ON TYPICAL MEANS OF FOURIER SERIES OF CONTINUOUS FUNCTIONS

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1. Introduction

Let $f(x)$ be continuous in $(-\pi, \pi)$ and periodic with period 2π and let

$$f(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) = \sum_{\nu=0}^{\infty} A_\nu(x),$$

$$R_\lambda^\alpha(x) = \sum_{\nu \leqq \lambda} \left(1 - \frac{\nu}{\lambda}\right)^\alpha A_\nu(x).$$

The aim of this paper is to prove the following theorem.

THEOREM. *Let* $\alpha > 0$, $a > 0$, $\phi_x(t) = f(x+t)+f(x-t)-2f(x)$, *and let* $\omega_2(h)$ *be the modulus of smoothness of* $f(x)$ *so that*

$$\omega_2(h) = \sup_{|t| \leqq h} ||\phi_x(t)|| = \sup_{|t| \leqq h} \max_x |\phi_x(t)|.$$

Then

$$R_\lambda^\alpha(x) - f(x) = \frac{\alpha}{\pi} \int_a^\infty \frac{\phi_x(t/\lambda)}{t^2}\, dt + O\left(\omega_2\left(\frac{1}{\lambda}\right)\right)$$

uniformly for all values of x.

It was proved in [2] and [3] that this theorem is true when α is a positive integer and any real number greater than 3.

It follows from this theorem that the Fourier series of the continuous and periodic function $f(x)$ is uniformly summable Riesz of any positive order to $f(x)$.

2. A lemma

For the proof of the above theorem, we require the following lemma.

LEMMA. *Let* $\alpha > 0$, *and let*

$$h_\alpha(z) = \int_0^1 (1-u)^{\alpha-1} e^{izu}\, du.$$

Then for $z = x+iy$, $x/|z| \geqq c > 0$.

$$h_\alpha(z) = \frac{\Gamma(\alpha)e^{i(z-\pi\alpha/2)}}{z^\alpha} + \frac{i}{z} + \frac{\alpha-1}{z^2} - \frac{i(\alpha-1)(\alpha-2)}{z^3} + O\left(\frac{1}{|z|^4}\right) \tag{1}$$

as $|z| \to \infty$.

PROOF. We have

$$h_\alpha(z) = \left(\int_0^{i\delta} + \int_{i\delta}^{1+i\delta} + \int_{1+i\delta}^{1}\right)(1-u)^{\alpha-1}e^{izu}\,du = A+B+C, \quad \text{say} \tag{2}$$

Now

$$\begin{aligned} A &= i\int_0^\delta (1-iu)^{\alpha-1}e^{-zu}\,du \\ &= i\int_0^\delta e^{-zu}\,du + (\alpha-1)\int_0^\delta ue^{-zu}\,du - \frac{(\alpha-1)(\alpha-2)i}{2!}\int_0^\delta u^2e^{-zu}\,du \\ &\quad + O\left(\int_0^\delta u^3e^{-xu}\,du\right) \\ &= i\int_0^\infty e^{-zu}\,du + (\alpha-1)\int_0^\infty ue^{-zu}\,du - \frac{(\alpha-1)(\alpha-2)i}{2!}\int_0^\infty u^2e^{-zu}\,du \qquad (3) \\ &\quad + O(e^{-\delta x/2}) + O\left(\frac{1}{x^4}\right) \\ &= \frac{i}{z} + \frac{\alpha-1}{z^2} - \frac{(\alpha-1)(\alpha-2)i}{z^3} + O\left(\frac{1}{|z|^4}\right). \end{aligned}$$

By a change of variable,

$$B = e^{-\delta z}\int_0^1 (1-i\delta-u)^{\alpha-1}e^{izu}du = e^{-\delta z}\int_0^1 O(e^{-yu})\,du = O(e^{-\delta x}) = O\left(\frac{1}{|z|^4}\right), \tag{4}$$

and

$$\begin{aligned} C &= (-i)^\alpha e^{iz}\int_0^\delta u^{\alpha-1}e^{-zu}\,du \\ &= (-i)^\alpha e^{iz}\int_0^\infty u^{\alpha-1}e^{-zu}\,du + O\left(\int_\delta^\infty u^{\alpha-1}e^{-xu}\,du\right) \\ &= \frac{(-i)^\alpha e^{iz}}{z^\alpha}\int_0^\infty u^{\alpha-1}e^{-u}\,du + O(e^{-\delta x/2}) \qquad (5) \\ &= \frac{\Gamma(\alpha)e^{i(z-\pi\alpha/2)}}{z^\alpha} + O\left(\frac{1}{|z|^4}\right). \end{aligned}$$

(1) follows from (2), (3), (4) and (5).

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3. Proof of the theorem

Let

$$C_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}\cos tu\,du \qquad (\alpha > 0),$$

$$\gamma_\alpha(t) = t^{-\alpha}C_\alpha(t).$$

Then (see [1] page 568)

$$R_\lambda^\alpha(x)-f(x) = \frac{\Gamma(\alpha+1)}{\pi}\int_0^\infty \gamma_{1+\alpha}(t)\phi_x\left(\frac{t}{\lambda}\right)dt.$$

It follows from the lemma that, for $t \geqq a$,

$$\gamma_{1+\alpha}(t) = \frac{\cos\left(t-\frac{\alpha\pi}{2}\right)}{t^{1+\alpha}} + \frac{1}{\Gamma(\alpha)t^2} + O\left(\frac{1}{t^4}\right).$$

Hence

$$R_\lambda^\alpha(x)-f(x) = \frac{\Gamma(\alpha+1)}{\pi}\left(\int_0^a \gamma_{1+\alpha}(t)\phi_x\left(\frac{t}{\lambda}\right)dt + \int_a^\infty \frac{\cos\left(t-\frac{\alpha\pi}{2}\right)}{t^{1+\alpha}}\phi_x\left(\frac{t}{\lambda}\right)dt\right.$$

$$\left. + \frac{1}{\Gamma(\alpha)}\int_a^\infty \frac{\phi_x\left(\frac{t}{\lambda}\right)}{t^2}dt + \int_a^\infty O\left(\frac{1}{t^4}\right)\phi_x\left(\frac{t}{\lambda}\right)dt\right)$$

$$= \frac{\Gamma(\alpha+1)}{\pi}(I_1+I_2+I_3+I_4).$$

Since $\gamma_{1+\alpha}(t) = 0(1)$ for $0 < t \leqq a$, we have

$$I_1 = O\left(\omega_2\left(\frac{a}{\lambda}\right)\right) = O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

Now

$$I_2 = \sum_{\nu=0}^\infty \int_a^{a+\pi} \cos\left(t-\frac{\alpha\pi}{2}\right)\left[\frac{\phi_x\left(\frac{t+2\nu\pi}{\lambda}\right)}{(t+2\nu\pi)^{\alpha+1}} - \frac{\phi_x\left(\frac{t+\overline{2\nu+1}\pi}{\lambda}\right)}{(t+\overline{2\nu+1}\pi)^{\alpha+1}}\right]dt$$

$$= \tfrac{1}{2}\sum_{\nu=0}^\infty \int_a^{a+\pi} \frac{\cos\left(t-\frac{\alpha\pi}{2}\right)}{(t+\overline{2\nu+1}\pi)^{\alpha+1}}\left[\phi_x\left(\frac{t+2\nu\pi}{\lambda}\right) - 2\phi_x\left(\frac{t+\overline{2\nu+1}\pi}{\lambda}\right)\right.$$

$$\left. + \phi_x\left(\frac{t+\overline{2\nu+2}\pi}{\lambda}\right)\right]dt$$

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$$+\int_a^{a+\pi} \cos\left(t-\frac{\alpha\pi}{2}\right)\phi_x\left(\frac{t}{\lambda}\right)\left[\frac{1}{t^{\alpha+1}}-\frac{1}{2(t+\pi)^{\alpha+1}}\right]dt$$

$$-\tfrac{1}{2}\sum_{\nu=1}^{\infty}\int_a^{a+\pi}\cos\left(t-\frac{\alpha\pi}{2}\right)\phi_x\left(\frac{t+2\nu\pi}{\lambda}\right)\left[\frac{1}{(t+\overline{2\nu-1}\pi)^{\alpha+1}}-\frac{2}{(t+2\nu\pi)^{\alpha+1}}\right.$$

$$\left.+\frac{1}{(t+\overline{2\nu+1}\pi)^{\alpha+1}}\right]dt$$

$$= J_1+J_2+J_3,$$

say. The expression in the square brackets in J_1 is equal to

$$\phi_y\left(\frac{\pi}{\lambda}\right)+\phi_z\left(\frac{\pi}{\lambda}\right),$$

where

$$y = x+\frac{t+(2\nu+1)\pi}{\lambda} \quad\text{and}\quad z = x-\frac{t+(2\nu+1)\pi}{\lambda}.$$

Hence it doe not exceed in modulus $2\omega_2(\pi/\lambda)$. Whence it follows that

$$J_1 = O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

It is clear that

$$J_2 = O\left(\omega_2\left(\frac{a+\pi}{\lambda}\right)\right) = O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

Next, the expression in the square brackets in J_3 is $0(1/\nu^{\alpha+3})$ uniformly in $a \leqq t \leqq a+\pi$. Also, uniformly in $a \leqq t \leqq a+\pi$,

$$\phi_x\left(\frac{t+2\nu\pi}{\lambda}\right) = O\left(\nu^2\omega_2\left(\frac{1}{\lambda}\right)\right).$$

Hence

$$J_3 = O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

Again, uniformly in $t \geqq a$,

$$\phi_x\left(\frac{t}{\lambda}\right) = O\left(t^2\omega\ \left(\frac{1}{\lambda}\right)\right),$$

so that
$$I_4 = O\left(\omega_2\left(\frac{1}{\lambda}\right)\int_a^\infty \frac{dt}{t^2}\right) = O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

Finally
$$R_\lambda^\alpha(x)-f(x) = \frac{\Gamma(\alpha+1)}{\pi} I_3+O\left(\omega_2\left(\frac{1}{\lambda}\right)\right)$$

$$= \frac{\alpha}{\pi}\int_a^\infty \frac{\phi_x(t/\lambda)}{t^2}\,dt+O\left(\omega_2\left(\frac{1}{\lambda}\right)\right).$$

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