \in \mathbb{N}_s$ we denote by $S(s)$ the set $\{k \in \mathbb{N} : s \ {}_s\geqslant k\}$. Now we can state the Sharkovskiĭ Theorem [**14**].

Theorem 1.1. *For every interval map f there exists $s \in \mathbb{N}_s$ such that* $\operatorname{Per}(f) = S(s)$*. Conversely, for every $s \in \mathbb{N}_s$ there exists an interval map f such that* $\operatorname{Per}(f) = S(s)$.

Now we shall describe the structure of the set of periods of a circle map having fixed points (see [**10, 8, 7, 2, 4**]). To this end we shall denote by $\mathbb{N}_\infty$ the set $\mathbb{N} \cup \{\infty\}$. Then for each $b \in \mathbb{N}$ we set $B(b) = \{k \in \mathbb{N} : b \leqslant k\}$. Additionally we set $B(\infty) = \emptyset$.

Theorem 1.2. *Let f be a circle map of degree d. Then we have*

(a) *If $d \notin \{-2, -1, 0, 1\}$ then* $\operatorname{Per}(f) = \mathbb{N}$.
(b) *If $d = -2$ then* $\operatorname{Per}(f)$ *is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$, and both possibilities can occur.*
(c) *Suppose that $d \in \{-1, 0\}$. Then there exists $s \in \mathbb{N}_s$ such that* $\operatorname{Per}(f) = S(s)$*. Conversely, for every $s \in \mathbb{N}_s$ there exists a circle map g with degree either -1 or 0 such that* $\operatorname{Per}(g) = S(s)$.
(d) *Suppose $d = 1$ and $1 \in \operatorname{Per}(f)$. Then there exists $s \in \mathbb{N}_s$ and $b \in \mathbb{N}_\infty$ such that* $\operatorname{Per}(f) = S(s) \cup B(b)$*. Conversely, for every $s \in \mathbb{N}_s$ and $b \in \mathbb{N}_\infty$ there exists a circle map g with degree 1 such that* $\operatorname{Per}(g) = S(s) \cup B(b)$.

The complete description of the sets of periods of circle maps of degree one was given by Misiurewicz by using the notion of rotation interval (see [**13**]).

After Theorem 1.2 we see that the degree of a circle map plays a crucial role in the description of its set of periods. For cylinder maps of dimension greater than one this role will be played by the degree vector which we define as follows.

Let $f = (f^1, f^2, \dots, f^n)$ be a cylinder map on $\prod_{i=1}^{n} X_i$. If X_i is an interval then we set $d_i = 0$. Suppose now that X_i is a circle. Since f (and consequently $f^i(x_1, x_2 \dots, x_{i-1}, \cdot)$) is continuous, the degree d_i of $f^i(x_1, x_2 \dots, x_{i-1}, \cdot)$ depends continuously on $(x_1, x_2 \dots, x_{i-1})$. But, since the degree of a circle map is an integer, then d_i is independent on $(x_1, x_2 \dots, x_{i-1})$. Then we can define the *degree vector* of f as $d = (d_1, d_2, \dots, d_n) \in \mathbf{Z}^n$. In the particular case of a triangular map f on the n-dimensional torus $\mathbf{T}^n$ it is not difficult to see that if $(d_1, d_2, \dots, d_n)$ is the degree vector of f, then the homomorphism f_{*1} induced by f on the first homological group of $\mathbf{T}^n$ is given by the following $n \times n$ upper triangular matrix of integers:

$$\begin{pmatrix} d_1 & & * \\ & \ddots & \\ & 0 & d_n \end{pmatrix}.$$

We shall also be interested in the following property of triangular maps. A triangular map $f = (f^1, f^2, \dots, f^n)$ will be called *closed* if for each $i = 1, 2, \dots, n$ and for each $(x_1, x_2, \dots, x_{i-1}) \in \prod_{j=1}^{i-1} X_j$ the map $f^i(x_1, x_2 \dots, x_{i-1}, \cdot)$ belongs to a class which is closed under composition and such that each map from this class has a fixed point.

It is well-known that any interval map and any circle map of degree different from one has fixed points. On the other hand, the class of circle maps with degree different from -1 and 1 is closed under composition. Therefore, each cylinder map whose degree vector has all components different from ± 1 (with the exception of the first one which can also be -1) is an example of a closed triangular map.

Later on (in Section 3; Lemma 3.2) we shall see that a closed triangular map always has fixed points. However, it is not difficult to see that the converse is not true.

For closed cylinder maps we get the following nice characterisation of the sets of periods.

Theorem 1.3. *Let f be a closed cylinder map. Then the following hold.*

(a) *If some component of the degree vector is different from -2, -1, 0 and 1 then* $\operatorname{Per}(f) = \mathbb{N}$.

(b) *If some component of the degree vector is -2 then* $\operatorname{Per}(f)$ *is either* $\mathbb{N}$ *or* $\mathbb{N} \setminus \{2\}$*, and both possibilities can occur.*

(c) *If the degree vector of f is either $(0,0,\dots,0)$ or $(-1,0,\dots,0)$ then there exists $s \in \mathbb{N}_s$ such that* $\mathrm{Per}(f) = S(s)$. *Conversely, for every $s \in \mathbb{N}_s$ there exists a closed cylinder map g with degree vector either $(0,0,\dots,0)$ or $(-1,0,\dots,0)$ such that* $\mathrm{Per}(g) = S(s)$.

Statement (c) of Theorem 1.3 in the particular case in which X is an n-dimensional rectangle was proved by Kloeden (see [**11**]).

Theorem 1.3 will be proved in Section 3.

After Theorem 1.3 it is not difficult to characterise the set of periods of cylinder maps (without being closed) but, in this case, the characterisation we get is not so nice. As an example, in Section 3, we provide this characterisation in the particular case of the 2-dimensional cylinders (see Theorem 3.3).

It is important to note that the fact that for cylinder maps we can characterise the set of periods depends crucially on the fact that the structure of the sets of periods for interval and circle maps is well-known and has an easy description. Therefore, whenever the structure of the set of periods is known for some class of maps, we can extend the results of this paper to appropriate classes of triangular maps on the corresponding spaces. Thus, the results from [**3, 6, 9, 1, 12**], for instance, can be extended to appropriate classes of triangular maps where the spaces X_i can be the Y, the m-od, the m-sphere, the m-torus, and the Klein bottle.

2. Triangular maps

In this section we shall study the set of periods of general triangular maps. We shall start by studying the particular case of triangular maps on the cartesian products of two sets.

First of all we have to fix the notation. Let $f = (g,h)$ be a triangular self-map from $Y \times Z$ and let $P = \{y_0, y_1, \dots, y_{r-1}\}$ be an r-periodic orbit of g such that $g(y_j) = y_{j+1}$ for $j = 0,1,\dots,r-2$ and $g(y_{r-1}) = y_0$. Then we define $h_P : Z \to Z$ by

$$h(y_{r-1}, h(y_{r-2}, \dots, h(y_1, h(y_0, \cdot))\dots)).$$

If, in addition, $Q = \{z_0, z_1, \dots, z_{s-1}\}$ is an s-periodic orbit of h_P such that $h_P(z_j) = z_{j+1}$ for $j = 0,1,\dots,s-2$ and $h_P(z_{s-1}) = z_0$ then we define the *product of P by Q*, denoted by $P \cdot Q$, as follows. First we define a sequence of rs points in Z by setting

$$t_{ir+j} = \begin{cases} z_i & \text{if } j = 0, \\ h(y_{j-1}, t_{ir+j-1}) & \text{for } j = 1,2,\dots,r-1, \end{cases}$$

for $i = 0,1,\dots,s-1$. Now we define $P \cdot Q = \{(y_j, t_{ir+j}) : j = 0,1,\dots,r-1 \text{ and } i = 0,1,\dots,s-1\}$. Notice that $P \cdot Q \subset Y \times Z$ and it has cardinality rs.

https://doi.org/10.1017/S0004972700012247 Published online by Cambridge University Press

The following result is the basis of our study. Its proof is based on the ideas of Kloeden (see [**11**]).

Lemma 2.1. *Let $f=(g,h): Y\times Z \to Y\times Z$ be a triangular map. Then the following hold.*

(a) *If g has a periodic orbit P and h_P has a periodic orbit Q, then $P\cdot Q$ is a periodic orbit of f.*

(b) *Conversely, each periodic orbit of f can be obtained as a product of a periodic orbit P of g by a periodic orbit of h_P.*

Proof: Suppose that P and Q are as in the hypotheses of statement (a). We shall use the notation from the definition of $P\cdot Q$. Then, from the definition of the points t_j we know the image by h of all points of $P\cdot Q$ except for the points (y_{r-1}, t_{ir+r-1}) for $i=0,1,\dots,s-1$. Now we compute the images of these points. For $i=0,1,\dots,s-2$ we have

$$\begin{aligned} h(y_{r-1}, t_{ir+r-1}) &= h(y_{r-1}, h(y_{r-2}, t_{ir+r-2})) \\ &\dots \\ &= h(y_{r-1}, h(y_{r-2},\dots, h(y_1, h(y_0, t_{ir}))\dots)) \\ &= h_P(t_{ir}) = h_P(z_i) = z_{i+1} = t_{(i+1)r}. \end{aligned}$$

In a similar way we obtain that $h\big(y_{r-1}, t_{(s-1)r+r-1}\big) = t_0$.

The last step in the proof of (a) is to compute the image by f of the points of $P\cdot Q$. We have

$$\begin{aligned} f(y_j, t_{ir+j}) &= (y_{j+1}, t_{ir+j+1}) \quad \text{for } j=0,1,\dots,r-2 \text{ and } i=0,1,\dots,s-1; \\ f(y_{r-1}, t_{ir+r-1}) &= \big(y_0, t_{(i+1)r}\big) \quad \text{for } i=0,1,\dots,s-2; \\ f\big(y_{r-1}, t_{(s-1)r+r-1}\big) &= (y_0, t_0). \end{aligned}$$

Therefore, $P\cdot Q$ is an rs-periodic orbit of f.

Now we prove (b). Let $R=\{(y_0,z_0),(y_1,z_1),\dots,(y_{m-1},z_{m-1})\}$ be an m-periodic orbit of f such that $f((y_k,z_k)) = (y_{k+1}, z_{k+1})$ for $k=0,1,\dots,m-2$ and $f((y_{m-1},z_{m-1})) = (y_0,z_0)$. Since $g(y,z)=g(y)$, y_0 is an r-periodic point of g with r a divisor of m. Then $P=\{y_0,y_1,\dots,y_{r-1}\}$ is an r-periodic orbit of g such that $g(y_j)=y_{j+1}$ for $j=0,1,\dots,r-2$ and $g(y_{r-1})=y_0$. We note that $y_{ir+j} = g(y_{ir+j-1}) = g^2(y_{ir+j-2}) = \dots = g^{ir}(y_j) = y_j$ for all $j=0,1,\dots,r-1$ and $i=0,1,\dots,s-1$ where $s=m/r$. Therefore, since R is a periodic orbit of f we get $z_{ir+j} = h(y_{ir+j-1}, z_{ir+j-1}) = h(y_{j-1}, z_{ir+j-1})$ for all $j=0,1,\dots,r-1$ and $i=0,1,\dots,s-1$. Then, to prove (b), it only remains to show that $Q=\{z_{ir}: i=0,1,\dots,s-1\}$ is an s-periodic orbit of h_P.

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For $i = 0, 1, \dots, s-2$ we have

$$\begin{aligned} z_{(i+1)r} &= h(y_{ir+r-1}, z_{ir+r-1}) = h(y_{r-1}, z_{ir+r-1}) \\ &= h(y_{r-1}, h(y_{ir+r-2}, z_{ir+r-2})) = h(y_{r-1}, h(y_{r-2}, z_{ir+r-2})) \\ &\quad \dots \\ &= h(y_{r-1}, h(y_{r-2}, \dots, h(y_1, h(y_0, z_{ir}))\dots)) \\ &= h_P(z_{ir}). \end{aligned}$$

Moreover, in a similar way we obtain

$$z_0 = h(y_{m-1}, z_{m-1}) = h\bigl(y_{(s-1)r+r-1}, z_{(s-1)r+r-1}\bigr) = h_P\bigl(z_{(s-1)r}\bigr).$$

Hence Q is a t-periodic orbit h_P with t dividing s. Suppose that $t < s$. Then, in view of (a), $P \cdot Q$ would be an rt-periodic orbit of f contained in a periodic orbit of f of larger period; a contradiction. Therefore Q has period s. ∎

Now we are going to consider the general case. The following result is the natural extension of Lemma 2.1 to this setting.

Proposition 2.2. *Let $f = (f^1, f^2, \dots, f^n)$ be a triangular map. Then the following hold.*

(a) *If f^1 has a periodic orbit P_1 and $f^i_{P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_{i-1}} = f^i_{(\ldots((P_1 \cdot P_2) \cdot P_3) \cdot \ldots \cdot) P_{i-1}}$ has a periodic orbit P_i for $i = 2, 3, \dots, n$, then $P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_n$ is a periodic orbit of f.*

(b) *Conversely, each periodic orbit of f can be obtained as a product $P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_n$ of a periodic orbit P_1 of f^1 by periodic orbits P_i of $f^i_{P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_{i-1}}$ for $i = 2, 3, \dots, n$.*

Proof: We shall only prove statement (a). Statement (b) follows similarly. The proof is by induction on n.

If $n = 2$ then, by Lemma 2.1(a), we get that $P_1 \cdot P_2$ is a periodic orbit of the triangular map $(f^1, f^2) : X_1 \times X_2 \to X_1 \times X_2$. Now assume that $P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_{n-1}$ is a periodic orbit of $(f^1, f^2, \dots, f^{n-1}) : \prod_{j=1}^{n-1} X_j \to \prod_{j=1}^{n-1} X_j$. Again by Lemma 2.1(a) we get that $P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_n$ is a periodic orbit of $(f^1, f^2, \dots, f^n) : \prod_{j=1}^{n} X_j \to \prod_{j=1}^{n} X_j$ (here we take $g = (f^1, f^2, \dots, f^{n-1})$ and $h = f^n$). ∎

The following corollary follows immediately from Proposition 2.2 and gives the characterisation of the set of periods for triangular maps. Prior to stating it we shall introduce new notation. Let $g : Y \to Y$ be a map. We shall denote the set of all periodic orbits of g by $\mathrm{Orb}(g)$. Also, if P is a periodic orbit of g, then the period of P will be denoted by $|P|$. Let $A \subset \mathbb{N}$ and $r \in \mathbb{N}$. We shall denote by rA the set $\{ra : a \in A\}$.

Corollary 2.3. *Let $f = (f^1, f^2, \dots, f^n)$ be a triangular map. Then*

$$\operatorname{Per}(f) = \cup_{P_1 \in \operatorname{Orb}(f^1),\ P_2 \in \operatorname{Orb}\left(f^2_{P_1}\right), \dots,\ P_n \in \operatorname{Orb}\left(f^n_{P_1 \cdot P_2 \cdot \dots \cdot P_{n-1}}\right)} \prod_{i=1}^{n} |P_i|.$$

We shall end this section with a weaker but simpler version of the previous corollary for closed triangular maps.

Corollary 2.4. *Let $f = (f^1, f^2, \dots, f^n)$ be a closed triangular map and let $P_1 \cdot P_2 \cdot \dots \cdot P_n$ be a periodic orbit of f where $P_1 \in \operatorname{Orb}(f^1)$ and $P_i \in \operatorname{Orb}\left(f^i_{P_1 \cdot P_2 \cdot \dots \cdot P_{i-1}}\right)$ for $i = 2, 3, \dots, n$. Then*

$$\operatorname{Per}(f) \supset \operatorname{Per}(f^1) \cup \left[\bigcup_{i=2}^{n} \left(\prod_{j=1}^{i-1} |P_j|\right) \operatorname{Per}\left(f^i_{P_1 \cdot P_2 \cdot \dots \cdot P_{i-1}}\right)\right].$$

Proof: The proof is by induction on n. If $n = 1$ then the statement is trivially true. Now assume that the statement holds for $n-1$ and prove it for n. By Proposition 2.2(a), the triangular map $g = (f^1, f^2, \dots, f^{n-1})$ has $P_1 \cdot P_2 \cdot \dots \cdot P_{n-1}$ as a periodic orbit. Therefore, by the induction hypotheses we get

$$\operatorname{Per}(g) \supset \operatorname{Per}(f^1) \cup \left[\bigcup_{i=2}^{n-1} \left(\prod_{j=1}^{i-1} |P_j|\right) \operatorname{Per}\left(f^i_{P_1 \cdot P_2 \cdot \dots \cdot P_{i-1}}\right)\right].$$

If Q is a periodic orbit of g then the map f^n_Q is a composition of maps of the form $f^n(\cdot,\cdot) : X_n \to X_n$ with the first argument in $\prod_{i=1}^{n-1} X_i$. Therefore, since f is closed, the map f^n_Q has a fixed point. Hence, in view of Corollary 2.3, we get

$$\operatorname{Per}(f) = \operatorname{Per}((g, f^n)) = \cup_{Q \in \operatorname{Orb}(g),\ R \in \operatorname{Orb}\left(f^n_Q\right)} |Q| \cdot |R| \supset \cup_{Q \in \operatorname{Orb}(g)} |Q| \cdot 1 = \operatorname{Per}(g).$$

On the other hand, since f is triangular so is the map $(g, f^n) : \left(\prod_{i=1}^{n-1} X_i\right) \times X_n \to \left(\prod_{i=1}^{n-1} X_i\right) \times X_n$. Thus, again by Corollary 2.3, we get

$$\begin{aligned}
\operatorname{Per}(f) &= \cup_{Q \in \operatorname{Orb}(g),\ R \in \operatorname{Orb}\left(f^n_Q\right)} |Q| \cdot |R| \\
&\supset \cup_{R \in \operatorname{Orb}\left(f^n_{P_1 \cdot P_2 \cdot \dots \cdot P_{n-1}}\right)} |P_1 \cdot P_2 \cdot \dots \cdot P_{n-1}| \cdot |R| \\
&= |P_1| \cdot |P_2| \cdot \dots \cdot |P_{n-1}| \cdot \operatorname{Per}\left(f^n_{P_1 \cdot P_2 \cdot \dots \cdot P_{n-1}}\right).
\end{aligned}$$

This ends the proof of the corollary. ∎

3. Closed cylinder maps

The goal of this section is to prove Theorem 1.3. Prior to doing this we shall prove some preliminary results. The first one states a number-theoretical property of the Sharkovskiĭ ordering.

Lemma 3.1. *For each $a, b \in \mathbb{N}$ we have $S(ab) \subset S(a) \cup aS(b)$.*

Proof: Let $a = 2^l t$ with $l \geqslant 0$ and $t \geqslant 1$ odd. If $t > 1$ then ab is of the form $2^{l+r}s$ with s odd, $s \geqslant t$ and $r \geqslant 0$. So $a \,{}_s\!\geqslant ab$ and, consequently, $S(ab) \subset S(a) \subset S(a) \cup aS(b)$. Now, consider the case $t = 1$; that is, $a = 2^l$ with $l \geqslant 0$.

If $b = 2^r$ with $r \geqslant 0$ then

$$\begin{aligned} S(a) \cup aS(b) &= \{1, 2, 2^2, \dots, 2^l\} \cup 2^l\{1, 2, 2^2, \dots, 2^r\} \\ &= \{1, 2, 2^2, \dots, 2^{l+r}\} = S(ab). \end{aligned}$$

If $b = 2^r s$ with $r \geqslant 0$ and $s > 1$ odd, then

$$S(b) = \{2^i : i \geqslant 0\} \cup \{2^i k : i > r \text{ and } k > 1 \text{ odd}\} \cup \{2^r k : k \geqslant s \text{ odd}\}.$$

Therefore,

$$aS(b) = \{2^i : i \geqslant l\} \cup \{2^i k : i > r + l \text{ and } k > 1 \text{ odd}\} \cup \{2^{r+l} k : k \geqslant s \text{ odd}\}.$$

Thus,

$$\begin{aligned} S(a) \cup aS(b) &= \{2^i : i \geqslant 0\} \cup \{2^i k : i > r + l \text{ and } k > 1 \text{ odd}\} \cup \{2^{r+l} k : k \geqslant s \text{ odd}\} \\ &= S(ab). \end{aligned}$$

∎

Lemma 3.2. *Let f be a closed triangular map. Then f has a fixed point.*

Proof: Let $f = (f^1, f^2, \dots, f^n)$. Since f is closed then f^1 has a fixed point. Let P_1 be the periodic orbit consisting of this fixed point. Then, again since f is closed, $f^2_{P_1}$ has a fixed point. Let P_2 be the periodic orbit consisting of this fixed point. By iterating this process we get a sequence P_i of periodic orbits which consist of a fixed point of the map $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ for $i = 2, 3, \dots, n$. In view of Proposition 2.2(a), f has a fixed point. ∎

Proof of Theorem 1.3: Let $f = (f^1, f^2, \dots, f^n)$ and let $d = (d_1, d_2, \dots, d_n)$ be the degree vector of f. By Lemma 3.2 and Proposition 2.2(b) the map f has a periodic orbit $P_1 \cdot P_2 \cdot \ldots \cdot P_n$ consisting of a fixed point such that P_1 is a periodic orbit of f^1 and P_i is a periodic orbit of $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ for $i = 2, 3, \dots, n$, and all these orbits consist of a fixed point of the corresponding map.

Suppose there exists $d_i \notin \{-2,-1,0,1\}$ for some $i \in \{1,2,\dots,n\}$. If $i=1$ then, by Theorem 1.2(a) and Corollary 2.4, we obtain $\mathbb{N} = \operatorname{Per}(f^1) \subset \operatorname{Per}(f)$. Now assume that $i \geqslant 2$. Since the class of circle maps with degree different from -2, -1, 0 and 1 is closed under composition and the map $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ is a composition of such maps we get that $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ has degree different from -2, -1, 0 and 1. Thus, since $|P_1| \cdot |P_2| \cdot \ldots \cdot |P_{i-1}| = 1$, again by Theorem 1.2(a) and Corollary 2.4, we get that $\mathbb{N} = \operatorname{Per}\left(f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}\right) \subset \operatorname{Per}(f)$. This ends the proof of (a).

In a similar way we prove that if some component of the degree vector is -2 then $\operatorname{Per}(f)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$. To end the proof of (b) we still have to show that both possibilities can occur. We shall show this by constructing examples of both situations. In view of Theorem 1.2(b) there exist circle maps g^1 and h^1 of degree -2 such that $\operatorname{Per}(g^1) = \mathbb{N}$ and $\operatorname{Per}(h^1) = \mathbb{N} \setminus \{2\}$. Now we construct closed cylinder maps $g = (g^1, g^2, \dots, g^n)$ and $h = (h^1, h^2, \dots, h^n)$ on $\mathbf{S}^1 \times I \times \ldots \times I$ by setting $g^i(z_1, z_2, \dots, z_n) = h^i(z_1, z_2, \dots, z_n) = (z_1, z_2, \dots, z_i)$ for all $(z_1, z_2, \dots, z_n) \in \mathbf{S}^1 \times I \times \ldots \times I$ and $i = 2, 3, \dots, n$. Clearly, in view of Corollary 2.3, we get that $\operatorname{Per}(g) = \mathbb{N}$ and $\operatorname{Per}(h) = \mathbb{N} \setminus \{2\}$. Hence (b) is proved.

Now we prove (c). To prove the first statement of (c) we only have to show that if $m \in \operatorname{Per}(f)$ then $S(m) \subset \operatorname{Per}(f)$. By Proposition 2.2(b) the map f has an m-periodic orbit $P_1 \cdot P_2 \cdot \ldots \cdot P_n$ such that P_1 is a periodic orbit of f^1 and P_i is a periodic orbit of $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ for $i = 2, 3, \dots, n$. Moreover, $|P_1| \cdot |P_2| \cdot \ldots \cdot |P_n| = m$. By using again the same argument of the proof of (a) we get that $f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}$ is either an interval map or a circle map of degree 0 for all $i = 2, 3, \dots, n$. Therefore, by Theorems 1.1 and 1.2(c), we have $S(|P_1|) \subset \operatorname{Per}(f_1)$ and $S(|P_i|) \subset \operatorname{Per}\left(f^i_{P_1 \cdot P_2 \cdot \ldots \cdot P_{i-1}}\right)$ for $i = 2, 3, \dots, n$. Thus, by Corollary 2.4 we obtain

$$\operatorname{Per}(f) \supset S(|P_1|) \cup [\bigcup_{i=2}^{n} \left(\prod_{j=1}^{i-1} |P_j| \right) S(|P_i|)].$$

Then, by a repetitive use of Lemma 3.1 we have $S(m) = S(|P_1| \cdot |P_2| \cdot \ldots \cdot |P_n|) \subset \operatorname{Per}(f)$. In short, the first statement of (c) is proved.

The proof of the second statement of (c) follows in a similar way to the proof of the second statement of (b). $\square$

From Theorem 1.3 and Corollary 2.3 we obtain the following result which gives the characterisation of the set of periods of the triangular maps on a 2-dimensional generalised cylinder.

Theorem 3.3. *Let $f = (f^1, f^2)$ be a triangular map on a 2-dimensional generalised cylinder X. Then the following hold.*

(a) *If X is a rectangle then there exists $s \in \mathbb{N}_s$ such that* $\operatorname{Per}(f) = S(s)$.

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(b) *Suppose* $X = \mathbf{S}^1 \times I$ *and let* $(d,0)$ *be the degree vector of* f. *Then* $\mathrm{Per}(f)$ *is*

$$\begin{array}{ll} \mathbb{N} & \text{if } d \notin \{-2,-1,0,1\}, \\ \mathbb{N} \text{ or } \mathbb{N}\setminus\{2\} & \text{if } d=-2, \\ S(s) \text{ for some } s \in \mathbb{N}_s & \text{if } d \in \{-1,0\}, \\ \displaystyle\bigcup_{P_1\in \mathrm{Orb}(f^1),\ P_2 \in \mathrm{Orb}\left(f^2_{P_1}\right)} |P_1|\cdot|P_2| & \text{if } d=1. \end{array}$$

(c) *Suppose* $X = I \times \mathbf{S}^1$ *and let* $(0,d)$ *be the degree vector of* f. *Then* $\mathrm{Per}(f)$ *is*

$$\begin{array}{ll} \mathbb{N} & \text{if } d \notin \{-2,-1,0,1\}, \\ \mathbb{N} \text{ or } \mathbb{N}\setminus\{2\} & \text{if } d=-2, \\ S(s) \text{ for some } s \in \mathbb{N}_s & \text{if } d=0, \\ \displaystyle\bigcup_{P_1\in \mathrm{Orb}(f^1),\ P_2 \in \mathrm{Orb}\left(f^2_{P_1}\right)} |P_1|\cdot|P_2| & \text{if } d=\pm 1. \end{array}$$

(d) *Suppose that* X *is the torus and let* (d_1,d_2) *be the degree vector of* f. *Then* $\mathrm{Per}(f)$ *is*

$$\begin{array}{ll} \mathbb{N} & \text{if } \{d_1,d_2\} \not\subset \{-2,-1,0,1\}, \\ \mathbb{N} \text{ or } \mathbb{N}\setminus\{2\} & \text{if } -2 \in \{d_1,d_2\}, \\ S(s) \text{ for some } s \in \mathbb{N}_s & \text{if } (d_1,d_2) \in \{(-1,0),(0,0)\}, \\ \displaystyle\bigcup_{P_1\in \mathrm{Orb}(f^1),\ P_2 \in \mathrm{Orb}\left(f^2_{P_1}\right)} |P_1|\cdot|P_2| & \text{if } |d_1| \leqslant 1, |d_2| \leqslant 1 \\ & \text{and } (d_1,d_2) \notin \{(-1,0),(0,0)\}. \end{array}$$

Moreover, each set of one of the above forms can occur as a set of periods of a cylinder map on a 2-dimensional generalised cylinder.

It is important to notice that the fact that the Sharkovskiĭ Theorem extends to cylinder maps with degree vector $(0,0,\dots,0)$ and $(-1,0,\dots,0)$ is due to the following two facts. First, to the structure of the periodic orbits of triangular maps (see Proposition 2.2 and Corollary 2.3) and second, to the number-theoretical property of the Sharkovskiĭ ordering stated in Lemma 3.1. However, these two facts play different roles in this extension, namely, the special structure of the periodic orbits of triangular maps allows us to use the known periodic structure of the underlying one dimensional maps (in the case of cylinder maps the interval and circle maps) to obtain the periodic structure of the class of triangular maps under consideration. However, the number-theoretical property of the Sharkovskiĭ ordering is the responsible for the fact that the periodic structure is preserved when going from one dimensional maps to closed triangular maps. It would be very nice if this situation were general. That is, that the "underlying periodic structure" were always preserved when considering closed triangular maps. Unfortunately, in general this is not true. We shall see this in the next section.

4. An example where the periodic structure is not preserved

Let C be the class of all continuous maps from $Y = \{z \in \mathbb{C} : z^3 \in [0,1]\}$. Of course this class is closed under composition and each map from C has a fixed point because Y is a contractible space. The characterisation of the set of periods for this class of maps is known (see [**6**, **3**, **5**]). We shall describe it for completeness.

The set of periods of a map from C can be expressed as a union of three sets. One of them is a set of the form $S(s)$ with $s \in \mathbb{N}_S = \mathbb{N}_s$ and the other two sets are constructed similarly by using the following two total orderings. The first one, called the *green ordering*, is the following ordering of $\mathbb{N}_G = \mathbb{N} \setminus \{2\} \cup \{3 \cdot 2^\infty\}$:

$$\begin{gathered}
5 \;{}_g{>}\, 8 \;{}_g{>}\, 4 \;{}_g{>}\, 11 \;{}_g{>}\, 14 \;{}_g{>}\, 7 \;{}_g{>}\, 17 \;{}_g{>}\, 20 \;{}_g{>}\, 10 \;{}_g{>}\, 23 \;{}_g{>}\, 26 \;{}_g{>}\, 13 \;{}_g{>} \\
\dots \;{}_g{>}\, 3\cdot 3 \;{}_g{>}\, 3\cdot 5 \;{}_g{>}\, 3\cdot 7 \;{}_g{>} \dots \;{}_g{>}\, 3\cdot 3\cdot 2 \;{}_g{>}\, 3\cdot 5\cdot 2 \;{}_g{>}\, 3\cdot 7\cdot 2 \;{}_g{>} \\
\dots \;{}_g{>}\, 3\cdot 3\cdot 4 \;{}_g{>}\, 3\cdot 5\cdot 4 \;{}_g{>}\, 3\cdot 7\cdot 4 \;{}_g{>} \dots \;{}_g{>} \dots \;{}_g{>}\, 3\cdot 3\cdot 2^n \;{}_g{>}\, 3\cdot 5\cdot 2^n \;{}_g{>}\, 3\cdot 7\cdot 2^n \;{}_g{>} \\
\dots \;{}_g{>}\, 3\cdot 2^\infty \;{}_g{>} \dots \;{}_g{>}\, 3\cdot 2^n \;{}_g{>} \dots \;{}_g{>}\, 3\cdot 16 \;{}_g{>}\, 3\cdot 8 \;{}_g{>}\, 3\cdot 4 \;{}_g{>}\, 3\cdot 2 \;{}_g{>}\, 3\cdot 1 \;{}_g{>}\, 1.
\end{gathered}$$

The rule for writing the first part of this ordering is simple. Write the arithmetic progression: $5, 8, 11, 14, 17, 20, 23, \dots$ and after each even number n insert $n/2$. The second part is the same as the Sharkovskiĭ ordering but all numbers are multiplied by 3. At the very end of the ordering there is 1. From now on, given $t \in \mathbb{N}_G$ we shall denote by $G(t)$ the set $\{k \in \mathbb{N} : t \;{}_g{\geqslant}\, k\}$.

The second one, called the *red ordering*, is the following ordering of $\mathbb{N}_R = \mathbb{N} \setminus \{2,4\} \cup \{3 \cdot 2^\infty\}$:

$$\begin{gathered}
7 \;{}_r{>}\, 10 \;{}_r{>}\, 5 \;{}_r{>}\, 13 \;{}_r{>}\, 16 \;{}_r{>}\, 8 \;{}_r{>}\, 19 \;{}_r{>}\, 22 \;{}_r{>}\, 11 \;{}_r{>}\, 25 \;{}_r{>}\, 28 \;{}_r{>}\, 14 \;{}_r{>} \\
\dots \;{}_r{>}\, 3\cdot 3 \;{}_r{>}\, 3\cdot 5 \;{}_r{>}\, 3\cdot 7 \;{}_r{>} \dots \;{}_r{>}\, 3\cdot 3\cdot 2 \;{}_r{>}\, 3\cdot 5\cdot 2 \;{}_r{>}\, 3\cdot 7\cdot 2 \;{}_r{>} \\
\dots \;{}_r{>}\, 3\cdot 3\cdot 4 \;{}_r{>}\, 3\cdot 5\cdot 4 \;{}_r{>}\, 3\cdot 7\cdot 4 \;{}_r{>} \dots \;{}_r{>} \dots \;{}_r{>}\, 3\cdot 3\cdot 2^n \;{}_r{>}\, 3\cdot 5\cdot 2^n \;{}_r{>}\, 3\cdot 7\cdot 2^n \;{}_r{>} \\
\dots \;{}_r{>}\, 3\cdot 2^\infty \;{}_r{>} \dots \;{}_r{>}\, 3\cdot 2^n \;{}_r{>} \dots \;{}_r{>}\, 3\cdot 16 \;{}_r{>}\, 3\cdot 8 \;{}_r{>}\, 3\cdot 4 \;{}_r{>}\, 3\cdot 2 \;{}_r{>}\, 3\cdot 1 \;{}_r{>}\, 1.
\end{gathered}$$

This ordering is obtained in the same way as the green ordering, the only difference being that we start from 7 instead of from 5. From now on, given $r \in \mathbb{N}_R$ we shall denote by $R(r)$ the set $\{k \in \mathbb{N} : r \,_r\!\geqslant k\}$.

For each map $\varphi \in \mathcal{C}$ there exist $s \in \mathbb{N}_S$, $t \in \mathbb{N}_G$ and $r \in \mathbb{N}_R$ such that $\operatorname{Per}(\varphi) = S(s) \cup G(t) \cup R(r)$. Furthermore, each set of this type is the set of periods of some continuous map from $\mathcal{C}$.

Then the natural extension of Theorem 1.3 to this setting would be the following result. *Let $\mathcal{T}$ be the class of triangular maps $f = (f^1, f^2, \dots, f^n)$ such that $f^1 \in \mathcal{C}$ and the map $f^i(x_1, x_2 \dots, x_{i-1}, \cdot) \in \mathcal{C}$ for each $(x_1, x_2, \dots, x_{i-1}) \in Y \times Y \times \dots \times Y$ and $i = 2, 3, \dots, n$ (note that each map from $\mathcal{T}$ is closed). Then, for each $f \in \mathcal{T}$ there exist $s \in \mathbb{N}_S$, $t \in \mathbb{N}_G$ and $r \in \mathbb{N}_R$ such that $\operatorname{Per}(f) = S(s) \cup G(t) \cup R(r)$. Furthermore, each set of this type is the set of periods of some continuous map from $\mathcal{T}$.* Unfortunately, this statement in general is not true. Let us see why.

If we try to prove the above assertion, we can proceed as in the proof of Theorem 1.3(c) and we shall find that $\operatorname{Per}(f)$ contains a set of the form (here we assume $n = 2$ for simplicity):

$$(S(|P|) \cup G(|Q|) \cup R(|T|)) \cup \left(|P| \operatorname{Per}(f_P^2) \cup |Q| \operatorname{Per}(f_Q^2) \cup |T| \operatorname{Per}(f_T^2)\right),$$

which can be written as

$$\begin{aligned} &S(s_1) \cup G(t_1) \cup R(r_1) \\ &\cup s_1 \left(S(s_2^P) \cup G(t_2^P) \cup R(r_2^P)\right) \\ &\cup t_1 \left(S\left(s_2^Q\right) \cup G\left(t_2^Q\right) \cup R\left(r_2^Q\right)\right) \\ &\cup r_1 \left(S(s_2^T) \cup G(t_2^T) \cup R(r_2^T)\right). \end{aligned}$$

Then we need a result to play the role of Lemma 3.1 in this case; it is the following one. *Assume that $\Gamma, \Lambda \in \{S, G, R\}$. Then, for each $a \in \mathbb{N}_\Gamma$ and $b \in \mathbb{N}_\Lambda$ there exists $\Psi \in \{S, G, R\}$ such that $ab \in \mathbb{N}_\Psi$ and $\Psi(ab) \subset \Gamma(a) \cup a\Lambda(b)$.* This assertion is not true and this is actually the reason why we cannot assure that the periodic structure of $\mathcal{C}$ is preserved in $\mathcal{T}$. To see this, consider a set of the form $S(2^n) \cup 2^n G(k)$ with $n \geqslant 0$ and $k > 2$. Since $2 \notin \mathbb{N}_G$, clearly we get that $2^{n+1} \notin S(2^n) \cup 2^n G(k)$. On the other hand it is also clear that $3 \notin S(2^n) \cup 2^n G(k)$. However, $2^{n+1} \in S(2^n k)$, $3 \in G(2^n k)$ and $3 \in R(2^n k)$. Therefore, there does not exist $\Psi \in \{S, G, R\}$ such that $\Psi(2^n k) \subset S(2^n) \cup 2^n G(k)$.

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