A GENERALIZATION OF RADON’S THEOREM II

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A new proof is given of the following result:

Let \( m \) and \( d \) be positive integers, and let a set of \( md + m - d \) points be given in \( d \)-dimensional space. Then the set can be partitioned into \( m \) sets such that the \( m \) convex polytopes spanned by the sets have a non-empty intersection.

Let \( n \)-set mean a set of \( n \) points in \( \mathbb{R}^d \). We shall say that an \( n \)-set is \( m \)-divisible if it can be divided into \( m \) sets in such a way that the convex hulls of the \( m \) sets have a non-empty intersection. In 1964, [6], I proved the following:

THEOREM. Any \( (m(d+1)-d) \)-set is \( m \)-divisible.

The proof has been regarded as difficult and it is therefore a pleasure to be able to present a much simpler proof below. It is also a pleasure to acknowledge my debt to Imre Bárány, whose proof of Theorem 2.2 in [I] inspired the present one.

The proof is by induction on \( m \). The case \( m = 1 \) is trivial and so, assuming that the theorem holds for \( m = k > 0 \), we are to prove that it holds for \( m = k + 1 \). Put \( K = (k+1)(d+1) - d \) and let a \( K \)-set \( \Omega_0 = \{p_1^0, \ldots, p_K^0\} \) be given. If the theorem is false for \( \Omega_0 \), then there is an \( \varepsilon > 0 \) such that it is also false for any set \( \Omega = \{p_1, \ldots, p_K\} \),

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where \(|p_i - p_i^0| < \varepsilon, i = 1, \ldots, K\). We choose \(\Omega\) strongly independent, as defined by Reay [4], and prove the theorem for \(\Omega\), which suffices.

Strong independence of \(\Omega\) means that for each \(t\) and for any \(t\) affine subspaces \(B_1, \ldots, B_t\) of \(\mathbb{R}^d\), spanned by pairwise disjoint subsets of \(\Omega\), \(\dim(B_1 \cap \ldots \cap B_t) = \max\{-1, \dim B_1 + \ldots + \dim B_t - (t-1)d\}\).

Consider a partition in \(\Omega\), that is a partition of a subset of \(\Omega\), consisting of disjoint non-empty sets \(\Omega_0, \ldots, \Omega_k\), having the property that \(\text{conv} \ \Omega_1 \cap \ldots \cap \text{conv} \ \Omega_k \neq \emptyset\). Such partitions exist by the case \(m = k\) of the theorem. Choose such a partition for which the distance from \(\text{conv} \ \Omega_0\) to \(\text{conv} \ \Omega_1 \cap \ldots \cap \text{conv} \ \Omega_k\) is minimal. If this distance is zero, we are through, and so we assume it to be positive. A contradiction will now be obtained by showing that some other partition in \(\Omega\) will make the distance considered smaller.

Let \(q \in \text{conv} \ \Omega_0\) and \(r \in \text{conv} \ \Omega_1 \cap \ldots \cap \text{conv} \ \Omega_k\) be a pair of points realizing the distance. By Carathéodory's theorem there is a simplex, with vertex set \(\Omega_0' \subseteq \Omega_0\), such that \(q\) is in the relative interior of \(\text{conv} \ \Omega_0'\). Replacing \(\Omega_0\) by \(\Omega_0'\), we may thus assume that \(q\) is in the relative interior of \(\text{conv} \ \Omega_0\). Similarly, we may assume that each \(\text{conv} \ \Omega_i\), \(i = 1, \ldots, k\), is a simplex with \(r\) in its relative interior. Putting \(A_i = \text{aff} \ \Omega_i\), the affine hull of \(\Omega_i\), we then get \(\dim A_i = |\Omega_i| - 1\), \(i = 0, \ldots, k\). Furthermore, by the condition of strong independence, \(\dim(A_0 \cap \ldots \cap A_k) = \dim A_0 + \ldots + \dim A_k - (k-1)d\)

\[= |\Omega_0 \cup \ldots \cup \Omega_k| - k + d - kd.\]

We now want to prove that \(\Omega_0 \cup \ldots \cup \Omega_k\) is a proper subset of \(\Omega\), which will leave us some point \(p_j\) to add to a suitable \(\Omega_i\) so as to lower the distance in question. Consider the parallel hyperplanes \(H_q\) through \(q\) and \(H_r\) through \(r\), both orthogonal to \(q - r\). The open
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A slab between them clearly separates \( \text{conv } \Omega_0 \) from \( \text{conv } \Omega_1 \cap \ldots \cap \text{conv } \Omega_k \). Furthermore \( \Omega_0 \subset H_q \), while \( \text{conv } \Omega_1 \cap \ldots \cap \text{conv } \Omega_k \subset H_r \). The first of these inclusions holds true because \( H_q \) is a supporting hyperplane of \( \text{conv } \Omega_0 \) in \( q \) and \( q \) is in the relative interior of the simplex \( \text{conv } \Omega_0 \). The second one holds because \( r \) is in the relative interior of each simplex \( \text{conv } \Omega_1, \ldots, \text{conv } \Omega_k \), so that some neighbourhood of \( r \) in \( A_1 \cap \ldots \cap A_k \) is contained in \( \text{conv } \Omega_1 \cap \ldots \cap \text{conv } \Omega_k \), and is thus supported by \( H_r \) in \( R \). (If it was not, it would meet the open slab mentioned above.) These two inclusions show that \( A_0 \subset H_q \) and \( A_1 \cap \ldots \cap A_k \subset H_r \), which, by the strong independence in \( \Omega \), can happen only if
\[
\dim A_0 + \dim (A_1 \cap \ldots \cap A_k) < d.
\]
Thus
\[
|\Omega_0| - 1 + |\Omega_1 \cup \ldots \cup \Omega_k| - k + d - kd < d,
\]
so that
\[
|\Omega_0 \cup \ldots \cup \Omega_k| < k(d+1) + 1 = |\Omega|.
\]
It is no restriction to assume that \( p_1 \notin \Omega_0 \cup \ldots \cup \Omega_k \).

The easier case is when \( p_1 \) is in that open halfspace, bounded by \( H_q \), in which \( H_r \) lies. The segment \( qp_1 \) will then be in \( \text{conv}(\Omega_0 \cup \{p_1\}) \) and for any point \( q' \) on it sufficiently near \( q \), but not equal to \( q \), we will have \( |q'-r| < |q-r| \). Thus the distance from \( \text{conv}(\Omega_0 \cup \{p_1\}) \) to \( \text{conv } \Omega_1 \cap \ldots \cap \text{conv } \Omega_k \) will be smaller than that from \( \text{conv } \Omega_0 \).

In the second, more difficult, case, \( p_1 \) is separated (weakly) from \( H_r \) by \( H_q \). We shall see that for some \( i \in \{1, \ldots, k\} \) there is a ray from \( r \), contained in \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k \), which passes through the simplex \( \text{conv}(\Omega_i \cup \{p_1\}) \) and also lies in the halfspace bounded by \( H_r \).
and containing $H_q$. Any point $r'$ (≠ $r$) on this ray, sufficiently near $r$, will then satisfy $|q-r'| < |q-r|$ and also be in 

$$\text{conv } \Omega_1 \cap \ldots \cap \text{conv}\{\Omega_i \cup \{p\} \} \cap \ldots \cap \text{conv } \Omega_k,$$

so that the desired contradiction will be obtained once again.

For the proof of the existence of the ray just mentioned, we introduce an affine coordinate system in which $r = (0, \ldots, 0)$, $p_1 = (1, \ldots, 1)$, $A_1$ is given by $x_j = 0$, $j \in C_1, \ldots, A_k$ by $x_j = 0$, $j \in C_k$. Here $C_1, \ldots, C_k$, form a partition in $\{1, \ldots, d\}$ (with possibly some empty parts). The disjointness of the $C_i$ is a consequence of the strong independence in $\Omega$. Furthermore $H_r$ is given by $a_1x_1 + \ldots + a_dx_d = 0$, where $a_j = 0$ for $j \notin (C_1 \cup \ldots \cup C_k)$, as we have seen that $A_1 \cap \ldots \cap A_k \subset H_r$. The $a_j$'s are normalized so that $p_1$ (which is not in $H_r$) lies in the hyperplane $a_1x_1 + \ldots + a_dx_d = 1$. Thus $H_q$ will have the equation $a_1x_1 + \ldots + a_dx_d = a$, with $0 < a \leq 1$.

Now consider the flat $\text{aff}\{\Omega_1 \cup \{p\}\} \cap A_2 \cap \ldots \cap A_S$. It is no restriction to assume that $C_1 = \{1, \ldots, |C_1|\}$, $C_2 = \{|C_1|+1, \ldots, |C_1|+|C_2|\}$ and so on. Then the point $p_1 = (1, \ldots, 1, 0, \ldots, 0) (|C_1| 1's)$ is in the flat, as it equals $p_1 + (0, \ldots, 0, -1, \ldots, -1)$. Furthermore $p_1'$ and $p_1$ are in the same open halfspace of $\text{aff}\{\Omega_1 \cup \{p\}\}$, bounded by $A_1$, as

$$(0, \ldots, -1, \ldots, -1)$$

is in $A_1$. Hence the ray from $r$ through $p_1'$ passes through the simplex $\text{conv}\{\Omega_1 \cup \{p\}\}$. It will also lie in the halfspace bounded by $H_r$ and containing $H_q$, provided $(a_1, \ldots, a_d) \cdot (1, \ldots, 1, 0, \ldots, 0) = a_1 + \ldots + a|C_1| > 0$. Thus, if $a_1 + \ldots + a|C_1| > 0$, we have what we want. Similarly, we shall be satisfied with $a|C_1|+1 + \ldots + a|C_1|+|C_2| > 0$, and so on. But one of these equalities must hold, as $a_1 + \ldots + a|C_1|+\ldots+|C_k| = 1$.
Doignon and Valette [2] have proved that our theorem remains valid in any affine space over an ordered division ring. The proof given above can be modified so as to show this. Finally I would like to call the reader's attention to the recent survey papers by Eckhoff [3] and Reay [5], which give a lot of information on Radon's theorem and related matters.

References


