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Tail Bounds for the Stable Marriage of Poisson and Lebesgue

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Abstract. Let Ξ be a discrete set in $\mathbb{R}^d$. Call the elements of Ξ *centers*. The well-known Voronoi tessellation partitions $\mathbb{R}^d$ into polyhedral regions (of varying volumes) by allocating each site of $\mathbb{R}^d$ to the closest center. Here we study allocations of $\mathbb{R}^d$ to Ξ in which each center attempts to claim a region of equal volume α.

We focus on the case where Ξ arises from a Poisson process of unit intensity. In an earlier paper by the authors it was proved that there is a unique allocation which is *stable* in the sense of the Gale–Shapley marriage problem. We study the distance X from a typical site to its allocated center in the stable allocation.

The model exhibits a phase transition in the appetite α. In the critical case $\alpha = 1$ we prove a power law upper bound on X in dimension $d = 1$. (Power law lower bounds were proved earlier for all d). In the non-critical cases $\alpha < 1$ and $\alpha > 1$ we prove exponential upper bounds on X.

1 Introduction

The following model was studied in [3]. Let $d \geq 1$. We call the elements of $\mathbb{R}^d$ *sites*. We write $|\cdot|$ for the Euclidean norm and $\mathcal{L}$ for Lebesgue measure or *volume* on $\mathbb{R}^d$. Let $\Xi \subset \mathbb{R}^d$ be a discrete set. We call the elements of Ξ *centers*. Let $\alpha \in [0,\infty]$ be a parameter, called the *appetite*. An *allocation* (of $\mathbb{R}^d$ to Ξ with appetite α) is a measurable function $\psi\colon \mathbb{R}^d \to \Xi \cup \{\infty, \Delta\}$ such that $\mathcal{L}\psi^{-1}(\Delta) = 0$, and $\mathcal{L}\psi^{-1}(\xi) \leq \alpha$ for all $\xi \in \Xi$. We call $\psi^{-1}(\xi)$ the *territory* of the center ξ. We say that ξ is *sated* if $\mathcal{L}\psi^{-1}(\xi) = \alpha$, and *unsated* otherwise. We say that a site x is *claimed* if $\psi(x) \in \Xi$, and *unclaimed* if $\psi(x) = \infty$.

The following definition of stability is an adaptation of that introduced by Gale and Shapley [2].

Definition 1.1 Let ξ be a center and let x be a site with $\psi(x) \notin \{\xi, \Delta\}$. We say that x *desires* ξ if $|x - \xi| < |x - \psi(x)|$ or x is unclaimed. We say that ξ *covets* x if

$$|x - \xi| < |x' - \xi| \text{ for some } x' \in \psi^{-1}(\xi), \text{ or } \xi \text{ is unsated.}$$

We say that a site-center pair (x, ξ) is *unstable* for the allocation ψ if x desires ξ and ξ covets x. An allocation is *stable* if there are no unstable pairs. Note that no stable allocation may have both unclaimed sites and unsated centers.

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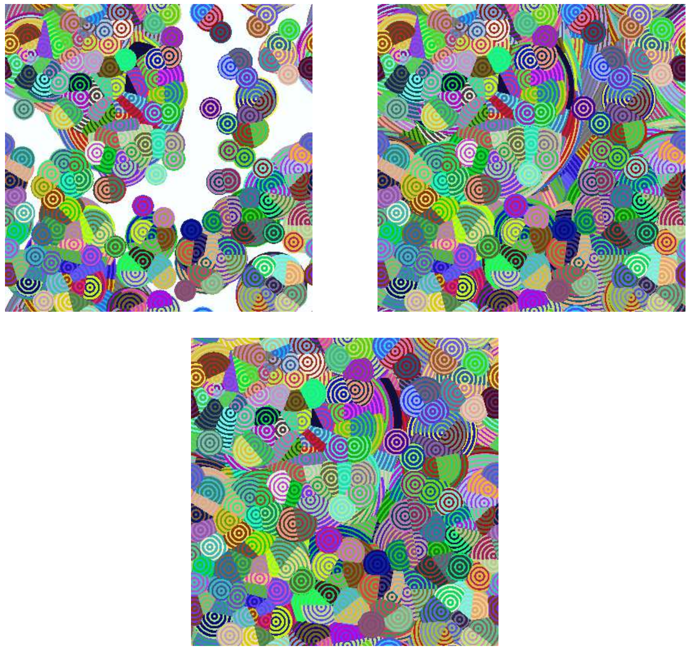

Figure 1: Stable allocations with appetites $\alpha = 0.8, 1, 1.2$. The centers are chosen uniformly at random in a 2-torus, with one center per unit area. Each territory is represented by concentric annuli in two colors.

Now let Π be a translation-invariant, ergodic, simple point process on $\mathbb{R}^d$, with intensity $\lambda \in (0,\infty)$, law $\mathbf{P}$, and expectation operator $\mathbf{E}$. Our main focus will be on the case when Π is a Poisson process of intensity $\lambda = 1$. The *support* of Π is the random set $[\Pi] = \{z \in \mathbb{R}^d : \Pi(\{z\}) = 1\}$. We consider stable allocations of the random set of centers $\Xi = [\Pi]$.

It was proved [3] that for any ergodic point process Π with intensity $\lambda \in (0,\infty)$ and any appetite $\alpha \in (0,\infty)$ there is a $\mathcal{L}$-a.e. unique stable allocation $\Psi = \Psi_\Pi$ from $\mathbb{R}^d$ to $[\Pi]$, which can be chosen to be an isometry-equivariant factor of Π. Furthermore we have the following phase transition phenomenon.

(i) If $\lambda\alpha < 1$ *(subcritical)*, then a.s. all centers are sated but there is an infinite volume of unclaimed sites.
(ii) If $\lambda\alpha = 1$ *(critical)*, then a.s. all centers are sated and $\mathcal{L}$-a.a. sites are claimed.
(iii) If $\lambda\alpha > 1$ *(supercritical)*, then a.s. not all centers are sated but $\mathcal{L}$-a.a. sites are claimed.

See Figure 1 for an illustration. For further information and more pictures see [3]. The critical model was applied in [5] to the construction of certain shift-couplings.

While the results in (i)–(iii) above suggest the subcritical/critical/supercritical terminology, the typical signature of a critical phenomenon in statistical physics is exponential decay (of correlations, cluster sizes, or large deviation probabilities) in the

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subcritical and supercritical regimes, and sub-exponential decay (usually given by a power law) at criticality. We will establish such a phenomenon for the stable allocation model when the centers are distributed as a Poisson process.

One natural quantity to consider is the distance from the origin to its center:

$$X = |\Psi(0)|,$$

where we take $X = \infty$ if 0 is unclaimed. Another natural quantity is the *radius* of the territory $\Psi^{-1}(\xi)$:

$$R(\xi) = R_\Psi(\xi) = \operatorname*{ess\,sup}_{x\in\Psi^{-1}(\xi)} |\xi - x|.$$

Suppose Π is a Poisson process. We introduce the point process Π^* with law $\mathbf{P}^*$ and expectation operator $\mathbf{E}^*$ obtained from Π by adding an extra center at the origin:

$$[\Pi^*] = [\Pi] \cup \{0\}. \tag{1.1}$$

Define the radius for a typical center thus: $R^* = R_{\Psi_{\Pi^*}}(0)$. In the subcritical and critical phases, the conditional law of X, given that it is finite, is dominated by the law of R^*; see Lemma 1.5 in the remarks below.

Theorem 1.2 **(Critical upper bound)** *Let Π be a Poisson process with intensity $\lambda = 1$. For $d = 1$ and $\alpha = 1$ we have $\mathbf{E}^*(R^*)^{1/18} < \infty$.*

Theorem 1.3 **(Non-critical upper bounds)** *Let Π be a Poisson process with intensity $\lambda = 1$.*

(i) *For all d and $\alpha > 1$ we have $\mathbf{E}e^{cX^d} < \infty$;*
(ii) *For all d and $\alpha < 1$ we have $\mathbf{E}^* e^{c(R^*)^d} < \infty$,*

for some $c = c(d, \alpha) > 0$.

We shall also prove the following, which answers a question posed by Lincoln Chayes (personal communication).

Theorem 1.4 **(Supercritical rigidity)** *Let Π be a Poisson process with intensity $\lambda = 1$, and consider the stable allocation to the process Π^*. As $\alpha \searrow 1$ we have*

$$\mathbf{P}^*(0 \text{ is an unsated center}) \to 0.$$

Remarks. In the case when Π is a Poisson process, the process Π^* defined by (1.1) is the Palm process associated with Π; it may be thought of as Π conditioned to have a center at 0. The center at 0 may be thought of as playing the role of a "typical" center in the original process Π. (The Palm process Π^* may also be defined for general point processes, but (1.1) is no longer a correct description; see [3, 6] for more information).

The following simple result relates the random variables X and R^*.

Lemma 1.5 **(Site to center comparison)** *Let Π be a Poisson process of intensity λ and suppose $\lambda\alpha \le 1$. Then for all $r \in [0, \infty)$ we have $\mathbf{P}(X > r \mid X < \infty) \le \mathbf{P}^*(R^* > r)$.*

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Thus, in the subcritical and critical phases, upper bounds for R^* yield corresponding upper bounds for X.

In particular, applying Lemma 1.5 to Theorem 1.2, we obtain the following. Let Π be a Poisson process with intensity $\lambda = 1$. Then for $d = 1$ and $\alpha = 1$ we have $\mathbf{E}X^{1/18} < \infty$. It is immediate from [3, Theorem 5(i)] that $R^* < \infty$ a.s. in all dimensions, but we have been unable to prove any quantitative upper bound on R^* or X in the critical case in dimensions $d \geq 2$.

The following lower bounds for the critical phase were proved in [3]. By Lemma 1.5, they imply the analogous lower bounds for R^*. Let Π be a Poisson process with intensity $\lambda = 1$.

(i) For $d = 1, 2$ and $\alpha = 1$ we have $\mathbf{E}X^{d/2} = \infty$.
(ii) For $d \geq 3$ and $\alpha = 1$ we have $\mathbf{E}X^{d} = \infty$.

Applying Lemma 1.5 to Theorem 1.3(ii) we obtain the following. Let Π be a Poisson process with intensity $\lambda = 1$. Then for all d and $\alpha < 1$ we have

$$\mathbf{E}(e^{cX^d}\,; X < \infty) < \infty.$$

We conjecture that $(R^*)^d$ has a finite exponential moment in the supercritical case $\alpha > 1$ also. It is straightforward to check that the exponential bounds obtained are tight up to the value of c. Indeed, denoting the ball $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$, consider the event that $B(0, r)$ contains centers lying approximately on a densely packed lattice, while $B(0, 2r) \setminus B(0, r)$ contains no centers. Such an event has probability decaying at most exponentially in r^d (for any α), and it guarantees that $X > r$ and $R^* > r$.

Our proof of Theorem 1.3 does not in general yield any explicit bound on the exponential decay constant $c(d, \alpha)$. However, such a bound is available in each of the following cases:

(i) $d \geq 1$ and $\alpha > 2^d$;
(ii) $d \geq 1$ and $\alpha < 2^{-d}$;
(iii) $d = 1$ and $\alpha \neq 1$.

For the precise statements see Propositions 4.1 and 4.2. The proofs of these results are considerably simpler than that of Theorem 1.3, and are based on standard large deviation bounds for the Poisson process.

To what extent are stable allocations robust to changes in the parameters? There are several natural ways to formulate such a question precisely. We shall prove one such formulation, Theorem 1.6 below, which, roughly speaking, states that if we change the set of centers Ξ far away from the origin, then near the origin the stable allocation ψ changes only on a small volume. This result will be a key ingredient in the proofs of Theorems 1.3 and 1.4.

In order to state Theorem 1.6 precisely, we need the following conventions (to be used only in Sections 5 and 9). We will work with various sets of centers, and we want to ensure that they have various almost sure properties enjoyed by point processes. We call an allocation ψ to a set of centers Ξ *canonical* if, for any $z \in \mathbb{R}^d$ and $\zeta \in \Xi \cup \{\infty\}$, whenever $\mathcal{L}[B(z, r) \setminus \psi^{-1}(\zeta)] = 0$ for some $r > 0$, then $\psi(z) = \zeta$. We call a set of centers Ξ *benign* if it satisfies the following contitions.

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(i) Ξ has a $\mathcal{L}$-a.e. unique stable allocation.
(ii) Ξ has a unique canonical allocation, which we denote ψ_Ξ.

By [3, Theorems 1, 3, 24], for any ergodic point process Π we know that $[\Pi]$ is almost surely a benign set. (But it appears hard to describe simple properties of $[\Pi]$ which ensure that it is benign). If Ξ is benign, then ψ_Ξ has all territories open and the unclaimed set open. Furthermore it is the unique minimizer of the set $\psi^{-1}(\Delta)$ in the class of stable allocations ψ of Ξ with those properties.

For sets of centers $\Xi_1, \Xi_2, \dots$, and Ξ we write $\Xi_n \Rightarrow \Xi$ if for any compact $K \subseteq \mathbb{R}^d$ there exists N such that for $n > N$ we have $\Xi_n \cap K = \Xi \cap K$. For allocations $\psi_1, \psi_2, \dots$, and ψ we write $\psi_n \to \psi$ a.e. if for $\mathcal{L}$-a.e. $x \in \mathbb{R}^d$ we have $\psi_n(x) \to \psi(x)$ in the one-point compactification $\mathbb{R}^d \cup \{\infty\}$.

Theorem 1.6 *(Continuity) Fix α. Let $\Xi_1, \Xi_2, \dots$, and Ξ be benign sets of centers, and write $\psi_n = \psi_{\Xi_n}$ and $\psi = \psi_\Xi$ for their canonical allocations. If $\Xi_n \Rightarrow \Xi$, then $\psi_n \to \psi$ a.e.*

We shall refer extensively to results from the companion article [3]. We adopt the convention that "Theorem I–x" refers to Theorem x of [3].

2 Site to Center Comparison

Proof of Lemma 1.5 Our proof applies in the more general context when Π is an ergodic point process of intensity $\lambda \in (0,\infty)$, and Π^* is the Palm process (see [3, 6] for more details). First note that by Theorem I–4, $\lambda\alpha \le 1$ implies that all centers are sated a.s.

We claim that for any $r \in (0,\infty)$,

$$\mathbf{P}(R(\Psi_\Pi(0)) > r \mid X < \infty) = \mathbf{P}^*(R^* > r), \tag{2.1}$$

so $R(\Psi_\Pi(0))$ conditioned on 0 being claimed is equal in distribution to R^*. Once this is proved, the result follows, because clearly $X \le R(\Psi_\Pi(0))$, $\mathbf{P}$-a.s.

We shall use the mass-transport principle (Lemma I–17). For $z \in \mathbb{Z}^d$, let $Q_z = [0,1)^d + z \subset \mathbb{R}^d$ and define

$$m(u,v) = \mathbf{E}\mathcal{L}\{x \in Q_u : \Psi_\Pi(x) \in Q_v,\ R(\Psi_\Pi(x)) > r\}.$$

Using Fubini's Theorem and translation invariance we have

$$\sum_{v\in\mathbb{Z}^d} m(0,v) = \mathbf{E}\mathcal{L}\{x \in Q_0 : \Psi_\Pi(x) \neq \infty \text{ and } R(\Psi_\Pi(x)) > r\}$$
$$= \mathbf{P}(X < \infty \text{ and } R(\Psi_\Pi(0)) > r).$$

On the other hand, since all centers are sated, and by a standard property of the Palm process (see [6, Chapter 11, (1); Lemma 11.2]),

$$\sum_{u\in\mathbb{Z}^d} m(u,0) = \alpha\mathbf{E}\#(\xi \in [\Pi] \cap Q_0 : R(\xi) > r) = \alpha\lambda\mathbf{P}^*(R^* > r).$$

Lemma I–17 states that $\sum_{v\in\mathbb{Z}^d} m(0,v) = \sum_{u\in\mathbb{Z}^d} m(u,0)$, and by Proposition I–20 we have $\mathbf{P}(X<\infty) = \alpha\lambda$, so (2.1) follows. ■

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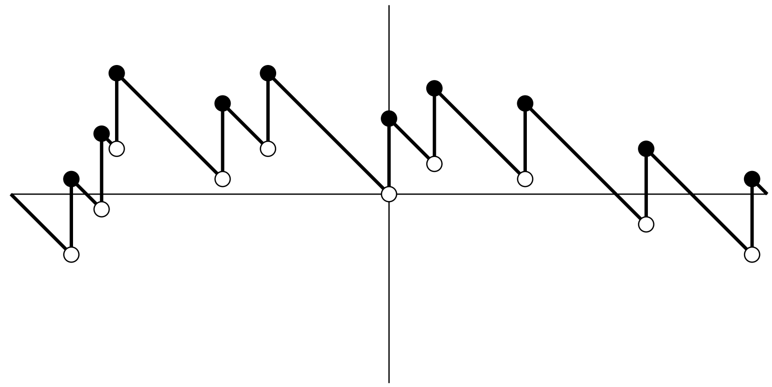

Figure 2: The function F jumps up by 1 at each center, and has slope -1 elsewhere.

3 One Dimensional Critical Bound

In this section we deduce Theorem 1.2 from a more general result. Let Π be a stationary renewal process, and let Π^* be its Palm version. Write the support $[\Pi^*] = \{\xi_j : j \in \mathbb{Z}\}$, where (ξ_j) is an increasing sequence. Thus (ξ_j) is a two-sided random walk, with $\xi_0 = 0$. We assume that the i.i.d. increments $\xi_j - \xi_{j-1}$ have mean 1 and finite variance σ^2. In the (critical) stable allocation with $\alpha = 1$, our goal is to prove a power law tail bound for $R^* = R_{\Psi_{\Pi^*}}(0)$.

Theorem 3.1 *With the assumptions above, there exists a constant $C < \infty$ that depends on the law of $\xi_j - \xi_{j-1}$, such that for all $r > 1$, $\mathbf{P}^*(R^* > r) \le Cr^{-1/17.6}$.*

Proof of Theorem 1.2 This is immediate from Theorem 3.1. ■

Given $\Xi \subseteq \mathbb{R}$, write $N(s,t] = \#(\Xi \cap (s,t])$. We introduce the function $F \colon \mathbb{R} \to \mathbb{R}$ defined by $F(0) = 1$; $F(t) - F(s) = N(s,t] - (t - s)$ for $s < t$. See Figure 2 for an illustration. Note that for all $\xi \in \Xi$, we have $F(\xi+) = F(\xi) = F(\xi-) + 1$.

We will prove that if F has certain properties, then R^* cannot be too large. On the other hand, we can analyze the behavior of F using the technology of random walks.

Proposition 3.2 **(Measure-preserving map)** *Let $\Xi \subseteq \mathbb{R}$ be a discrete set of centers and let ψ be a stable allocation to Ξ with appetite $\alpha = 1$ in which all centers are sated. For each center ξ, the restriction of F to $\psi^{-1}(\xi)$ is a measure-preserving map into $[F(\xi-), F(\xi)]$ (where on both sides the measure is $\mathcal{L}$).*

We will prove the above proposition from the following.

Lemma 3.3 *Let $\Xi \subseteq \mathbb{R}$ be a discrete set of centers and let ψ be a stable allocation to Ξ with appetite $\alpha = 1$.*

(i) *Suppose that $t > \xi$ satisfies $\psi(t) = \xi$ and $\mathcal{L}(\psi^{-1}(\xi) \cap [\xi, t]) = \beta$. Then $F(t) = F(\xi) - \beta$.*

(ii) *Suppose that $s < \xi$ satisfies $\psi(s) = \xi$ and $\mathcal{L}(\psi^{-1}(\xi) \cap [s, \xi]) = \gamma$. Then $F(s) = F(\xi-) + \gamma$.*

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Proof By symmetry, it suffices to prove (i).

Case A: Suppose $F(t) > F(\xi) - \beta$. Then $N(\xi, t] > t - \xi - \beta$, so there is a center $\eta \in (\xi, t]$ and a set of positive length of sites $s \notin (\xi, t]$ with $\psi(s) = \eta$. If $s > t$ or $\eta - s > t - \eta$, then (t, η) is an unstable pair, so we must have $s \in [2\eta - t, \xi)$ for almost every such s. But then (s, ξ) is an unstable pair.

Case B: Suppose that $F(t) < F(\xi) - \beta$. Then $N(\xi, t] < t - \xi - \beta$, so there exists a center $\eta \notin [\xi, t]$ and a set of positive length of sites $s \in (\xi, t)$ with $\psi(s) = \eta$. If $\eta < \xi$ or $\eta - s > s - \xi$, then s, ξ is an unstable pair, so we must have $\eta \in [t, 2s - \xi)$ for almost every such s. But then t, η is an unstable pair. ■

Proof of Proposition 3.2 Denote $\Gamma_+ = \psi^{-1}(\xi) \cap (\xi, \infty)$ and $\gamma_+ = \mathcal{L}(\Gamma_+)$. Lemma 3.3(i) implies that the restriction of F to Γ_+ is a monotone decreasing, measure-preserving map into $[F(\xi) - \gamma_+, F(\xi)]$. Lemma 3.3(ii) implies that the restriction of F to $\Gamma_- = \psi^{-1}(\xi) \cap (-\infty, \xi)$ is a monotone decreasing, measure-preserving map into $[F(\xi-), F(\xi-) + \gamma_-]$, where $\gamma_- = \mathcal{L}(\Gamma_-)$. Since $\gamma_- + \gamma_+ = \alpha = 1$, this completes the proof. ■

Lemma 3.4 *Under the assumptions of Proposition* 3.2*, suppose that* $0 \in \Xi$ *and that* $x > 0$ *is such that* $F(t) < 0$ *for* $t \in (x, 2x)$*. Then* $R(0) \leq x$*.*

Proof Let $r \leq x$ be maximal such that $F(r-) = 0$. Denote

$$D = \{y \in (0, r) : 0 < F(y) < 1\}.$$

Consider two cases.

Case I: There exists a site $t \in D$ such that $\psi(t) < 0$ or $\psi(t) > 2x$. In this case, since there is a center at 0, stability of the pair $(0, t)$ implies that 0 must be sated by distance r.

Case II: $\psi(D) \subseteq [0, 2x]$. Then Lemma 3.3 implies that $\psi(D) \subseteq [0, r]$. Since $F(r) = F(0-) = 0$, the upcrossings and downcrossings of $[0, 1]$ by F from 0 to r must equalize, *i.e.,*

$$\sum_{\xi \in \Xi \cap [0,r]} [(F(\xi) \wedge 1) - (F(\xi-) \vee 0)] = \mathcal{L}(D). \tag{3.1}$$

By Proposition 3.2, for every center $\xi \in [0, r]$ we have

$$(F(\xi) \wedge 1) - (F(\xi-) \vee 0) \geq \mathcal{L}(D \cap \psi^{-1}\xi),$$

and the identity in (3.1) implies that for each ξ the above inequality must be an equality. In particular, for $\xi = 0$ this shows that $R(0) \leq r \leq x$. ■

The following random walk lemma will provide the tail estimate needed to prove Theorem 3.1.

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Lemma 3.5 **(Random walk estimate)** *Let $\{X_j\}_{j\geq 1}$ be i.i.d. random variables, with mean zero and variance $\sigma^2 < \infty$. Suppose $X_j \geq -1$ a.s. Write $S_k = \sum_{j=1}^{k} X_j$ and denote $A_m := \{S_j > 1$ for all $j \in [2^{3m-1}, 2^{3m})\}$. Then $\mathbf{P}(\bigcap_{k=1}^{m} A_k^c) \leq C_1\theta^m$ for some $\theta < 8^{-1/35.2}$ and $C_1 < \infty$.*

Proof Write $D_m = \bigcap_{k=1}^{m} A_k^c$. It suffices to show that for all m sufficiently large,

$$\mathbf{P}(A_{m+1}^c \mid D_m) \leq \theta. \tag{3.2}$$

Fix $m > 1$, and denote $M := 2^{3m}$. Consider the event

$$G_m := \{\exists j \in [M, 2M] : S_j \geq -1\}.$$

We will first show that

$$\mathbf{P}(G_m \mid D_m) \geq 1 - 2^{-1/2} + o(1) \text{ as } m \to \infty. \tag{3.3}$$

It clearly suffices to show this when D_m on the right-hand side is replaced by $D_m \cap \{S_M < 0\}$. To do so, let τ be the largest integer $j \leq M$ such that $S_j \geq 0$. Denote by τ^* the index of the last maximum for the walk in $(\tau, M]$, so that $S_i \leq S_{\tau^*}$ for $i \in (\tau, \tau^*]$, and $S_{\tau^*} > S_j$ for $j \in (\tau^*, M]$. Note that $S_{\tau^*} \geq -1$, since all $X_j \geq -1$. We will derive (3.3) from the uniform estimate

$$\mathbf{P}(G_m \mid D_m, \tau^*, S_{\tau^*}) \geq 1 - 2^{-1/2} + o(1). \tag{3.4}$$

Observe that conditional on $\tau^* = \ell$, the sequence $\{S_{\ell+i} - S_\ell\}_{i=0}^{\infty}$ has the same law as the sequence $\{S_i\}_{i=0}^{\infty}$ conditioned to stay negative for the interval $i \in [1, M-\ell]$, and this also applies when we condition further on D_m and on the value of S_ℓ. By [1, Chapter XII (8.8)], $\mathbf{P}(S_i < 0 \text{ for } i \in [1,k)) = (c + o(1))k^{-1/2}$ as $k \to \infty$, and furthermore the probability is non-zero for all $k \geq 1$. Therefore,

$$\begin{aligned} \mathbf{P}\big(S_{\ell+i} < S_\ell \;\forall i \in (M-\ell, 2M-\ell] \;\big|\; S_{\ell+i} < S_\ell \;\forall i \in [1, M-\ell]\big) \\ \leq \frac{(c+o(1))(2M-\ell)^{-1/2}}{(c+o(1))(M-\ell-1)^{-1/2}} \\ \leq 2^{-1/2} + o(1) \end{aligned}$$

uniformly in ℓ as $M = 2^{3m} \to \infty$. This proves (3.4) and hence (3.3).

Next, we show that $\mathbf{P}(A_{m+1} \mid G_m \cap D_m) \geq (1/\pi)\arcsin 3^{-1/2} - o(1)$. Indeed by the strong Markov property, it suffices to show that

$$\mathbf{P}\big(A_{m+1} \;\big|\; S_j \geq -1\big) \geq (1/\pi)\arcsin 3^{-1/2} - o(1).$$

holds uniformly in $j \in [M, 2M]$. By Donsker's Theorem [6, Theorem 14.9], this, in turn, is a consequence of the following inequality for standard Brownian motion

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$B(\cdot)$. For all $u \in [1,2]$ and $b \geq 0$,

$$\begin{aligned}\mathbf{P}\Big(\min_{4\leq t\leq 8} B(t) \geq 0 \;\Big|\; B(u)=b\Big) &\geq \mathbf{P}\Big(\min_{4\leq t\leq 8} B(t)\geq 0 \;\Big|\; B(2)=0\Big)\\ &= \mathbf{P}\Big(\min_{1/3\leq t\leq 1} B(t)\geq 0\Big)\\ &= (1/\pi)\arcsin 3^{-1/2}.\end{aligned}$$

The latter follows from the arcsine law for the last zero of Brownian motion on an interval [6, Theorem 13.16].

In conclusion, we obtain (3.2), with any θ such that $1-\theta < (1-2^{-1/2})(1/\pi)\arcsin 3^{-1/2}$. This is valid if $\theta > 8^{-1/35.2}$, proving the lemma. ■

Proof of Theorem 3.1 Let $X_j = \xi_j - \xi_{j-1} - 1$ so that $S_j = \xi_j - j$. On the event A_m defined in Lemma 3.5, we have $\xi_j > j+1$ for all $j \in [M/2, M)$, where $M = 2^{3m}$. Therefore, on A_m we have $F(t) < 0$ for all $t \in [M/2, M)$, whence $R(0) < M/2$ by Lemma 3.4. Therefore $\{R(0) \geq M/2\} \subseteq \bigcap_{k=1}^{m} A_k^c$. So far, we have only considered the centers on the positive axis, and our estimates hold uniformly over the positions of centers on the negative axis. By considering the symmetrical events on the negative axis, we obtain

$$\mathbf{P}^*[R(0)\geq M/2] \leq \mathbf{P}^*(D_m)^2 \leq C_0 8^{-m/17.6} \leq C_0 M^{-1/17.6}.$$

Given any $r > 1$, we can choose m maximal so that $M/2 = 2^{3m-1} \leq r$. Since $C_0 M^{-1/17.6} \leq C r^{-1/17.6}$ for a suitable C, the theorem follows. ■

4 Explicit Exponential Bounds

In this section we prove exponential upper bounds involving explicit constants in several cases. Denote $q(x) = [x - 1 - \log(x)]/x$ so that $q(x) > 0$ for all positive $x \neq 1$. Write ω_d for the volume of the unit ball in $\mathbb{R}^d$.

***Proposition 4.1* (explicit bounds for extreme α)** *Let Π be a Poisson process on $\mathbb{R}^d$ with intensity 1.*

(i) *For any $\alpha > 2^d$ we have $\mathbf{E}e^{cX^d} < \infty$ provided $c < \omega_d q(\alpha/2^d)$.*

(ii) *For any $\alpha < 2^{-d}$ we have $\mathbf{E}^* e^{c(R^*)^d} < \infty$ provided $c < 2^d\omega_d q(\alpha 2^d)$.*

***Proposition 4.2* (explicit bounds for $d=1$)** *Let Π be a Poisson process on $\mathbb{R}$ with intensity 1. For any $\alpha \neq 1$, we have $\mathbf{E}(e^{cX}; X<\infty) < \infty$ and $\mathbf{E}^* e^{cR^*} < \infty$ provided $c < q(\alpha)$.*

(In fact, the proofs of Propositions 4.1 and 4.2 give explicit upper bounds on the tail probabilities $\mathbf{P}(X > r)$ and $\mathbf{P}^*(R^* > r)$.)

Proof of Proposition 4.1 We first note a standard large deviation estimate. If Z is a Poisson random variable with mean γ, we have:

$$\mathbf{P}(Z \geq b) \leq e^{-\gamma q(\gamma/b)} \quad \text{for } b > \gamma, \tag{4.1}$$

$$\mathbf{P}(Z \leq a) \leq e^{-\gamma q(\gamma/a)} \quad \text{for } 0 < a < \gamma. \tag{4.2}$$

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Indeed, (4.1) follows from setting $s = b/\gamma$ in $s^b\mathbf{P}(Z \geq b) \leq \mathbf{E}(s^Z) = e^{\gamma(s-1)}$, and (4.2) follows similarly from $(a/\gamma)^a\mathbf{P}(Z \leq a) \leq \mathbf{E}((a/\gamma)^Z)$. See [6, Chapter 27].

For (i), fix $\alpha > 2^d$ and let Z be the number of centers in $[\Pi] \cap B(0, r)$. Then Z is Poisson with mean $\omega_d r^d$. On the event that $Z > \omega_d r^d 2^d/\alpha$, there must be at least one center ξ in $[\Pi] \cap B(0, r)$ which is not sated within $B(0, 2r)$. Stability of the pair $(0, \xi)$ then implies that 0 must be allocated to some center no farther than ξ, whence $X \leq |\xi| < r$. Thus $\mathbf{P}(X > r) \leq \mathbf{P}(Z \leq \omega_d r^d 2^d/\alpha)$; an application of (4.2) completes the proof.

For (ii), fix $\alpha < 2^{-d}$ and let Z' be the number of centers in $[\Pi^*] \cap B(0, 2r)$. Then $Z' - 1$ is a Poisson with mean $\omega_d 2^d r^d$. On the event that $Z' < \omega_d r^d/\alpha$, there must be (a positive volume of) sites x in $B(0, r)$ that are not allocated to any center in $[\Pi] \cap B(0, 2r)$. Stability of such a site x and the center 0 implies that 0 must be sated within the closed ball $\overline{B(0, |x|)}$, whence $R^* \leq |x| < r$. Thus

$$\mathbf{P}^*(R^* > r) \leq \mathbf{P}(Z' - 1 \geq \omega_d r^d/\alpha - 1);$$

an application of (4.1) completes the proof. ■

In order to prove Proposition 4.2, it will be convenient to work with $\alpha = 1$ and $\lambda \neq 1$ and then rescale. Recall the definition of the function F from Section 3. The following states that sites are allocated to centers on the same level of F.

Lemma 4.3 *Let $\Xi \subseteq \mathbb{R}$ be a discrete set of centers and let ψ be a stable allocation to Ξ with appetite $\alpha = 1$. If $\psi(x) = \xi \in \Xi$ then $F(x) \in [F(\xi-), F(\xi)]$.*

Proof The result is immediate from Lemma 3.3, since for any interval I we have $\mathcal{L}(\psi^{-1}(\xi) \cap I) \in [0, 1]$. ■

Proof of Proposition 4.2 We start by noting the following standard large deviation estimates. If Π is a Poisson process with intensity λ on $\mathbb{R}$, then for any $r, a \geq 0$ we have

$$\mathbf{P}(\exists t \geq r : \Pi(0, t] \leq t + a) \leq \lambda^a e^{-q(\lambda)\lambda r} \quad \text{for} \lambda > 1, \tag{4.3}$$

$$\mathbf{P}(\exists t \geq r : \Pi(0, t] \geq t - a) \leq \lambda^{-a} e^{-q(\lambda)\lambda r} \quad \text{for} \lambda < 1. \tag{4.4}$$

To prove the above facts, consider the martingale $M(t) := e^{t(\lambda-1)}\lambda^{-\Pi(0,t]}$. If $\lambda > 1$, consider the stopping time $\tau = \inf\{t \geq r : \Pi(0, t] \leq t + a\}$. On the event $\tau = t$, where $t < \infty$, we have $M(\tau) \geq e^{t(\lambda-1)}\lambda^{-(t+a)} = \lambda^{-a}e^{q(\lambda)\lambda t}$. Hence applying the optional stopping theorem to $\tau \wedge N$ yields $1 = \mathbf{E}M(\tau \wedge N) \geq \mathbf{P}(\tau < N)\lambda^{-a}e^{q(\lambda)\lambda r}$, and taking $N \to \infty$ yields (4.3). For (4.4) we apply similar reasoning to

$$\tau' = \inf\{t \geq r : \Pi(0, t] \geq t - a\}.$$

Now we prove exponential bounds on X and R^* in the case when $d = 1$, $\alpha = 1$, and $\lambda \neq 1$; then we will rescale $\mathbb{R}$.

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First, let $\Xi = [\Pi]$. By Lemma 4.3, on the event that $r < X < \infty$, there exists some center $\xi \in [\Pi] \setminus [-r, r]$ with $F(\xi) \in [0, 1]$. Recalling the definition of F, taking $t = |\xi|$ and using (4.3) and (4.4), we therefore obtain

$$\begin{aligned}\mathbf{P}(r < X < \infty) &\le \mathbf{P}\big(\exists t > r : \Pi(0, t] \in [t-1, t]\big)\\ &\qquad + \mathbf{P}\big(\exists t > r : \Pi(-t, 0] \in [t-1, t]\big)\\ &\le 2(1 \vee \lambda^{-1})e^{-q(\lambda)\lambda r}.\end{aligned}$$

Secondly, let $\Xi = [\Pi^*]$. By Lemma 4.3, on the event that $R^* > r$, there exists $x \in \mathbb{R} \setminus [-r, r]$ with $F(x) \in [0, 1]$, so we obtain similarly

$$\begin{aligned}&\mathbf{P}^*(R^* > r)\\ &\quad \le \mathbf{P}^*(\exists t > r : \Pi^*(0, t] \in [t-1, t]) + \mathbf{P}^*(\exists t > r : \Pi^*(-t, 0] \in [t-1, t])\\ &\quad \le 2(1 \vee \lambda^{-2})e^{-q(\lambda)\lambda r}.\end{aligned}$$

Finally, rescaling $\mathbb{R}$ by a factor of λ changes the intensity to 1 and the appetite to λ, while scaling X and R^* by a factor of λ. Thus we obtain the desired results. ■

5 Continuity

Recall the continuity result, Theorem 1.6, stated in the introduction. In this section we deduce some consequences which will be used in the proofs of Theorems 1.3 and 1.4. The proof of Theorem 1.6 is deferred until the end of the article.

We shall apply Theorem 1.6 as follows. Roughly speaking, given an almost sure local property of stable allocations, we may find some large box such that with high probability the property holds throughout the box, whatever the configuration of centers outside. More precisely, we apply this to the notions of replete sets and decisive sets as described below.

In what follows we take $\alpha = 1$, and Π to be a Poisson process of intensity λ, with associated probability measure and expectation $\mathbf{P}_\lambda$, $\mathbf{E}_\lambda$. Lemma 5.1 and Corollary 5.2 below apply to the critical and subcritical models, that is, to $\lambda \le 1$. The critical case will be used to prove Theorem 1.4 and the subcritical case will be used to prove Theorem 1.3(ii).

Recall that ψ_Ξ denotes the canonical allocation of the benign set of centers Ξ. Given a benign set $\Xi \subseteq \mathbb{R}^d$ and a measurable $A \subseteq \mathbb{R}^d$, let Ξ'_A be a random set of centers which is the union of $\Xi \cap A$ and a Poisson process of intensity λ in $\mathbb{R}^d \setminus A$. Write $\mu_{\lambda,A}$ for the law of Ξ'_A. For $\xi \in \Xi$, say that A is Ξ-*replete* for ξ if for every $\lambda \in (0, \infty)$ we have for $\mu_{\lambda,A}$-a.e. Ξ'_A that Ξ'_A is benign and $\mathcal{L}(\psi^{-1}_{\Xi'_A}(\xi) \cap A) = 1$, that is, ξ is sated within A whatever happens outside A.

Define the box $Q(L) = [-L, L)^d$.

Lemma 5.1 *Let $\alpha = 1$ and let Π be a Poisson process of intensity $\lambda \le 1$. Let G be the event that for every $\xi \in [\Pi]$ there exists $L < \infty$ such that $\xi + Q(L)$ is $[\Pi]$-replete for ξ. Then $\mathbf{P}_\lambda(G) = 1$.*

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***Corollary 5.2* (Replete boxes)** *Let $\alpha = 1$ and let Π be a Poisson process of intensity $\lambda \le 1$. For any $\epsilon > 0$ there exists M such that*

$$\mathbf{E}_\lambda \#\{\xi \in [\Pi] \cap Q(M) : Q(M) \textit{ is not } [\Pi]\textit{-replete for } \xi\} < \epsilon (2M)^d.$$

Now given benign Ξ, we say that a measurable set $A \subseteq \mathbb{R}^d$ is Ξ-*decisive* for a site x if for every $\lambda \in (0, \infty)$ we have $\mu_{\lambda,A}$-a.s. that Ξ'_A is benign and $\psi_{\Xi'_A}(x) = \psi_\Xi(x)$. (That is, if $\psi(x)$ can be determined by looking only at $\Xi \cap A$). Note that if A is Ξ-decisive for x, then $\psi_\Xi(x)$ cannot be a center outside A.

The supercritical case below will be used to prove Theorem 1.3(i).

Lemma 5.3 *Let $\alpha = 1$ and let Π be a Poisson process of intensity $\lambda \ge 1$. Then $\mathbf{P}_\lambda$-a.s. there exists $L < \infty$ such that $Q(L)$ is $[\Pi]$-decisive for 0.*

***Corollary 5.4* (Decisive boxes)** *Let $\alpha = 1$ and let Π be a Poisson process of intensity $\lambda \ge 1$. For any $\epsilon > 0$ there exists $M < \infty$ such that*

$$\mathbf{E}_\lambda \mathcal{L}[x \in Q(M) : Q(M) \textit{ is not } [\Pi]\textit{-decisive for } x] < \epsilon (2M)^d.$$

Next we turn to the proofs of the four results above.

Lemma 5.5 *Suppose $\Xi_n \Rightarrow \Xi$ and $\psi_n \to \psi$ a.e. are as in Theorem 1.6. If there is a set A of positive volume such that every $z \in A$ desires ξ under ψ, then for n sufficiently large, ξ is sated in ψ_n, and*

$$\limsup_{n\to\infty} R_{\psi_n}(\xi) \le \operatorname*{ess\,inf}_{z\in A} |z - \xi|, \quad (< \infty).$$

Proof As the set A has positive volume, Theorem 1.6 implies that there exists $z \in A$ such that $\psi_n(z) \to \psi(z)$. Thus for n sufficiently large, z desires ξ under ψ_n. By stability ξ does not covet z, and the result follows. ■

Proof of Lemma 5.1 On G^c, there exists a center $\xi \in [\Pi]$ such that for each L there is a benign set of centers Ξ_L agreeing with $[\Pi]$ on $\xi + Q(L)$ and satisfying

$$\mathcal{L}\big(\psi_{\Xi_L}^{-1}(\xi) \cap (\xi + Q(L))\big) < 1 \tag{5.1}$$

for each L. Suppose that $\mathbf{P}_\lambda(G^c) > 0$ and write $\psi_L = \psi_{\Xi_L}$. Since $\Xi_L \Rightarrow [\Pi]$, Theorem 1.6 implies that $\psi_L \to \psi_{[\Pi]}$ a.e. Furthermore, Lemma 5.5 applies to ξ (by Theorem I–4(i) if $\lambda < 1$ or Theorem I–6(i) if $\lambda = 1$), so almost surely for L sufficiently large ξ is sated in each ψ_L, and the radii $R_{\psi_L}(\xi)$ are bounded as $L \to \infty$. This contradicts (5.1). We conclude that $\mathbf{P}_\lambda(G^c) = 0$. ■

Proof of Corollary 5.2 For $A \subseteq \mathbb{R}^d$, let $\Pi^L(A)$ denote the number of $\xi \in [\Pi] \cap A$ such that $\xi + Q(L)$ is not $[\Pi]$-replete for ξ. Lemma 5.1 and the monotone convergence theorem imply that $\mathbf{E}_\lambda \Pi^L(Q(1)) \to 0$ as $L \to \infty$. Thus we can choose an $L < \infty$ so that the translation-invariant point process Π^L has intensity less than $\epsilon/2$. Observe

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that for $M > L$ and $\xi \in [\Pi] \cap Q(M-L)$, if $Q(M)$ is not $[\Pi]$-replete for ξ, then $\xi \in [\Pi^L]$. Therefore

$$\mathbf{E}_\lambda \#\{\xi \in [\Pi] \cap Q(M) : Q(M) \text{ is not } [\Pi]\text{-replete for } \xi\} \\ < (\epsilon/2)(2M-2L)^d + (2M)^d - (2M-2L)^d,$$

which is smaller than $\epsilon(2M)^d$ if M is sufficiently large. ■

In order to prove Lemma 5.3 we need the following enhancement of Theorem 1.6, in which we (partially) specify the set on which a.e. convergence occurs. The proof is deferred until the end of the article.

Proposition 5.6 *Suppose $\Xi_n \Rightarrow \Xi$ and $\psi_n \to \psi$ a.e. are as in Theorem* 1.6. *If $\psi(z) = \xi \in \Xi$ and z is not equidistant from any two centers of Ξ, then $\psi_n(z) \to \xi$.*

Proof of Lemma 5.3 Since $\lambda \geq 1$, by Theorem I–4, 0 is claimed a.s. And a.s. 0 is not equidistant from any two centers.

Now on the complement of the event in the lemma, for each L there exists a benign Ξ_L agreeing with $[\Pi]$ on $Q(L)$ such that $\psi_{\Xi_L}(0) \neq \psi_{[\Pi]}(0)$.

But by Proposition 5.6, when all the events mentioned above occur, we have $\psi_{\Xi_L}(0) \to \psi_{[\Pi]}(0)$ as $L \to \infty$, a contradiction. ■

Proof of Corollary 5.4 Fix $\epsilon > 0$. Let U^L be the (random) set of sites x for which $Q(L) + x$ is not $[\Pi]$-decisive. Then the process U^L is translation-invariant in law, and by Lemma 5.3, we may fix L large enough so that it has intensity less than $\epsilon/2$. Now if M is sufficiently large, then $\mathbf{E}_\lambda \mathcal{L}[U^L \cap Q(M)] < (\epsilon/2)(2M)^d$, whence

$$\mathbf{E}_\lambda \mathcal{L}[x \in Q(M) : Q(M) \text{is not } [\Pi]\text{-decisive for } x] \\ < (\epsilon/2)(2M)^d + (2M)^d - (2M-2L)^d,$$

which is less than $\epsilon(2M)^d$ if M is sufficiently large. ■

6 Supercritical Rigidity

In this section we prove Theorem 1.4.

Lemma 6.1 **(Coupling)** *For any set $A \subseteq \mathbb{R}^d$ of finite volume and any $\delta > 0$ there exist $\lambda > 1$ and a coupling (Π_1, Π_λ) of two Poisson processes of respective intensities* $1, \lambda$, *such that* $\mathbf{E}[\Pi_\lambda(A)\,;\,\Pi_\lambda \neq \Pi_1 \text{ on } A] < \delta$.

Proof We take $\Pi_\lambda = \Pi_1 + \Pi_\beta$ where Π_1, Π_β are independent Poisson processes of intensities $1, \beta$ with $1 + \beta = \lambda$. Then we have

$$\begin{aligned} \mathbf{E}[\Pi_\lambda(A) : \Pi_\lambda \neq \Pi_1 \text{ on } A] &= \mathbf{E}[\Pi_1(A) + \Pi_\beta(A) : \Pi_\beta(A) > 0] \\ &= \mathbf{E}[\Pi_1(A)]\mathbf{P}[\Pi_\beta(A) > 0] + \mathbf{E}[\Pi_\beta(A)] \\ &= (\mathcal{L}A)(1 - e^{-\beta \mathcal{L} A}) + \beta \mathcal{L} A \to 0 \text{ as } \beta \to 0. \end{aligned}$$ ■

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Proof of Theorem 1.4 By rescaling $\mathbb{R}^d$, the required result is equivalent to the same limiting statement as $\lambda \searrow 1$ with $\alpha = 1$.

Given any $\epsilon > 0$, choose M by Corollary 5.2 so that, writing $\Pi_1 = \Pi$,

$$\mathbf{E}_1\#\{\xi \in [\Pi_1] \cap Q(M) : Q(M) \text{ is not } [\Pi_1]\text{-replete for } \xi\} < \epsilon(2M)^d. \tag{6.1}$$

Then choose λ and a coupling (Π_1, Π_λ) by Lemma 6.1 so that

$$\mathbf{E}\Big(\#([\Pi_\lambda] \cap Q(M))\ ;\ \Pi_\lambda \neq \Pi_1 \text{ on } Q(M)\Big) < \epsilon(2M)^d. \tag{6.2}$$

Note that if $\Xi = \Xi'$ on $Q(M)$, then $Q(M)$ is Ξ'-replete for a center if and only if it is Ξ-replete. Therefore, using (6.1) and (6.2),

$$\begin{aligned}
&\mathbf{E}_\lambda\#\{\xi \in [\Pi_\lambda] \cap Q(M) : Q(M) \text{ is not } [\Pi_\lambda]\text{-replete for } \xi\}\\
&\quad\leq \mathbf{E}\Big(\#\big\{\,\xi \in [\Pi_1] \cap Q(M) : Q(M) \text{ is not } [\Pi_1]\text{-replete for } \xi\big\}\,;\\
&\qquad\qquad\qquad \Pi_1 = \Pi_\lambda \text{ on } Q(M)\Big)\\
&\qquad\quad + \mathbf{E}\Big(\#([\Pi_\lambda] \cap Q(M))\ ;\ \Pi_1 \neq \Pi_\lambda \text{ on } Q(M)\Big)\\
&\quad\leq 2\epsilon(2M)^d.
\end{aligned}$$

So in particular $\mathbf{E}_\lambda\#(\xi \in [\Pi_\lambda] \cap Q(M)\ ;\ \xi \text{ is unsated}) \leq 2\epsilon(2M)^d$, and therefore since Π^* is the Palm process, $\mathbf{P}^*_\lambda(0 \text{ is unsated}) \leq 2\epsilon$. ∎

7 Supercritical Bound

In this section we prove Theorem 1.3(i). Let $\alpha = 1$ and let Π be a Poisson process of rate λ with law $\mathbf{P}_\lambda$.

Theorem 7.1 *Let $\alpha = 1$ and let Π be a Poisson process of intensity λ. For any $\lambda > 1$ there exist $C, c \in (0, \infty)$ such that for all $r > 0$, $\mathbf{P}_\lambda(X > r) < Ce^{-cr^d}$.*

Proof of Theorem 1.3(i) By rescaling $\mathbb{R}^d$, the required result is equivalent to the same statement with $\alpha = 1$ and $\lambda > 1$, and this is immediate from Theorem 7.1. ∎

Proof of Theorem 7.1 First observe that if

$$\text{there exists } \xi \in [\Pi] \cap B(0, r) \text{ with } \mathcal{L}[\Psi^{-1}(\xi) \cap B(0, 2r)] < 1, \tag{7.1}$$

then $X \leq r$. This is because ξ must covet some $z \notin B(0, 2r)$, so $|\xi - z| > r$; but $|0 - \xi| < r$, so $(0, \xi)$ would be unstable if $X > r$.

So it is enough to show that the probability that (7.1) fails decays exponentially in r^d. Given λ, let

$$\epsilon = \frac{\lambda - 1}{10 \cdot 2^d} \wedge 1,$$

and let $M = M(\lambda, \epsilon)$ be as in Corollary 5.4. Note that ϵ and M do not depend on r.

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Now for any $r > 0$ we tile the shell $B(0, 2r) \setminus B(0, r)$ with disjoint copies of the box $Q(M)$. Recall that $Q(M) = [-M, M)^d$. For $z \in \mathbb{Z}^d$ write $Q_z = Q(M) + 2Mz$, and define the random variable

$$Y_z = \mathcal{L}(x \in Q_z : Q_z \text{ is not } [\Pi]\text{-decisive for } x).$$

Let $I = I(r) = \{z \in \mathbb{Z}^d : Q_z \subseteq B(0, 2r) \setminus B(0, r)\}$ be the index set of the boxes lying entirely in the shell, and let $S = S(r) = [B(0, 2r) \setminus B(0, r)] \setminus \bigcup_{z \in I} Q_z$ be the remainder of the shell.

Observe that if r is sufficiently large, then

$$\mathcal{L}S < \epsilon \mathcal{L}B(0, r). \tag{7.2}$$

Also consider the events

$$E = \Big\{ \Pi(B(0, r)) > (\lambda - \epsilon)\mathcal{L}B(0, r) \Big\}, \quad G = \Big\{ \sum_{z \in I} Y_z < 4\epsilon 2^d \mathcal{L}B(0, r) \Big\}.$$

We claim that if E and G occur and (7.2) holds, then (7.1) is satisfied. To verify this claim, note that given those assumptions,

$$\begin{aligned} \mathcal{L}[x \in B(0, 2r) : \Psi(x) \in B(0, r)] &\leq \sum_{z \in I} Y_z + \mathcal{L}B(0, r) + \mathcal{L}S \\ &\leq (4\epsilon 2^d + 1 + \epsilon)\mathcal{L}B(0, r) \\ &< (\lambda - \epsilon)\mathcal{L}B(0, r) \\ &< \Pi(B(0, r)). \end{aligned}$$

(Here the third inequality holds because by the choice of ϵ we have $4\epsilon 2^d + 2\epsilon < 10\epsilon 2^d \leq \lambda - 1$.) Then, recalling that $\alpha = 1$, we see that (7.1) must indeed hold.

Finally, we must show that $\mathbf{P}(E^C)$ and $\mathbf{P}(G^C)$ each decay at least exponentially in r^d as $r \to \infty$. For E^C this is a standard large deviations bound since $\Pi(B(0, r))$ is Poisson with mean $\lambda \mathcal{L}B(0, r) = \Theta(r^d)$ as $r \to \infty$. Turning to G^C, note that the random variables $(Y_z)_{z \in I}$ are i.i.d. with mean less than $\epsilon(2M)^d$, by Corollary 5.4. We have $\#I = \Theta(r^d)$, while $(\#I)(2M)^d \leq \mathcal{L}B(0, 2r) \leq 2 \cdot 2^d \mathcal{L}B(0, r)$, and hence

$$\mathbf{E}_\lambda\Big(\sum_{z \in I} Y_z\Big) \leq (\#I)\epsilon(2M)^d \leq 2\epsilon 2^d \mathcal{L}B(0, r).$$

Furthermore, each random variable Y_z is bounded by $(2M)^d$. Therefore by the Chernoff bound [6, Corollary 27.4], $\mathbf{P}(G^C)$ decays exponentially in r^d. ■

8 Subcritical Bound

In this section we prove Theorem 1.3(ii) via the following theorem.

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Theorem 8.1 *Let $\alpha = 1$ and let Π be a Poisson process of intensity λ. For any $\lambda < 1$ there exist $C, c > 0$ such that for all $r > 0$,*

$$\mathbf{P}_\lambda\big(\exists \xi \in [\Pi] \cap B(0,1) \text{ such that } R(\xi) > r\big) < Ce^{-cr^d}.$$

Proof of Theorem 1.3(ii) First note that, by rescaling $\mathbb{R}^d$, it suffices to prove the same statement for $\alpha = 1$ and $\lambda < 1$. Let C, c be as in Theorem 8.1. Let Y be the number of centers $\xi \in [\Pi] \cap B(0,1)$ with $R(\xi) > r$, and note that by a standard property of the Palm process, $\mathbf{E}(Y) = \lambda \mathcal{L}B(0,1)\mathbf{P}^*(R^* > r)$, so it is enough to prove that $\mathbf{E}(Y)$ decays exponentially in r^d. Let $u = e^{cr^d/2}$. Then note that

$$\begin{aligned}\mathbf{E}(Y) &= \mathbf{E}(Y : 0 < Y \le u) + \mathbf{E}(Y : Y > u)\\ &\le u\mathbf{P}(Y > 0) + \mathbf{E}[\Pi(B(0,1)) : \Pi(B(0,1)) > u].\end{aligned}$$

From Theorem 8.1 we have $\mathbf{P}(Y > 0) \le Ce^{-cr^d}$, while the second term is bounded above by $\mathbf{E}(\Pi(B(0,1))^2)/u$. Thus both terms decay exponentially in r^d, hence so does $\mathbf{E}$(Y). ∎

Proof of Theorem 8.1 Fix $\lambda < 1$. First observe that if

(8.1) there exists $y \in B(0,r)$ with $\Psi(y) \notin B(0, 2r+1)$

then $R(\xi) < r+1$ for all $\xi \in [\Pi] \cap B(0,1)$. This is because otherwise we would have $|y - \xi| < r+1$ and $|y - \Psi(y)| > r+1$, and so (y, ξ) would be unstable.

So it is enough to show that the probability that (8.1) fails decays exponentially in r^d. Let

$$\epsilon = \frac{1-\lambda}{10 \cdot 2^d},$$

and let $M = M(\lambda, \epsilon)$ be as in Corollary 5.2. Note that ϵ and M do not depend on r.

Now for any $r > 0$ we tile the shell $B(0, 2r+1) \setminus B(0,r)$ with disjoint copies of the box $Q(M)$. For $z \in \mathbb{Z}^d$ write $Q_z = Q(M) + 2Mz$, and define the random variable

$$W_z = \#(\xi \in [\Pi] \cap Q_z : Q_z \text{ is not } [\Pi]\text{-replete for } \xi).$$

Let $I = I(r) = \{z \in \mathbb{Z}^d : Q_z \subseteq B(0, 2r+1) \setminus B(0,r)\}$ be the index set of the boxes lying entirely in the shell, and let $S = S(r) = [B(0, 2r+1) \setminus B(0,r)] \setminus \bigcup_{z \in I} Q_z$ be the remainder of the shell.

Consider the events

$$E = \big\{\Pi(B(0,r)) < (\lambda + \epsilon)\mathcal{L}B(0,r)\big\}, \quad F = \big\{\Pi(S) < \epsilon \mathcal{L}B(0,r)\big\},$$
$$G = \Big\{\sum_{z \in I} W_z < 4\epsilon 2^d \mathcal{L}B(0,r)\Big\}.$$

We claim that if E, F, and G all occur, then (8.1) is satisfied. To verify this claim, recall that $\alpha = 1$, so that on E we have

$$\begin{aligned}\mathcal{L}\{y \in B(0,r) : \psi(y) \notin B(0,r)\} &\ge \mathcal{L}B(0,r) - (\lambda + \epsilon)\mathcal{L}B(0,r)\\ &\ge 9\epsilon 2^d \mathcal{L}B(0,r).\end{aligned}$$

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(The second inequality holds because $9\epsilon 2^d + \epsilon \le 10\epsilon 2^d = 1 - \lambda$ by the choice of ϵ). On F we clearly have $\mathcal{L}\{y \in B(0,r) : \psi(y) \in S\} < \epsilon \mathcal{L}B(0,r)$, while on G, by the definition of replete we have

$$\mathcal{L}\{y \in B(0,r) : \psi(y) \in \bigcup_{z\in I} Q_z\} < 4\epsilon 2^d \mathcal{L}B(0,r).$$

Therefore since $B(0,2r+1) = S \cup B(0,r) \cup \bigcup_{z\in I} Q_z$, on $E \cap F \cap G$, we have

$$\mathcal{L}\{y \in B(0,r) : \psi(y) \notin B(0,2r+1)\} \ge (9\epsilon 2^d - 4\epsilon 2^d - \epsilon)\mathcal{L}B(0,r) > 0,$$

establishing the claim.

Finally, we must show that $\mathbf{P}(E^C), \mathbf{P}(F^C), \mathbf{P}(G^C)$ each decay at least exponentially in r^d as $r \to \infty$. For E^C this is a standard large deviations bound since $\#([\Pi] \cap B(0,r))$ is Poisson with mean $\lambda \mathcal{L}B(0,r) = \Theta(r^d)$. For F^C it also follows from the standard large deviations bound on noting that $\mathcal{L}S < (\epsilon/2)\mathcal{L}B(0,r)$ for r sufficiently large. Turning to G^C, note that the random variables $(W_z)_{z\in I}$ are i.i.d. with mean less than $\epsilon(2M)^d$, by Corollary 5.2. We have $\#I = \Theta(r^d)$, while

$$(\#I)(2M)^d \le \mathcal{L}B(0,2r+1) \le 2 \cdot 2^d \mathcal{L}B(0,r),$$

and hence

$$\mathbf{E}_\lambda\Big(\sum_{z\in I} W_z\Big) \le (\#I)\epsilon(2M)^d \le 2\epsilon 2^d \mathcal{L}B(0,r).$$

Furthermore, we have $W_z \le \Pi(Q_z)$ so each random variable W_z has exponentially decaying tails. Therefore by the Chernoff bound [6, Corollary 27.4], $\mathbf{P}(G^C)$ decays exponentially in r^d. ■

9 Proofs of Continuity Results

Proof of Theorem 1.6 We can find a countable dense set $X \subseteq \mathbb{R}^d$ such that $\psi_n(x) \ne \Delta$ for each $x \in X$ and for all n. We can choose a subsequence (n_j) such that $\psi_{n_j}(x)$ converges in the compact space $\Xi \cup \infty$ for all $x \in X$. We define the map ψ_∞ by $\psi_\infty(z) = \lim_{j\to\infty} \psi_{n_j}(z)$ for all z where the limit exists. Thus ψ_∞ exists on X and perhaps elsewhere.

We define $\widetilde{R}_\infty(\xi) = \sup\big\{ |x - \xi| : x \in X \text{ and } \psi_\infty(x) = \xi \big\}$. Let

$$Z = \bigcup_\xi \big\{ w \in \mathbb{R}^d : |w - \xi| = \widetilde{R}_\infty(\xi) \big\} \cup \bigcup_{\xi \ne \xi'} \big\{ w \in \mathbb{R}^d : |w - \xi| = |w - \xi'| \big\},$$

where the first and second unions are over all centers and all pairs of centers in Ξ respectively. And let $D = \bigcup_n \psi_n^{-1}(\Delta)$. The sets Z and D are $\mathcal{L}$-null a.s.

For $z \in \mathbb{R}^d$ let

$$S(z) = \big\{ \xi \in \Xi \cup \infty : \ \exists x_1, x_2, \ldots \in X \text{ such that } x_j \to z \text{ and } \psi_\infty(x_j) \to \xi \big\}.$$

By the compactness of $\Xi \cup \infty$, for any z the set $S(z)$ is not empty. We make the following claim.

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Claim 1 If $z \notin Z \cup D$, then $\psi_\infty(z)$ exists and $S(z) = \{\psi_\infty(z)\}$.

To prove this, we take $z \notin Z \cup D$ and consider two cases.

Case I. Suppose that $\xi \in S(z) \setminus \{\infty\}$. Since $\xi \in S(z)$, we have that $|\xi - z| \le \widetilde{R}_\infty(\xi)$, and as $z \notin Z$, we deduce $|\xi - z| < \widetilde{R}_\infty(\xi)$. Hence we can pick $x \in \psi_\infty^{-1}(\xi) \cap X$ such that $|x - \xi| > |z - \xi|$. Since $\psi_\infty(x)$ exists, there is N such that we have $\psi_{n_j}(x) = \xi$ for all $n_j > N$, so ξ covets z under ψ_{n_j}. Since (z, ξ) is stable for ψ_{n_j}, we deduce that $|\psi_{n_j}(z) - z| \le |z - \xi|$ for all $n_j > N$.

Label $\Xi \cap \overline{B(z, |z - \xi|)} = \{\xi_1, \xi_2, \dots, \xi_\ell\}$ in such a way that

$$|z - \xi_1| < |z - \xi_2| < \cdots < |z - \xi_\ell|.$$

This is possible as $z \notin Z$. (Note that $\xi = \xi_\ell$). Furthermore since $z \notin Z$ there exists $r > 0$ such that for all $y \in B(z, r)$,

- $r < \min_i \bigl| \widetilde{R}_\infty(\xi_i) - |z - \xi_i| \bigr|$,
- $|y - \xi_1| < |y - \xi_2| < \cdots < |y - \xi_\ell|$,
- $|y - \eta| > |y - \xi|$ for all $\eta \in \Xi \setminus \{\xi_1, \dots, \xi_\ell\}$.

We will show that for $\mathcal{L}$-a.e. $y \in B(z, r)$ we have $\psi_{n_j}(y) \to \xi$.

Let $L = \min\{i : \widetilde{R}_\infty(\xi_i) > |z - \xi_i|\}$. We first show that $\psi_{n_j}(y) \to \xi_L$ for all $y \in B(z, r) \setminus D$. By the definition of r there exists $w \in X$ and N such that for all $n_j > N$, $\psi_{n_j}(w) = \xi_L$ and

$$|z - \xi_L| + r < |w - \xi_L|. \tag{9.1}$$

For $n_j > N$ and for every $y \in B(z, r)$ with $\psi_{n_j}(y) \ne \Delta$, from the stability of (y, ξ_L) under ψ_{n_j}, and by (9.1) we have $|y - \psi_{n_j}(y)| < |y - \xi_L| \le |w - \xi_L|$. Therefore, for all $y \in B(z, r) \setminus D$ we have $\psi_{n_j}(y) = \xi_{i_j}$ for $n_j > N$, where $i_j = i_j(y) \le L$. Our next task is to show that in fact $i_j < L$ is impossible for j sufficiently large.

Suppose on the contrary that there exists $I < L$ and a subsequence (n_{j_k}) and sites (y_{j_k}) such that for all k, $\psi_{n_{j_k}}(y_{j_k}) = \xi_I$ and $|y_{j_k} - \xi_I| > |z - \xi_I| - r$. Then there exists $u \in X \cap B(z, r)$ such that for all k $|u - \xi_I| < |y_{j_k} - \xi_I|$. Since $u \in X$, the sequence $\psi_{n_{j_k}}(u)$ converges to some ξ_i. By stability of (u, ξ_I) under $\psi_{n_{j_k}}$ and by the choice of r we must have $i \le I < L$. Thus $\widetilde{R}_\infty(\xi_i) \ge |u - \xi_i| > |z - \xi_i| - r$. By the choice of r, the previous line implies $\widetilde{R}_\infty(\xi_i) \ge |z - \xi_i| + r$. This contradicts the definition of L, so there is no $I < L$ as described.

We have shown that for all $y \in B(z, r) \setminus D$ the sequence $\psi_{n_j}(y)$ converges to the same center ξ_L, and since $\xi \in S(z)$, this center must be ξ. Since $z \notin D$, we have that $\psi_\infty(z) = \xi$. Hence we have proved the claim in Case I.

Case II. Suppose $S(z) \cap \Xi = \varnothing$; then $S(z) = \{\infty\}$, and we want to show that $\psi_\infty(z) = \infty$. We work by contradiction. Suppose there exists $\xi \in \Xi$ and a subsequence (n_{j_k}) such that $\psi_{n_{j_k}}(z) \to \xi$. Then there exists $r > 0$ such that for all $x \in X \cap B(z, r)$, $|\psi_\infty(x) - x| > |\xi - z|$. As $z \notin \Xi$ we may further choose $x \in X \cap B(z, r)$ such that $|x - \xi| < |z - \xi|$. Then there exists j_k such that (x, ξ) is an unstable pair for $\psi_{n_{j_k}}$. Hence we have proved the claim in Case II also.

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We have proved that ψ_∞ is defined almost everywhere. It is straightforward to show that if $\Xi_n \Rightarrow \Xi$ and $\psi_n \to \psi_\infty$, then ψ_∞ is a stable allocation to Ξ (the main step is to show that $\psi^{-1}(\xi) = \liminf \psi_n^{-1}(\xi) = \limsup \psi_n^{-1}(\xi)$ a.e.). Since Ξ is benign it has an a.e. unique stable allocation, so ψ_∞ must agree with ψ a.e. Thus we have $\psi_{n_j} \to \psi$ a.e.

Finally we prove convergence of the entire sequence. We claim that for all $z \in \mathbb{R}^d \setminus D$ satisfying $\psi(z) \neq \Delta$ and

$$|z-\xi| \neq |z-\xi'| \text{ for all } \xi \neq \xi' \in \Xi, \tag{9.2}$$

we have $\psi_n(z) \to \psi(z)$. Suppose this does not hold for some z where $\psi(z) = \zeta \in \Xi \cup \{\infty\}$, say. Then there exists (n_j) such that

$$\psi_{n_j}(z) \neq \zeta \text{ for all } n_j. \tag{9.3}$$

Also since ψ is a canonical allocation, we have

$$\psi(y) = \zeta \text{ for all } y \text{ in a neighborhood of } z. \tag{9.4}$$

We will deduce a contradiction.

First suppose $\zeta \neq \infty$. As before, using (9.2) we can label $\Xi \cap \overline{B(z, |z-\xi|)} = \{\xi_1, \xi_2, \dots, \xi_\ell\}$ with $\xi_\ell = \zeta$ and choose $r > 0$ so that for all $y \in B(z,r)$ we have

$$|y-\xi_1| < |y-\xi_2| < \cdots < |y-\xi_\ell|. \tag{9.5}$$

By (9.4) and the subsequential convergence proved earlier there exist $x_0, \dots, x_\ell \in B(z,r)$ and a subsequence (n_{j_k}) such that

(i) $\psi_{n_{j_k}}(x_i) = \zeta$ for all j_k and $i = 0, \dots, \ell$,
(ii) $|x_i - \xi_i| < |z - \xi_i|$ for $i = 1, \dots, \ell$,
(iii) $|x_0 - \zeta| > |z - \zeta|$.

By (i) ($i = 0$), (iii) and stability we have for all j_k, $|\psi_{n_{j_k}}(z) - z| \leq |\zeta - z|$. By (i) ($i > 0$), (ii), (9.5) and stability we have

$$\psi_{n_{j_k}}(z) \neq \xi_i \text{ for any } i = 1, \dots, \ell - 1.$$

Thus for all j_k we have $\psi_{n_{j_k}}(z) = \zeta$ which contradicts (9.3).

Finally suppose $\zeta = \infty$. If $\psi_n(z)$ does not converge to ∞, then there exists a subsequence ψ_{n_j} and center ξ such that $\psi_{n_j}(z) = \xi$ for all j. By (9.4) and the subsequential convergence proved earlier there exist x and a further subsequence n_{j_k} such that

$$|x-\xi| < |z-\xi| \quad \text{and} \quad \psi_{n_{j_k}}(x) \to \infty.$$

Thus for k large enough we have that $|\psi_{n_{j_k}}(x) - x| > |\xi - x|$. By stability for these k we have that $\psi_{n_{j_k}}(z) \neq \xi$. This is a contradiction. ∎

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Proof of Proposition 5.6 Assume that $\psi(z) = \xi$. Label $\Xi \cap \overline{B(z, |z-\xi|)}$ as

$$\xi_1, \dots, \xi_\ell = \xi$$

such that $|z-\xi_1| < |z-\xi_2| < \cdots < |z-\xi_\ell|$. As $\psi^{-1}(\xi)$ is open, there exists $r > 0$ such that

$$B(z, r) \subset \psi^{-1}(\xi). \tag{9.6}$$

As $|z-\xi'| \neq |z-\xi''|$ for all $\xi' \neq \xi''$, we can choose r such that (9.6) is satisfied and for all $y \in B(z, r)$, $|y-\xi_1| < |y-\xi_2| < \cdots < |y-\xi_\ell|$.

As ψ_n converges a.e. we can find $y_1, \dots, y_{\ell-1} \in B(z, r)$ such that for all $i \in \{1, \dots, \ell-1\}$ we have $\psi_n(y_i) \to \xi$ and $|y_i - \xi_i| < |z - \xi_i|$. We can also find y_ℓ such that $\psi_n(y_\ell) \to \xi$ and $|y_\ell - \xi_\ell| > |z - \xi_\ell|$. Since for all i the sequence $\psi_n(y_i)$ converges to ξ, there exists N such that $\psi_n(y_i) = \xi$ for all $i \in \{1, \dots, \ell\}$ and all $n > N$.

There exists $r' > 0$ such that for all $y \in B(z, r')$

(i) $|y_i - \xi_i| < |y - \xi_i|$ for all $i < \ell$,
(ii) $|y_\ell - \xi| > |y - \xi|$,
(iii) $|y - \eta| > |y - \xi|$ for all $\eta \in \Xi \setminus \{\xi_1, \dots, \xi_\ell\}$.

Claim 2 For all $n > N$ and all $y \in B(z, r')$ we have that $\psi_n(y) = \xi$ or $\psi_n(y) = \Delta$.

Suppose that the claim does not hold for some y and $n > N$. If $\psi_n(y) = \infty$ or if $\psi_n(y) = \eta \in \Xi \setminus \{\xi_1, \dots, \xi_\ell\}$, then (y, ξ) would be unstable by (ii) and (iii) above. On the other hand, if $\psi_n(y) = \xi_i$ where $i < \ell$, then by (i) and (ii) (ξ_i, y_i) would be an unstable pair. Thus the claim is established.

As for every n the set $\psi_n^{-1}(\xi)$ is open and $\mathcal{L}\psi_n^{-1}(\Delta) = 0$, we deduce from the claim and the fact that ψ is a canonical allocation that $\psi_n(y) = \xi$ for all $y \in B(z, r')$ and for all $n > N$. Thus $\psi_n(z) \to \xi$. ■

Open Problems

Critical behavior in dimension two and higher. What is the tail behavior of X or R^* for the critical Poisson model? In particular, give any quantitative upper bound on $\mathbf{P}(X > r)$ as $r \to \infty$ for $d \geq 2$.

Critical behavior in one dimension. Can the critical model be analyzed exactly in the case $d = 1$? Which moments of X are finite? The variant model in which each site is only allowed to be allocated to a center to its right can be analyzed exactly via of the function F from Section 3. The method may be found in [7], in a slightly different context. For this model, $\mathbf{E}X^\nu < \infty$ if and only if $\nu < 1/2$.

Explicit non-critical bounds. Give explicit bounds on the exponential decay rates for the subcritical and supercritical models for general appetite and dimension.

Supercritical radius. Does $(R^*)^d$ have exponentially decaying tail for the supercritical model in dimension $d \geq 2$?

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Added in proof: Substantial progress on the first and second problems has been made [4]. Specifically, power law upper bounds on $\mathbf{P}(X > r)$ are obtained for all d, and it is proved that the power in $d = 1$ is $1/2$.

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