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A NOTE ON EQUICONTINUOUS FAMILIES OF VOLUMES WITH AN APPLICATION TO VECTOR MEASURES

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Let V denote a ring of subsets of an abstract space X, let R denote the real numbers, and let N denote the positive integers. Denote by $a(V, R)$ (respectively $ca(V, R)$) the space of real valued, finitely additive (respectively countably additive) functions on the ring V and denote by $ab(V, R)$ the subspace consisting of those members of the space $a(V, R)$ with finite variation on each set in the ring V. Members of the space $a(V, R)$ are referred to as charges and members of the space $ab(V, R)$ are referred to as locally bounded charges. We denote by $cab(V, R)$ the intersection of the spaces $ab(V, R)$ and $ca(V, R)$. Elements of this space are called volumes and nonnegative elements are called positive volumes.

Let W denote any family of subsets of the space X. A sequence $A_n \in V$, $n \in N$, is said to be W-dominated if there exists a set $B \in W$ such that $A_n \subseteq B$, for $n = 1, 2, \ldots$. A charge $w \in a(V, R)$ is said to be Rickart on the ring V relative to the class W if for each W-dominated, disjoint sequence $A_n \in V$, $n \in N$, we have

$$\lim_n w(A_n) = 0.$$

This condition is an abstraction of the condition of strong boundedness introduced by Rickart in [**15**] for finitely additive set functions on a σ-algebra into a Banach space.

A family of charges $M \subset a(V, R)$ is said to be uniformly Rickart on the ring V relative to the class W if for each W-dominated, disjoint sequence $A_n \in V$, $n \in N$, $\lim_n w(A_n) = 0$ uniformly with respect to the charges $w \in M$.

Each charge $w \in ab(V, R)$ is Rickart on the ring V relative to the class

$$b(w) = \{A \in V_\sigma : |w|(A) < \infty\}$$

where V_σ denotes the family of all countable unions of members of the ring V and $|w|(\cdot)$ denotes the variation of the charge $w \in ab(V, R)$ defined on all subsets as in [**9**]. For a volume $w \in cab(V, R)$, the class $b(w)$ coincides with the sets $A \in V_\sigma$ for which the sequence

$$|w| \left(\bigcup_{k=1}^{n} A_k \right) \qquad n = 1, 2, 3, \ldots$$

is bounded for some representation $A = \bigcup_n A_n$. If the ring V is an algebra of sets or a σ-ring of sets, then each member of the space $ab(V, R)$ is Rickart on the ring V relative to the classes $W = \{X\}$, and $W = V$ respectively.

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In this paper it is shown that a family of volumes $M \subset cab(V, R)$ is uniformly Rickart on the ring V relative to a class $b(w_0)$, corresponding to a nonnegative volume $w_0 \in cab(V, R)$, if and only if there exists a nonnegative volume $v \in cab(V, R)$ such that the set M is v-equicontinuous. As a consequence, it is shown that for a class of vector measures on a general ring of sets, weak continuity is equivalent to strong continuity.

The proof of the main theorem requires the following equivalent formulation of the Rickart condition which was first used by T. Ando, [**1**]. A discussion of additional equivalent formulations of the Rickart condition for finitely subadditive real valued functions may be found in [**3**] and [**13**].

PROPOSITION 1. *Let V be a ring of subsets of the space X. A family $M \subset cab(V, R)$ is uniformly Rickart on the ring V relative to a class W if and only if for each W-dominated sequence $A_n \in V$, $n \in N$, and each number $\epsilon > 0$, there exists a strictly increasing map j from the set of positive integers N into itself such that for all indices $n \in N$ and charges $w \in M$, we have*

$$|w|(B_n \backslash C_n) < \epsilon$$

where for $t \in N$

$$B_t = A_t \cup A_{t+1} \cup \ldots \cup A_{j(t)}$$

and

$$C_t = \bigcap_{s=1}^{t} A_s.$$

THEOREM 1. *Let V be a ring of subsets of the space X. A pointwise bounded family $M \subset cab(V, R)$ is uniformly Rickart on the ring V relative to the class $b(w_0)$, corresponding to a positive volume $w_0 \in cab(V, R)$, if and only if there exists a positive volume $v \in cab(V, R)$ such that the set M is v-equicontinuous. In addition if each member of the set M is w_0-continuous, then the set M is w_0-equicontinuous.*

Proof. Since the sufficiency is clear, only a proof of the necessity is given. We show first that for each number $\epsilon > 0$, there exists a number $\delta > 0$ and a finite set $u_1, \ldots, u_n \in M \cup \{w_0\}$ such that $A \in V$ and $|u_k|(A) < \delta$, $k = 1, 2, 3, \ldots, n$ yields $|u(A)| < \epsilon$ for all charges $u \in M \cup \{w_0\}$. If the contrary is assumed, then beginning with the charge $w_0 \in M \cup \{w_0\}$, it is possible to choose a sequence $w_n \in M \cup \{w_0\}$, $n = 0, 1, 2, \ldots$ and a sequence $A_n \in V$, $n \in N$, such that for each index $n = 1, 2, 3, \ldots$ we have

$$|w_k|(A_n) < 1/2^n \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

and

$$|w_n(A_n)| > \epsilon.$$

The relation

$$|w_0|\left(\bigcup_{k=1}^{n} A_k\right) \leqq \sum_{k=1}^{n} |w_0|(A_k) \leqq 1 \quad \text{for } n = 1, 2, 3, \ldots$$

insures that we have $A = \bigcup_n A_n \in b(w_0)$.

Using the Ando formulation of the Rickart condition, there exist sequences $B_n \in V$ and $C_n \in V$, $n \in N$ such that for each index $n = 1, 2, 3, \ldots$

(1) $B_n = A_n \cup A_{n+1} \cup \ldots \cup A_{j(n)}$ where $j(n) < j(n+1)$,

(2) $B_n \supseteq C_n \supseteq C_{n+1}$,

(3) $|w|(B_n \backslash C_n) < \epsilon/2$ for all charges $w \in M$.

For each pair of indices k, n, with $k = 0, 1, 2, \ldots, n-1$ the relation

$$|w_k|(B_n) \leqq \sum_{j=n}^{j(n)} |w_k|(A_j) \leqq 1/2^{n-1}$$

insures that we have

$$\lim_n |w_k|(B_n) = 0$$

and from monotonicity

$$\lim_n |w_k|(C_n) = 0$$

for each index $k = 0, 1, 2, 3, \ldots$. Since the set $M \cup \{w_0\}$ is uniformly Rickart on the ring V, and the sequence C_n, $n \in N$, is $b(w_0)$-dominated and monotone, we conclude that the limit is uniform in the index $k = 0, 1, 2, \ldots$. Therefore, there exists a nonnegative integer $n(\epsilon)$ such that $n \geqq n(\epsilon)$ yields

$$|w_k|(C_n) < \epsilon/2$$

uniformly in the index $k = 0, 1, 2, 3, \ldots$. We then have for indices $k = 0, 1, 2, \ldots, n = 1, 2, 3, \ldots$.

$$|w_k|(A_n) \leqq |w_k|(B_n) \leqq |w_k|(C_n) + |w_k|(B_n \backslash C_n).$$

This yields for indices $n \geqq n(\epsilon)$

$$|w_k|(A_n) < \epsilon$$

uniformly in the index $k = 0, 1, 2, 3, \ldots$. This contradicts the constructed relation

$$|w_n(A_n)| > \epsilon.$$

Using the above established result, choose a sequence $w_m{}^n \in M \cup \{w_0\}$, $m = 1, \ldots, k(n)$, $n = 1, 2, 3, \ldots$ and a sequence $\delta_n > 0$, $n = 1, 2, 3, \ldots$ such that $A \in V$ and $|w_m{}^n|(A) < \delta_n$ for $m = 1, 2, \ldots, k(n)$ yields

$$|w(A)| < 1/2^n$$

for all charges $w \in M \cup \{w_0\}$.

We set

$$v(\cdot) = \sum_n \frac{1}{2^n} \sum_{m=1}^{k(n)} \frac{|w_m{}^n|(\cdot)}{k(n)}.$$

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Since the set M is pointwise bounded, the function $v(\cdot)$ is nonnegative, countably additive and the set $M \cup \{w_0\}$ is v-equicontinuous. Note that countable additivity follows from the fact that the family $w_m{}^n$, $m = 1, 2, \ldots, k(n)$, $n = 1, 2, \ldots$ is uniformly Rickart and each member is countably additive.

Assume that each member of the family M is w_0-continuous. If the family M is not w_0-equicontinuous, there exists a number $\epsilon > 0$, a sequence $A_n \in V$, $n \in \mathrm{N}$ and a sequence $w_n \in M$, $n \in N$ such that

$$w_0(A_n) < 1/2^n$$

and

$$|w_n(A_n)| > \epsilon$$

for all indices $n = 1, 2, 3, \ldots$. Consequently $A = \bigcup_n A_n \in b(w_0)$ and the assertion follows from an argument similar to the one given above.

Let Y be a real Banach space and let Y' denote the dual. Let $a(V, Y)$ denote the space of Y-valued, finitely additive functions on the ring V. Elements of the space $a(V, Y)$ are referred to as vector charges. For each vector charge $\mu \in a(V, Y)$, the semivariation $p(\cdot, \mu): P(X) \rightarrow [0, \infty]$ is defined for a set $E \in P(X)$ by the relation

$$p(E, \mu) = \sup\,(|\mu(A)| : A \in V, A \subseteq E).$$

The semivariation is increasing on the σ-algebra $P(X)$ and subadditive on the ring V. Denote by $ab(V, Y)$ the space of charges with finite semivariation on each set in the ring V. A vector charge $\mu \in a(V, Y)$ is said to be Rickart on the ring V if for each disjoint sequence $A_n \in V$, $n \in N$, for which $p(\bigcup_{k=1}^{\infty} A_k, \mu) < \infty$ we have

$$\lim_n \mu(A_n) = 0.$$

A special case of the Rickart condition introduced here for vector charges has proven to be quite useful in the study of decompositions of a vector charge into a countably additive and (weakly) purely finitely additive component (see Rickart [**15**], Brooks [**4**], and Uhl [**17**]), as well as in the study of a Vitali-Hahn-Saks-Nikodym theorem for vector charges (see Brooks and Jewett [**7**], and Oberle [**13**]).

Brooks [**5**; **6**] has shown that a Rickart vector charge on an algebra of sets admits a finitely additive control measure. Uhl [**17**] noticed that the existence of a finitely additive control measure is equivalent (on an algebra of sets) to the weak compactness of the range of a countably additive, Rickart vector charge. However, the characterization of unconditional summability of a series in terms of weak relative compactness of the net of unordered, finite sums (see McArthur [**12**], or Robertson [**16**]) insures that a vector charge $\mu \in a(V, Y)$ is Rickart on a ring V if and only if the set

$$\left\{ \sum_{k \in \Delta} \mu(A_k) : \Delta \in P(N),\ \Delta\text{-finite} \right\}$$

is weakly relatively compact for each disjoint sequence $A_n \in V$, $n \in \mathrm{N}$, with $p(\bigcup_{k=1}^{\infty} A_k, \mu) < \infty$. Consequently each weakly compact vector charge is Rickart (a vector charge $\mu \in ab(V, Y)$ is said to be weakly compact if the set $\{\mu(B): B \in V, B \subseteq A\}$ is weakly compact in the space Y for each set $A \in V_\sigma$ with $p(A, \mu) < \infty$). This observation may be established directly from the Orlicz-Pettis theorem (Day [8]). The converse has been noted by Uhl [17] for Rickart charges on an algebra of sets. The Uhl theorem may be used to show that each countably additive Rickart charge on a general ring of sets is also weakly compact. Indeed, one need only note that for a Rickart charge $\mu \in ab(V, Y)$ and each set $A \in V_\sigma$ with $p(A, \mu) < \infty$ and $A \notin V$, the net $\{\mu(B): B \in V, B \subset A\}$ is Cauchy in the space Y. Consequently, the restriction of the charge μ to the ring $V(A) = \{B \in V: B \subset A\}$ has an extension to a Rickart vector charge on the algebra

$$\mathscr{A}(V, A) = \{B \in P(A): B \in V \quad \text{or} \quad A \backslash B \in V\}$$

where $P(A)$ denotes the σ-algebra of all subsets of the set A. This extension is given on a set $B \in \mathscr{A}(V, A)$ by the formula

$$\mu_e(B) = \begin{cases} \mu(B) & \text{if } B \in V \\ \lim\ (\mu(C): C \in V,\ C \subset A) - \mu(A \backslash B) & \text{if } A \backslash B \in V. \end{cases}$$

Since the charge $\mu_e(\cdot)$ is Rickart on the algebra $\mathscr{A}(V, A)$, it admits a control measure (Brooks [6]). By the Uhl theorem, the set $\{\mu_e(B): B \in \mathscr{A}(V, A)\}$ is weakly relatively compact in the space Y so that the set $\{\mu(B): B \in V, B \subseteq A\}$ is weakly relatively compact in the space Y.

G. G. Gould [11] first noted that for a large class of Banach spaces, which includes the weakly complete spaces, each vector charge with finite semivariation on the ring V is Rickart. The results of Bessaga and Pelczynski [2] insure that the spaces considered by Gould are precisely those Banach spaces which do not contain the space c_0 of sequences of scalars converging to zero.

If a vector charge $\mu \in a(V, Y)$ has finite semivariation on the ring V and is Rickart on the ring V, then the set of charges $\{|y' \circ \mu|(\cdot): y' \in Y', |y'| = 1\}$ is uniformly Rickart on the ring V relative to the class

$$W = \{A \in V_\sigma: p(A, \mu) < \infty\}$$

where V_σ denotes the class of all countable unions of elements from the ring V. However, the Uniform Boundedness Principle, (Dunford-Schwartz [10]) insures that this is equivalent to saying that the set $\{|y' \circ \mu|(\cdot): y' \in Y', |y'| = 1\}$ is uniformly Rickart on the ring V relative to

$$\bar{W} = \bigcap \{b(y' \circ \mu(\cdot)): y' \in Y', |y'| = 1\}.$$

Indeed, consider any disjoint sequence $A_n \in V$, $n \in N$ such that $A = \bigcup_n A_n \in \bar{W}$. We then have

$$|y' \circ \mu|(A) < \infty$$

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for each functional $y' \in Y'$, $|y'| = 1$. Now for each set $B \in V$, $B \subseteq A$, the relation

$$T_B(y') = y'(\mu(B)) \quad \text{for } y' \in Y'$$

defines a net of linear continuous functionals on the space Y' which satisfy the relation

$$|T_B(y')| \leqq |y' \circ \mu|(A)$$

for each functional $y' \in Y'$, $|y'| \leqq 1$. The Uniform Boundedness Principle insures that there exists a constant $m > 0$ so that

$$|T_B(y')| \leqq m$$

for all sets $B \in V$, $B \subseteq A$ and all functionals $y' \in Y'$, $|y'| \leqq 1$. Consequently we have

$$p(A, \mu) < m.$$

We now apply the Rickart condition to conclude that

$$\lim_n \mu(A_n) = 0.$$

This observation and Theorem 1 yield the following characterization of continuity in terms of weak continuity for finitely additive vector charges.

THEOREM 2. *Let V be a ring of subsets of a space X, let Y be a Banach space and let $v \in cab(V, R)$ be nonnegative. A Rickart vector charge $\mu \in ab(V, Y)$ is v-continuous if and only if for each functional $y' \in Y'$, the charge $y' \circ \mu(\cdot)$ is v-continuous.*

Proof. The necessity is clear. To see the sufficiency, notice that the assumption that for each functional $y' \in Y'$, the charge $y' \circ \mu$ is v-continuous, insures that we have

$$b(v) \subset b(y' \circ \mu).$$

Indeed, we need only show that for two nonnegative volumes $w, v \in cab(V, R)$ the condition that w is v-continuous insures that $b(v) \subset b(w)$. To see this, let $\bar{w}$ denote the extension of the volume w to a (possibly infinite valued) measure on the σ-ring $\sigma(V)$ generated by the ring V. Assume that for a set $A \in V_\sigma$, we have

$$A \in b(v) \quad \text{and} \quad A \notin b(w).$$

From the countable additivity of the extension $\bar{w}$, there exists a set $A_1 \in V$, $A_1 \subset A$ such that $w(A_1) > 1$. From additivity, we have

$$\infty = \bar{w}(A) = w(A_1) + \bar{w}(A \backslash A_1).$$

Consequently

$$\bar{w}(A \backslash A_1) = \infty.$$

Continuing in this fashion, we may choose a disjoint sequence $A_n \subseteq A$, $A_n \in V$,

$n = 1, 2, 3, \ldots$ such that $w(A_n) > 1$, for $n = 1, 2, 3, \ldots$. However, $A \in b(v)$ yields

$$\lim_n v(A_n) = 0$$

so that by continuity

$$\lim_n w(A_n) = 0$$

which contradicts the constructed relation

$$w(A_n) > 1$$

for indices $n = 1, 2, 3, \ldots$. We now have

$$b(v) \subset \bigcap (b(y' \circ \mu) : y' \in Y', |y'| = 1).$$

Therefore the set $\{|y' \circ \mu|(\cdot) : y' \in Y', |y'| = 1\}$ is uniformly Rickart on the ring V relative to the class $b(v)$. Using Theorem 1, we conclude that the set $\{|y' \circ \mu|(\cdot) : y' \in Y', |y'| = 1\}$ is v-equicontinuous and hence the vector charge $\mu \in ab(V, Y)$ is v-continuous.

Theorem 2 extends the original formulation given by Pettis [**14**] for countably additive vector measures on a σ-algebra of sets which are defined by the Pettis integral of vector functions which are Pettis summable with respect to a finite measure.

COROLLARY. *Let V be a ring of subsets of an abstract space X and let Y be a real Banach space which does not contain the space c_0 of real sequences converging to zero. Then a charge $\mu \in ab(V, Y)$ is v-continuous with $v \in cab^+(V, R)$ (the cone of nonnegative volumes) if and only if the charges $y' \circ \mu$, $y' \in Y'$, $|y'| = 1$ are v-continuous.*

A nonnegative volume $v \in cab(V, R)$ is said to control a vector charge $\mu \in a(V, Y)$ if the charge μ is v-continuous on the ring V. Theorem 1 contains the essentials of the following existence theorem for control volumes.

THEOREM 3. *Let V be a ring of subsets of an abstract space X and let Y be a real Banach space. A countably additive, Rickart vector charge $\mu \in ab(V, Y)$ admits a control volume $v \in cab(V, R)$ if and only if there exists a nonnegative volume $w \in cab(V, R)$ with*

$$b(w) \subset \{A \in V_\sigma : p(A, \mu) < \infty\}.$$

Proof. If there exists a nonnegative volume $w \in cab(V, R)$ with $b(w) \subset \{A \in V_\sigma : p(A, \mu) < \infty\}$, then the family $\{y' \circ \mu : y' \in Y', |y'| = 1\}$ is uniformly Rickart on the ring V relative to the family $b(w)$. Theorem 1 gives the existence of the control volume. Conversely, if the charge $\mu \in ab(V, Y)$ is v-continuous for a non-negative volume $v \in cab(V, R)$, then an argument similar to the one given in the proof of Theorem 2 yields

$$b(v) \subset \{A \in V_\sigma : p(A, \mu) < \infty\}.$$

Indeed, to see that the argument applies, one need only note that for a countably additive charge $\mu \in ab(V, Y)$, the semivariation $p(\cdot, \mu)$ is subadditive on the family $\{A \in V_\sigma : p(A, \mu) < \infty\}$.

When the ring V is an algebra or a σ-ring of sets, the family V_σ coincides with the family $b(v)$ for each nonnegative volume $v \in cab(V, R)$. Consequently, each countably additive, Rickart vector charge $\mu \in ab(V, Y)$ satisfies the conditions of Theorem 3.

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