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A NEW CLASS OF RESTRICTED TYPE SPACES

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Abstract We find new properties for the space $R(X)$, introduced by Soria in the study of the best constant for the Hardy operator minus the identity. In particular, we characterize when $R(X)$ coincides with the minimal Lorentz space $\Lambda(X)$. The condition that $R(X) \neq \{0\}$ is also described in terms of the embedding $(L^{1,\infty} \cap L^{\infty}) \subset X$. Finally, we also show the existence of a minimal rearrangement-invariant Banach function space (RIBFS) X among those for which $R(X) \neq \{0\}$ (which is the RIBFS envelope of the quasi-Banach space $L^{1,\infty} \cap L^{\infty}$).

Keywords: rearrangement-invariant spaces; restricted type; Lorentz spaces

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1. Introduction

Let X be a rearrangement-invariant Banach function space (RIBFS) in $\mathbb{R}^n$, endowed with the Lebesgue measure, and with a function norm $\|\cdot\|_X$ (see [**2**] for further details). Consider $R(X)$ to be the class of all measurable functions such that

$$\|f\|_{R(X)} = \int_0^{\infty} v_n^{-1}\lambda_f(t) \left\| \frac{1}{v_n^{-1}\lambda_f(t) + |\cdot|^n} \right\|_X \mathrm{d}t < +\infty,$$

where $\lambda_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$ is the distribution function of f and v_n is the volume of the unit ball in $\mathbb{R}^n$. We recall that the non-increasing rearrangement of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\}.$$

The space $R(X)$ was introduced in [**10**] and appears naturally in the study of the norm of the Hardy operator minus the identity in the cone of radially decreasing functions (see also [**7**]). More precisely, if S_n is the Hardy operator in $\mathbb{R}^n$,

$$S_n f(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(y)\,\mathrm{d}y,$$

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it can be proved that, for a positive radially decreasing function f,

$$S_n f(x) - f(x) = \frac{1}{v_n|x|^n}\int_{f(x)}^{\infty} \lambda_f(t)\,\mathrm{d}t,$$

and hence, using Minkowski's integral inequality, we obtain

$$\|S_n f - f\|_X \leqslant \int_0^{\infty} v_n^{-1}\lambda_f(t)\left\|\frac{1}{v_n^{-1}\lambda_f(t) + |\cdot|^n}\right\|_X \mathrm{d}t = \|f\|_{R(X)},$$

and the inequality is sharp.

It is shown in [**10**, Theorem 13] that, in many cases, $R(X)$ coincides with the minimal Lorentz space $\Lambda(X)$. One of the main goals of this paper is to characterize, in terms of the upper fundamental index of X, the spaces where this happens (Theorem 2.5). More properties for $R(X)$ are considered in § 2.

In § 3, we study a general minimality property for quasi-Banach Lorentz spaces $\Lambda(X)$. In particular, we find the RIBFS envelope of the quasi-Banach space $L^{1,\infty} \cap L^{\infty}$, which is also minimal among the RIBFS for which $R(X) \neq 0$ (recall that $L^{1,\infty}$ is the weak-L^1 quasi-Banach space defined by $\|f\|_{L^{1,\infty}} = \sup_{s>0} sf^*(s) < \infty$).

We shall denote by $\bar{X}$ a rearrangement-invariant function space on $(\mathbb{R}^+, \mathrm{d}t)$ endowed with a function norm $\|\cdot\|_{\bar{X}}$ such that

$$\|f\|_X = \|f^*\|_{\bar{X}}.$$

By [**2**, Theorem II.4.10] such a representation of the space X always exists. Let φ_X denote the fundamental function of X; that is, the quasi-concave function on $(0, \infty)$ defined by

$$\varphi_X(t) = \|\chi_{[0,t)}\|_{\bar{X}}.$$

Let $g(x) = 1/(1 + v_n|x|^n)$, so that $g^*(s) = 1/(1 + s)$. Assuming that $g \in X$, define

$$W_X(t) = \left\|\frac{1}{1 + v_n|\cdot|^n/t}\right\|_X = \|E_{1/t}g^*\|_{\bar{X}},$$

where E_s denotes the usual dilation operator [**2**, Chapter III]. By the monotonicity of the norm on X, $W_X(t)$ is increasing and

$$\frac{W_X(t)}{t} = \left\|\frac{1}{t + v_n|\cdot|^n}\right\|_X$$

is non-increasing. That is, defining $W_X(0) = 0$, $W_X(t)$ is a quasi-concave function on $(0, \infty)$. Recall that, given such a function ϕ, the minimal Lorentz space Λ_ϕ [**2**, Definition II.5.12] is the RIBFS of all measurable functions f such that

$$\int_0^{\infty} \phi(\lambda_f(t))\,\mathrm{d}t = \int_0^{\infty} f^*(s)\,\mathrm{d}\phi(s) = \|f\|_{\Lambda_\phi} < +\infty. \tag{1.1}$$

In particular, if X is an RIBFS and φ_X denotes its fundamental function, let $\Lambda(X)$ be the space Λ_{φ_X}. In this way,

$$R(X) = \Lambda_{W_X}.$$

In particular, whenever $R(X)$ is not trivial, it is an RIBFS. It is easy to see that, for example, if $X = L^1$, then $R(X) = \{0\}$. We have assumed that $g \in X$ to obtain that $R(X)$ is not trivial. In the following proposition we shall show that the converse is true. To this end, we will need the following characterization of the intersection space $L^{1,\infty} \cap L^\infty$, which follows directly from the definition of the space.

Lemma 1.1. *$L^{1,\infty} \cap L^\infty$ is the quasi-Banach space of all measurable functions such that*

$$\sup_{s>0} f^*(s)(1+s) < +\infty.$$

Proposition 1.2. *Let X be an RIBFS. The following are equivalent:*

(i) *$R(X) \neq \{0\}$;*

(ii) *there exists $r > 0$ such that $W_X(r) < +\infty$;*

(iii) *$W_X(r) < +\infty$, for every $r > 0$;*

(iv) *$g^*(s) = 1/(1+s) \in \bar{X}$;*

(v) *$(L^{1,\infty} \cap L^\infty) \subset X$.*

Proof. If $R(X) \neq \{0\}$, by the lattice property, it contains the characteristic function of a set E of positive measure (say $|E| = r > 0$), and hence

$$\|\chi_E\|_{R(X)} = \int_0^\infty W_X(r\chi_{[0,1)}(s))\,\mathrm{d}s = W_X(r) < +\infty, \tag{1.2}$$

and (ii) follows from (i).

Assertion (iii) is easily seen to be equivalent to (ii), and can be proved by using the boundedness of the dilation operator E_s on X, which gives the existence, for every $r, s > 0$, of a constant $0 < C(r,s) < +\infty$ such that $W_X(s) \leqslant C(r,s)W_X(r)$.

Since $W_X(1) = \|g^*\|_{\bar{X}}$, (iii) implies (iv). Using Lemma 1.1, (v) is trivial from (iv) (in fact, they are equivalent). Using again this lemma, we find that (v) (or rather (iv)) implies that $W_X(1) < +\infty$, and hence $\chi_{(0,1)} \in \overline{R(X)}$, which shows (i). □

The equivalence between (iv) and (v) can be also found in [**9**, Proposition 2.7].

Another interesting property is the following.

Proposition 1.3. *Let X, Y be RIBFS. Then*

$$R(X \cap Y) = R(X) \cap R(Y).$$

Proof. We can assume that $1/(1+s) \in \overline{X \cap Y}$ since otherwise the result is trivial. Endowing $X \cap Y$ with the norm $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$, it easily follows that for any $t > 0$,

$$W_{X\cap Y}(t) = W_X(t) + W_Y(t).$$

Hence, $\|\cdot\|_{R(X\cap Y)} = \|\cdot\|_{R(X)} + \|\cdot\|_{R(Y)}$ and then the result follows. □

Let us recall that, for a quasi-concave function ϕ, the *Marcinkiewicz space* M_ϕ (see [**2**, Theorem II.5.13]) is the RIBFS that consists of all measurable functions f such that

$$\|f\|_{M_\phi} = \sup_t f^{**}(t)\phi(t) < +\infty, \tag{1.3}$$

where

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$$

is the maximal function of f. In particular, for any RIBFS X, the *maximal Lorentz space* $M(X)$ is defined as M_{φ_X}.

Lemma 1.4. *Let X be an RIBFS.*

(i) *If $1/(1+s) \in \bar{X}$, then for any $t > 0$*

$$\varphi_{R(X)}(t) = W_X(t),$$

and

$$\varphi_X(t) \leqslant \frac{W_X(t)}{\log 2}. \tag{1.4}$$

(ii) *The following embeddings hold:*

$$R(\Lambda(X)) \subset R(X) \subset R(M(X)) \subset \Lambda(X). \tag{1.5}$$

More precisely,

$$\log 2\|f\|_{\Lambda(X)} \leqslant \|f\|_{R(M(X))} \leqslant \|f\|_{R(X)} \leqslant \|f\|_{R(\Lambda(X))}.$$

Proof. The first equality is just (1.2). To prove (1.4) observe that, for any $r, t > 0$, and by Hölder's inequality [**2**, Theorem I.2.4]:

$$t\log\left(1+\frac{r}{t}\right) = \int_0^\infty \frac{\chi_{[0,r)}(s)}{1+s/t}\,\mathrm{d}s \leqslant W_X(t)\varphi_{X'}(r),$$

where X' is the *associate space* of X [**2**, Definition I.2.3]. Using the equality

$$\varphi_{X'}(r)\varphi_X(r) = r,$$

it follows that

$$\varphi_X(r) \leqslant \Phi\left(\frac{r}{t}\right) W_X(t),$$

where $\Phi(x) = x/\log(1+x)$. In particular, for $r = t$,

$$\varphi_X(t) \leqslant \Phi(1) W_X(t) = \frac{W_X(t)}{\log 2}.$$

To prove (ii), observe that since $\Lambda(M(X)) = \Lambda(X)$, by (1.4) it follows that

$$\log 2 \|f\|_{\Lambda(X)} \leqslant \|f\|_{R(M(X))}.$$

On the other hand, as $\|\cdot\|_{M(X)} \leqslant \|\cdot\|_X \leqslant \|\cdot\|_{\Lambda(X)}$, we trivially get

$$\|f\|_{R(M(X))} \leqslant \|f\|_{R(X)} \leqslant \|f\|_{R(\Lambda(X))}.$$

□

Remark 1.5. It is easy to see that $1/\log 2$ is the best possible constant in the inequality (1.4). In fact, taking $X = L^1 + L^\infty$, then $\varphi_X(t) = \min(1, t)$ and $W_X(t) = t \log(1 + 1/t)$. Hence, at $t = 1$ we get the equality

$$\frac{\varphi_X(1)}{W_X(1)} = \frac{1}{\log 2}.$$

Also, if we only assume that X is a quasi-Banach space (satisfying $1/(1+s) \in \bar{X}$), then it is easy to show that we can still prove an inequality like (1.4) replacing $1/\log 2$ by 2.

2. Main result

Before proving our main result (Theorem 2.5) we collect in Lemma 2.3 some of the embeddings between function spaces we shall use later. First, we need the following definition.

Definition 2.1. Given an increasing and positive function W, we define the weighted weak-type Lorentz space

$$\Lambda_W^{1,\infty} = \Big\{ f\colon \|f\|_{\Lambda_W^{1,\infty}} = \sup_{t>0} f^*(t) W(t) < \infty \Big\}.$$

Remark 2.2. If

$$W(t) = \int_0^t w(s)\,\mathrm{d}s,$$

where w is a weight on $\mathbb{R}_+$, then $\Lambda_W^{1,\infty} = \Lambda^{1,\infty}(w)$ (see [**4**] for the definition of this space). It was proved in [**9**, Theorem 3.1] that $\Lambda^{1,\infty}(w)$ is a Banach space if and only if $w \in B_1$, where

$$B_1 = \Big\{ w\colon \int_t^\infty \frac{w(s)}{s}\,\mathrm{d}s \leqslant \frac{C}{t} \int_0^t w(s)\,\mathrm{d}s \text{ for every } t > 0 \Big\}.$$

In this case,

$$\Lambda^{1,\infty}(w) = M_W;$$

see (1.3).

This class of weights was introduced in [**1**] and gives a characterization of the boundedness of the Hardy operator for decreasing functions in $L^1(w)$ (see also [**4**, **6**]).

Lemma 2.3. *Let V, W be two quasi-concave increasing functions. Then*

(i) $\Lambda_W \subset \Lambda_V$ *if and only if*

$$\sup_{t>0} \frac{V(t)}{W(t)} < \infty.$$

(ii) $\Lambda_W^{1,\infty} \subset M_V$ *if and only if*

$$\sup_{t>0} \frac{V(t)}{t} \int_0^t \frac{1}{W(s)}\, \mathrm{d}s < \infty.$$

(iii) $M_V \subset \Lambda_W^{1,\infty}$ *if and only if*

$$\sup_{t>0} \frac{V(t)}{t \sup_{s\geqslant t} s^{-1} W(s)} < \infty.$$

Proof. For the definition of the various spaces see (1.1), (1.3) and Definition 2.1. The embedding (i) is proved in [**5**, Corollary 2.7], (ii) is a consequence of [**9**, Theorem 4.1] and condition (iii) can be found in [**3**, Theorem 5.3]. □

Definition 2.4. For any RIBFS X, we define [**2**, pp. 177–178]

$$\bar{\varphi}_X(s) := \sup_{t>0} \frac{\varphi_X(st)}{\varphi_X(t)},$$

and the upper fundamental index

$$\bar{\beta}_X = \inf_{s>1} \frac{\log \bar{\varphi}_X(s)}{\log s}.$$

Theorem 2.5. *Let X be an RIBFS. The following are equivalent:*

(i) $\Lambda(X) = R(\Lambda(X))$;

(ii) $\Lambda(X) = R(X)$;

(iii) $\Lambda(X) = R(M(X))$;

(iv) $\bar{\beta}_X < 1$.

Proof. Using (1.5) it is clear that (i) implies (ii) and, similarly, (ii) implies (iii). Assume now that $\Lambda(X) = R(M(X))$ holds. Observe that

$$W_{M(X)}(t) = \sup_{s>0} \frac{\log(1+s)}{s} \varphi_X(st).$$

Hence, using Lemma 2.3 (i), $\Lambda(X) = \Lambda_{\varphi_X} \subset \Lambda_{W_{M(X)}} = R(M(X))$ if and only if

$$L := \sup_{t>0} \frac{W_{M(X)}(t)}{\varphi_X(t)} = \sup_{s>0} \frac{\log(1+s)}{s} \bar{\varphi}_X(s) < +\infty.$$

Thus, by the definition of the fundamental index, for any $s > 1$

$$s^{\bar{\beta}_X} \leqslant \bar{\varphi}_X(s) \leqslant L\frac{s}{\log(1+s)},$$

which implies that $\bar{\beta}_X < 1$.

Finally, assume that $\bar{\beta}_X < 1$. We have already seen that $\Lambda(X) \supset R(\Lambda(X))$. Integrating by parts, it follows that

$$W_{\Lambda(X)}(t) = t\lim_{s\to\infty}\frac{\varphi_X(s)}{s} + \int_0^\infty (\varphi_X(ts) - \varphi_X(0^+))\frac{\mathrm{d}s}{(1+s)^2}.$$

Thus,

$$\frac{W_{\Lambda(X)}(t)}{\varphi_X(t)} = \frac{t}{\varphi_X(t)}\lim_{s\to\infty}\frac{\varphi_X(s)}{s} + \int_0^\infty \frac{\varphi_X(ts) - \varphi_X(0^+)}{\varphi_X(t)}\frac{\mathrm{d}s}{(1+s)^2}.$$

Hence, since

$$\sup_{t>0}\frac{t}{\varphi_X(t)} = \lim_{t\to+\infty}\frac{t}{\varphi_X(t)},$$

we have

$$\begin{aligned}\sup_{t>0}\frac{W_{\Lambda(X)}(t)}{\varphi_X(t)} &\leqslant 1 + \sup_{t>0}\int_0^\infty \frac{\varphi_X(ts) - \varphi_X(0^+)}{\varphi_X(t)}\frac{\mathrm{d}s}{(1+s)^2} \\ &\leqslant 1 + \int_0^\infty \bar{\varphi}_X(s)\frac{\mathrm{d}s}{(1+s)^2},\end{aligned}$$

and the finiteness of the last term is equivalent to the finiteness of

$$\int_1^\infty \bar{\varphi}_X(s)\frac{\mathrm{d}s}{s^2}.$$

But, by [**2**, Lemma III.5.9] this is equivalent to $\bar{\beta}_X < 1$. That is, we have obtained that

$$\sup_{t>0}\frac{W_{\Lambda(X)}(t)}{\varphi_X(t)} < +\infty,$$

which, by Lemma 2.3 (i), implies that $\Lambda(X) \subset R(\Lambda(X))$. □

Remark 2.6. If X is an RIBFS and we define $D_X = \bar{\varphi}_X(2)$, it is known that $D_X \in [1, 2]$ and it is clear that

$$2^{\bar{\beta}_X} \leqslant D_X,$$

which implies that if $D_X < 2$, then $\bar{\beta}_X < 1$. This and the previous result recover [**10**, Theorem 13.ii].

Observe also that if $\bar{\beta}_X < 1$, then $R(X) = \Lambda(X) \neq \{0\}$.

Remark 2.7. If $\bar{\beta}_X = 1$, we know by Theorem 2.5 that $R(X) \subsetneq \Lambda(X)$. In this case, $R(X)$ can be either trivial or a proper subspace of $\Lambda(X)$. We now give some examples.

(i) If $X = L^1(\mathbb{R}^n)$, then $\bar{\beta}_X = 1$ and $R(X) = \{0\}$.

(ii) If $X = L^1 + L^\infty$, then

$$R(L^1 + L^\infty) = \Lambda_\Psi \subsetneq L^1 + L^\infty = \Lambda(L^1 + L^\infty),$$

where $\Psi(t) = t\log(1 + 1/t)$.

(iii) Consider the quasi-concave function $\Phi(t) = t/\log(1+t)$ and let $X = M_\Phi$. It is easy to see that $1/(1+s) \in \bar{X}$, which implies that $R(X) \neq \{0\}$. On the other hand, for any $t > 0$,

$$\bar{\varphi}_X(t) = t\sup_{s>0}\frac{\log(1+s)}{\log(1+st)} = \max(1,t),$$

whence it follows that $\bar{\beta}_X = 1$. As before, it is easy to see that $1/(1+s) \notin \Lambda(X)$, which implies that $R(R(X)) = \{0\}$.

It was proved in [**10**] that for $X = L^{p,q}$, the classical Lorentz spaces, $\|\cdot\|_{R(X)}$ is a multiple of $\|\cdot\|_{\Lambda(X)}$, and that this is not true for a general RIBFS. In the following proposition we show that, in order to have this relationship, it is necessary and sufficient that this condition holds for characteristic functions.

Proposition 2.8. *Let X be an RIBFS such that $1/(1+s) \in \bar{X}$. Then, there exists a constant $c > 0$ such that, for any measurable function f,*

$$\|f\|_{R(X)} = c\|f\|_{\Lambda(X)},$$

if and only if, for any $t > 0$,

$$W_X(t) = c\varphi_X(t). \tag{2.1}$$

Proof. The first assumption implies that $\Lambda(X) = R(X)$, which is equivalent, using that $R(X) = \Lambda_{W_X}$ and [**11**, Proposition 1], to the fact that the norms satisfy

$$\|f\|_{\Lambda(X)} \leqslant c_1\|f\|_{R(X)} \quad \text{and} \quad \|f\|_{R(X)} \leqslant c_2\|f\|_{\Lambda(X)},$$

where $c_1 = \sup_{t>0}\varphi_X(t)/W_X(t)$ and $c_2 = \sup_{t>0}W_X(t)/\varphi_X(t)$ are the best constants in the inequalities. Then

$$\|f\|_{R(X)} = c\|f\|_{\Lambda(X)},$$

if and only if $c = c_2 = c_1^{-1}$. That is, if and only if, for any $t > 0$,

$$W_X(t) = c\varphi_X(t).$$

□

Remark 2.9. Let $\Lambda^q(v)$ be the weighted Lorentz space, where v is a decreasing function and $q \geqslant 1$, endowed with the norm

$$\|f\|_{\Lambda^q(v)} = \left(\int_0^\infty f^*(t)^q v(t)\,\mathrm{d}t\right)^{1/q}$$

(see [**8**]). Let

$$V(t)=\int_0^t v(s)\,\mathrm{d}s.$$

Then, condition (2.1) holds for $X=\Lambda^q(v)$ if and only if $TV(t)=KV(t)$, where

$$Tg(t)=\int_0^\infty \frac{t^q}{(t+s)^{q+1}}g(s)\,\mathrm{d}s,$$

and $K=c^q/q$; that is, V is an increasing and concave eigenvector of the operator T. In particular, if $X=\Lambda^q(t^\alpha)$, $-1<\alpha\leqslant 0$ and $1<q<\infty$, then $X=L^{p,q}$, with $p=q/(1+\alpha)$, $1/(1+s)\in\overline{L^{p,q}}$, $V(t)=t^{\alpha+1}/(\alpha+1)$ is an eigenvector of T and

$$K=\int_0^\infty \frac{u^{1+\alpha}}{(1+u)^{q+1}}\,\mathrm{d}u=\frac{\Gamma(2+\alpha)\Gamma(-1-\alpha+q)}{\Gamma(1+q)},$$

which gives (see also [**10**, Proposition 8])

$$c=p^{-1/q'}\biggl(\frac{1}{\Gamma(1+q)}\Gamma\biggl(\frac{q}{p'}\biggr)\Gamma\biggl(\frac{q+p}{p}\biggr)\biggr)^{1/q}.$$

An example for which (2.1) does not hold is $X=\Lambda^q(v)$, for every weight v with compact support.

3. Minimal space

We shall show that there exists a minimal RIBFS M satisfying that $R(M)\neq 0$, and prove that, in fact, it is also minimal among the RIBFS M for which $(L^{1,\infty}\cap L^\infty)\subset M$. Thus, if X is an RIBFS such that $R(X)\neq\{0\}$ (and hence, by Proposition 1.2, $(L^{1,\infty}\cap L^\infty)\subset X$), then

$$(L^{1,\infty}\cap L^\infty)\subset M\subset X \tag{3.1}$$

(recall that $L^{1,\infty}\cap L^\infty$ is just a quasi-Banach space). We observe that if $\Phi(t)=t/\log(1+t)$, then $(L^{1,\infty}\cap L^\infty)\subset M_\Phi$ and

$$R(L^{1,\infty}\cap L^\infty)=R(M_\Phi)=(L^1\cap L^\infty)\subset R(X).$$

Thus, $R(M_\Phi)$ is minimal among all the non-trivial $R(X)$ spaces. Hence, a natural candidate for the minimal space in condition (3.1) is given by $M=M_\Phi$. We show in Proposition 3.3 that this is true. We will obtain this result as a consequence of a more general argument involving weighted weak-type Lorentz spaces. Observe that (3.1) tells us that M is the RIBFS envelope of $L^{1,\infty}\cap L^\infty$, which is characterized in terms of its second associate space [**3**, Theorem 9.1]. For the sake of completeness, we will give a direct proof of this result.

Theorem 3.1. *Let W be an increasing and positive function such that $1/W$ is locally integrable at zero, and let*

$$\tilde{W}(t)=\biggl(\frac{1}{t}\int_0^t\frac{1}{W(s)}\,\mathrm{d}s\biggr)^{-1}.$$

Then, $M_{\tilde{W}}$ is the minimal RIBFS containing $\Lambda_W^{1,\infty}$.

Proof. First of all, we prove that $\tilde{W}$ is a quasi-concave function: it is clear that $\tilde{W}$ is an increasing and positive function such that, for every $0 < t < \infty$, we have that $0 < \tilde{W}(t) < \infty$ and

$$\frac{\tilde{W}(t)}{t} = \biggl(\int_0^t \frac{1}{W(s)}\,\mathrm{d}s\biggr)^{-1}$$

is a decreasing function.

Let us see that $\Lambda_W^{1,\infty} \subset M_{\tilde{W}}$. Using Lemma 2.3 (ii), this embedding is equivalent to the condition

$$\sup_{t>0} \frac{\tilde{W}(t)}{t} \int_0^t \frac{1}{W(s)}\,\mathrm{d}s < \infty,$$

which is trivial by the definition of $\tilde{W}$.

We now show that if X is an RIBFS satisfying that $\Lambda_W^{1,\infty} \subset X$, then $M_{\tilde{W}} \subset X$. It is easy to see that $\Lambda_W^{1,\infty} \subset X$ is equivalent to $1/W \in X$, and hence, by duality,

$$\biggl\|\frac{1}{W}\biggr\|_{\bar{X}} = \sup_{f\in X'} \frac{1}{\|f\|_{X'}} \int_0^\infty \frac{f^*(t)}{W(t)}\,\mathrm{d}t < \infty,$$

that is, $X' \subset \Lambda^1(1/W)$. Again using duality, this turns out to be equivalent to $(\Lambda^1(1/W))' \subset X'' = X$. But, using [**11**, Proposition 1] and Lemma 2.3 (i), we have

$$\begin{aligned}
\|h\|_{(\Lambda^1(1/W))'} &= \sup_{g\in\Lambda^1(1/W)} \int_0^\infty h^*(t)g^*(t)\,\mathrm{d}t\biggl(\int_0^\infty \frac{g^*(t)}{W(t)}\,\mathrm{d}t\biggr)^{-1} \\
&= \sup_{r>0} \int_0^r h^*(t)\,\mathrm{d}t\biggl(\int_0^r \frac{1}{W(t)}\,\mathrm{d}t\biggr)^{-1} \\
&= \|h\|_{M_{\tilde{W}}},
\end{aligned}$$

which shows that $M_{\tilde{W}} \subset X$. □

Remark 3.2. As a consequence of Theorem 3.1, we observe that if

$$W(t) = \int_0^t w(s)\,\mathrm{d}s,$$

where w is a weight on $\mathbb{R}_+$, then $\Lambda^{1,\infty}(w) = M_{\tilde{W}}$ if and only if $\Lambda^{1,\infty}(w)$ is an RIBFS, which, by Remark 2.2, is equivalent to the condition $w \in B_1$. We can also give a direct proof of this fact by using Lemma 2.3 (iii) and [**9**, Theorem 2.5]:

$$\begin{aligned}
M_{\tilde{W}} \subset \Lambda^{1,\infty}(w) = \Lambda_W^{1,\infty} &\iff \sup_{t>0} \frac{W(t)}{t\sup_{s\geqslant t} s^{-1}\tilde{W}(s)} < \infty \\
&\iff \sup_{t>0} \frac{1}{t}\biggl(W(t)\int_0^t \frac{1}{W(s)}\,\mathrm{d}s\biggr) < \infty \\
&\iff w \in B_1.
\end{aligned}$$

Proposition 3.3. *Let X be an RIBFS such that $(L^{1,\infty} \cap L^\infty) \subset X$ and let Φ be the quasi-concave function given by $\Phi(t) = t/\log(1+t)$. Then*

$$(L^{1,\infty} \cap L^\infty) \subset M_\Phi \subset X. \tag{3.2}$$

Proof. By Lemma 1.1, $L^{1,\infty} \cap L^\infty = \Lambda_W^{1,\infty}$, with $W(t) = 1 + t$. Hence, (3.2) follows from Theorem 3.1, observing that

$$\tilde{W}(t) = \left(\frac{1}{t}\int_0^t \frac{1}{1+s}\,\mathrm{d}s\right)^{-1} = \frac{t}{\log(1+t)} = \Phi(t).$$

Therefore, $M_\Phi = M_{\tilde{W}} \subset X$. □

Remark 3.4. Propositions 1.2 and 3.3 show that M_Φ is the minimal RIBFS X satisfying the condition that $R(X) \neq \{0\}$. Hence, for any RIBFS X, either $R(X) = \{0\}$ or

$$R(M_\Phi) = (L^1 \cap L^\infty) \subset R(X) \subset \Lambda_\Psi = R(L^1 + L^\infty),$$

with $\Phi(t) = t/\log(1+t)$ and $\Psi(t) = t\log(1+1/t)$.

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