BULL. AUSTRAL. MATH. SOC. 42A20
VOL. 9 (1973), 337-342.

Uniform convergence and everywhere convergence of Fourier series. II

Masako Izumi and Shin-ichi Izumi

The first theorem shows that the subspaces of the space of functions with everywhere convergent Fourier series, defined in our previous paper, is a good subspace. The second theorem shows that convergence criterion in the previous paper is the proper generalization of Lebesgue's Convergence Criterion.

1. Introduction and theorems

By ec we denote the space of functions, periodic with period 2π, whose Fourier series converges everywhere.

LC is the space of functions f such that

$$\int_0^t \big(f(x+u)+f(x-u)-2f(x)\big)du = o(t) \quad \text{as} \quad t \to 0 \quad \text{for all} \quad x , \tag{1}$$

and M^p is the space of functions f such that

$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du = o\big(1/n^{p+1}\big) \quad \text{as} \quad n \to \infty , \tag{2}$$

where $p > 1$,

$$f_1(u) = \int_0^u f(v)dv \quad \text{for all} \quad u \quad \text{in} \quad (-\pi, \pi)$$

and

$$\Delta^2_{\pi/n} f_1(u) = f_1(u) - 2f_1(u-\pi/n) + f_1(u-2\pi/n) .$$

Received 16 July 1973.

Further we denote by N^p the class of functions f such that

$$\sum_{m=n}^{\infty}\left(|a_m|^p+|b_m|^p\right) = o\left(1/n^{p-1}\right) \quad \text{as} \quad n \to \infty ,$$

where a_m and b_m are the mth cosine and sine coefficients of the Fourier series of f .

In the previous paper [2], we have proved the following:

THEOREM A.

(i) $ec \supset LC \cap M^p$ *for any* $p \geq 1$,

(ii) $ec \supset LC \cap N^p$ *for any* $p > 1$.

On the other hand we know that

$$ec \cap \left(L^r - L^s\right) \neq \emptyset \quad (1 < r < s < \infty)$$

and

$$ec \cap \left(L^{\infty} - C\right) \neq \emptyset .$$

We shall prove that the subspaces $LC \cap M^p$ $(p > 1)$ and $LC \cap N^p$ $(p > 2)$ have the same property as above, that is,

THEOREM 1.

(i) $\left(LC \cap M^p\right) \cap \left(L^r - L^s\right) \neq \emptyset \quad (1 < p < r < s \leq \infty)$,

(ii) $\left(LC \cap M^p\right) \cap \left(L^{\infty} - C\right) \neq \emptyset \quad (p > 1)$,

(iii) $LC \cap N^p$ $(p \geq 2)$ *has the properties (i) and (ii).*

This theorem shows that $LC \cap M^p$ and $LC \cap N^p$ have the first character of "good subspace" (*cf.* [2]).

On the other hand, there is the well known convergence criterion of Lebesgue:

THEOREM B. *If* f *is an even function satisfying the conditions*

(1) $$f_1(t) = \int_0^t f(u)du = o(t) \quad \text{as} \quad t \to 0$$

https://doi.org/10.1017/S0004972700043331 Published online by Cambridge University Press

and

$$(4) \qquad \int_{2\pi/n}^{\pi} |\Delta_{\pi/n} f(u)| \frac{du}{u} = o(1) \quad as \quad n \to \infty ,$$

where $\Delta_{\pi/n} f(u) = f(u) - f(u-\pi/n)$, *then the Fourier series of* f *converges at the origin.*

We have proved the following generalization of Lebesgue's Convergence Criterion [2].

THEOREM C. *If* f *is an even function satisfying the conditions* (3) *and*

$$(5) \qquad \int_{3\pi/n}^{\pi} \left|\Delta_{\pi/n}^{2} f_1(u)\right| \frac{du}{u} = o(1/n) \quad as \quad n \to \infty ,$$

then the Fourier series of f *converges at the origin.*

Gergen [1] generalized Lebesgue's criterion as follows: the condition (3) is replaced by Cesàro continuity of any positive order i , that is,

$$(3') \qquad f_i(t) = o\left(t^i\right) \quad \text{as} \quad t \to 0$$

and the condition (4) by

$$(4') \qquad \int_{(j+1)\pi/n}^{\pi} \left|\Delta_{\pi/n} f(u)\right| \frac{du}{u} = \bar{o}(1) \quad \text{as} \quad n \to \infty ,$$

where $\bar{o}$ is the limit in the Pollard sense (*cf.* [1]).

Similarly we can generalize Theorem C as follows: the condition (3) can be replaced by (3') and the condition (5) can be replaced by

$$(5') \qquad \int_{(j+1)\pi/n}^{\pi} \left|\Delta_{\pi/n}^{j} f_k(u)\right| \frac{du}{u} = \bar{o}\left(1/n^k\right) \quad \text{as} \quad n \to \infty ,$$

where $\bar{o}$ is the limit in the Pollard sense.

We can easily see that (5') is more general than (4').

We shall prove that Theorem C is a proper generalization of Lebesgue's and the generalization of Theorem C is also a proper generalization of Gergen's:

THEOREM 2. *(i) There is a function f satisfying the conditions* (3) *and* (5), *but not* (4).

(ii) There is a function f satisfying the conditions (3') *and* (5'), *but not* (4').

2. Proof of Theorem 2 *(i)* and *(ii)*

Consider the even and periodic function f defined by

$$f(u) = u^{-\alpha}\sin 1/u \quad \text{on} \quad (0, \pi)\,, \tag{6}$$

where $0 \leq \alpha < 1$.

Then

$$f_1(u) = \int_0^u \frac{1}{v^\alpha}\sin\frac{1}{v}\,dv = \int_{1/u}^\infty \frac{\sin v}{v^{2-\alpha}}\,dv = O\left(u^{2-\alpha}\right) = o(u) \quad \text{as} \quad u \to 0 \tag{7}$$

and then the condition (3) is satisfied. On the other hand,

$$\begin{aligned} \Delta^2_{\pi/n}f_1(u) &= \left(\int_{1/u}^{1/(u-\pi/n)} - \int_{1/(u-\pi/n)}^{1/(u-2\pi/n)}\right)\frac{\sin v}{v^{2-\alpha}}\,dv \\ &= \int_{1/u}^{1/(u-\pi/n)}\left(\frac{\sin v}{v^{2-\alpha}} - \frac{\sin(v-\pi/n)}{(v-\pi/n)^{2-\alpha}}\right)dv \\ &= -\int_{1/u}^{1/(u-\pi/n)}\frac{\sin v}{v^{2-\alpha}(v-\pi/n)^{2-\alpha}}\{v^{2-\alpha}-(v-\pi/n)^{2-\alpha}\}dv \\ &\qquad + 2\sin\frac{\pi}{2n}\int_{1/u}^{1/(u-\pi/n)}\frac{\cos(v-\pi/2n)}{(v-\pi/n)^{2-\alpha}}\,dv \\ &= O\left(\frac{\pi/n}{u(u-\pi/n)}\cdot\frac{\pi}{n}\,u^{3-\alpha}\right) + O\left(\frac{\pi}{n}\cdot\frac{\pi/n}{u(u-\pi/n)}\cdot u^{2-\alpha}\right)\,; \end{aligned} \tag{8}$$

then

$$\begin{aligned} \int_{3\pi/n}^{\pi}\left|\Delta^2_{\pi/n}f_1(u)\right|\frac{du}{u} &= O\left(\frac{1}{n^2}\int_{3\pi/n}^{\pi}\frac{du}{u^\alpha}\right) + O\left(\frac{1}{n^2}\int_{3\pi/n}^{\pi}\frac{du}{u^{1+\alpha}}\right) \\ &= O\left(1/n^{2-\alpha}\right) = o(1/n) \quad \text{as} \quad n \to \infty\,. \end{aligned}$$

Therefore the condition (5) is also satisfied. Now we shall show that the Lebesgue condition (4) is not satisfied.

https://doi.org/10.1017/S0004972700043331 Published online by Cambridge University Press

$$\int_{2\pi/n}^{\pi} |f(u)-f(u-\pi/n)| \frac{du}{u} = \int_{2\pi/n}^{\pi} \left|\frac{1}{u^{\alpha}} \sin \frac{1}{u} - \frac{1}{(u-\pi/n)^{\alpha}} \sin \frac{1}{u-\pi/n}\right| \frac{du}{u}$$

$$\geq \int_{1/\sqrt{n}}^{\pi} \left|\sin \frac{-\pi/n}{(u-\pi/n)} \cos \frac{2u-\pi/n}{u(u-\pi/n)}\right| \frac{du}{u^{\alpha+1}} - \frac{A}{n} \int_{1/\sqrt{n}}^{\pi} \left|\frac{\sin 1/u}{u^{2+\alpha}}\right| du$$

$$\geq \frac{A}{n} \int_{1/\sqrt{n}}^{\pi} \frac{du}{u^{3+\alpha}} - \frac{A}{n} \int_{1/\sqrt{n}}^{\pi} \frac{du}{u^{2+\alpha}} \geq An^{\alpha/2} - \frac{A}{n^{(1-\alpha)/2}} \geq A \quad \text{for large } n .$$

Thus we have proved Theorem 2 *(i)*.

Theorem 2 *(ii)* can be proved by the same example.

We shall remark that the sine function of this example can be replaced by the cosine and or the exponential function.

3. Proof of Theorem 1 *(i)* and *(ii)*

We shall consider the even function f defined by (6). We take

$$1/s < \alpha < 1/r < 1 ,$$

then

$$\int_0^{\pi} |f(u)|^r du \leq \int_0^{\pi} u^{-\alpha r} du \leq A$$

and

$$\int_0^{\pi} |f(u)|^s du \geq A \int_0^{\pi} u^{-\alpha s} = \infty .$$

Therefore $f \in L^r$ but $f \notin L^s$. By (7), $f \in LC$. Now

$$\int_{-\pi}^{\pi} \left|\Delta^2_{\pi/n} f_1(u)\right|^p du \leq A \int_0^{3\pi/n} + A \int_{3\pi/n}^{\pi} = P + Q ,$$

where

$$P \leq A \int_0^{3\pi/n} |f_1(u)|^p du = A \int_0^{3\pi/n} u^{(2-\alpha)p} du \leq A/n^{(2-\alpha)p+1} = o\left(1/n^{p+1}\right)$$

by (7) and

https://doi.org/10.1017/S0004972700043331 Published online by Cambridge University Press

$$Q \le A \int_{3\pi/n}^{\pi} \left|\Delta_{\pi/n}^{2} f_1(u)\right|^p du \le A \frac{1}{n^{2p}} \int_{3\pi/n}^{\pi} \frac{du}{u^{\alpha p}} \le A/n^{(2-\alpha)p+1}$$
$$= o\left(1/n^{p+1}\right)$$

by (8). Therefore $f \in M^p$. Thus we have proved Theorem 1 *(i)*.

For the proof of Theorem 1 *(ii)*, we use the same function (6) with $\alpha = 0$. Then this function does not belong to the space C and belongs to the spaces $LC \cap M^p$ and L^∞ .

References

[1] J.J. Gergen, "Convergence and summability criteria for Fourier series", *Quart. J. Math. Oxford* 1 (1930), 252-275.

[2] Masako Izumi and Shin-ichi Izumi, "Uniform convergence and everywhere convergence of Fourier series. I", *Bull. Austral. Math. Soc.* 9 (1973), 321-335.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.