CHARACTERIZATIONS OF COMMUTATIVITY FOR C*-ALGEBRAS

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Let \( \mathcal{A} \) be a C*-algebra acting on the Hilbert space \( H \) and let \( \mathcal{S} \) be the self-adjoint elements of \( \mathcal{A} \). The following characterization of commutativity is due to I. Kaplansky (see Dixmier [3, p. 58]).

**Theorem 1.** \( \mathcal{A} \) is commutative if and only if 0 is the only nilpotent element of \( \mathcal{A} \).

In this note we use the above result of Kaplansky to give two numerical characterizations of commutativity. Ogasawara [5], Sherman [6], and Fukamiya, Misonou and Takeda [4] characterize commutativity for \( \mathcal{A} \) in terms of the usual order structure on \( \mathcal{S} \). We show that Kaplansky's theorem reduces the proofs of these order characterizations to simple computations.

1. Numerical characterizations. Taylor [7, Lemma 3.3] proves that, if \( A \) and \( B \) are self-adjoint elements of \( \mathcal{A} \) with \( 0 \neq \| A \| \leq \| B \| \), then

\[
\| A + B \| \leq \| A \| + k \| AB \|,
\]

where \( k \) may be taken as 2. If \( \mathcal{A} \) is commutative, the inequality holds with \( k = 1 \). Taylor asks if the converse is true; in Theorem 2 we prove this.

Note that an inequality of the form (1) can hold for all elements of a Banach algebra \( \mathcal{B} \) only if \( \mathcal{B} \) is commutative. For, setting \( B = A \) in (1), we obtain \( \| A \|^2 \leq k \| A^2 \| \) and thence \( \| A \| \leq kr(A) \), where \( r(A) \) is the spectral radius of \( A \). Thus \( \mathcal{B} \) is commutative (see, for example, [1, p. 33]).

A simple argument shows that inequality (1) holds if and only if it holds for self-adjoint \( A, B \) of norm 1.

**Remark.** We assume that \( \mathcal{A} \) has a unit element when there is no loss of generality in so doing.

**Theorem 2.** \( \mathcal{A} \) is commutative if and only if

\[
\| A + B \| \leq 1 + \| AB \|
\]

for all self-adjoint elements \( A, B \in \mathcal{A} \) with \( \| A \| = \| B \| = 1 \).

**Proof.** If \( \mathcal{A} \) is commutative, the result follows from the inequality

\[
(I - A)(I - B) \geq 0.
\]

Assume that \( \mathcal{A} \) is not commutative. By Theorem 1, there exists nonzero \( T \in \mathcal{A} \) such that \( T^2 = 0 \). Let \( H_1 \) be the subspace \((TH)^-\) and let \( H_2 \) be the orthogonal complement of \( H_1 \) in...
H. If we represent $H$ as $H_1 \oplus H_2$, $T, T^*$ are represented by the $2 \times 2$ matrices of operators

$$T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad T^* = \begin{bmatrix} 0 & 0 \\ S^* & 0 \end{bmatrix}. $$

We may suppose that $\|S\| = 1$. Let

$$A = TT^*, \quad B = \alpha TT^* + \alpha T^*T + \beta T + \beta T^*, $$

where $\alpha, \beta > 0$, $\alpha + \beta = 1$, so that $A, B \in \mathcal{A}$. Then

$$A = \begin{bmatrix} SS^* & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha SS^* & \beta S \\ \beta S^* & \alpha S S^* \end{bmatrix}. $$

Clearly $\|A\| = 1$. Since $\|S\| = 1$, there exist $x_n \in H_1$ such that $\|x_n\| = 1$ and $SS^*x_n - x_n \to 0$. To see this, note that $\|SS^*x_n - x_n\|^2 = \|SS^*x_n\|^2 - 2\|S^*x_n\|^2 + \|x_n\|^2 \leq 2(\|x_n\|^2 - \|S^*x_n\|^2)$, and choose $x_n$ such that $\|S^*x_n\| \to 1$. Hence

$$B(x_n + S^*x_n) - (x_n + S^*x_n) \to 0, $$

and so $\|B\| \geq 1$. But

$$\|B\| \leq \alpha \|TT^* + T^*T \| + \beta \|T + T^*\| \leq 1, $$

and so $\|B\| = 1$. Next,

$$\|AB\| = \sup \{\alpha \|SS^*x + \beta SS^*y\| : \|x\|^2 + \|y\|^2 = 1\} \leq \sup \{\alpha \|x\| + \beta \|y\| : \|x\|^2 + \|y\|^2 = 1\} = (\alpha^2 + \beta^2)^\lambda.$$

Let $\lambda = \alpha + \frac{1}{2} + (\frac{1}{2} + \beta^2)^\frac{1}{2}$, so that $\lambda$ satisfies the equation

$$(\lambda - \alpha)(\lambda - \alpha - 1) = \beta^2.$$

Let $x_n$ be as above and let $y_n = \beta(\lambda - \alpha)^{-1} S^*x_n$. Then

$$(A + B)(x_n + y_n) - \lambda(x_n + y_n) \to 0,$$

so that $\|A + B\| \geq \lambda$. If we choose $\alpha, \beta$ so that

$$\alpha + \frac{1}{2} + (\frac{1}{2} + \beta^2)^\frac{1}{2} > 1 + (\alpha^2 + \beta^2)^\frac{1}{2},$$

then we have $\|A + B\| > 1 + \|AB\|$. It is enough to take

$$\alpha = \frac{2}{3}, \quad \beta = \frac{1}{3}.$$

**Remark.** If $\mathcal{A}$ is commutative, we even have $\|A + B\| \leq 1 + \|AB\|$ for all elements $A, B \in \mathcal{A}$ with $\|A\| = \|B\| = 1$.

We recall that the numerical index $n(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$n(\mathcal{A}) = \inf \{w(A) : A \in \mathcal{A}, \|A\| = 1\},$$

where

$$w(A) = \sup\{\|Ax\| : x \in H, \|x\| = 1\},$$

and that $\frac{1}{2} \leq n(\mathcal{A}) \leq 1$ (see [1, pp. 43, 44]).
**Theorem 3.** $\mathcal{A}$ is commutative or not commutative according as $n(\mathcal{A})$ is 1 or $\frac{1}{2}$.

**Proof.** If $\mathcal{A}$ is commutative, each $A \in \mathcal{A}$ is normal and so has $w(A) = \|A\|$. If $\mathcal{A}$ is not commutative, then, by Theorem 1, there exists $T \in \mathcal{A}$, with $T \neq 0$, $T^2 = 0$. A result of Bouldin [2, Corollary 2, p. 214] shows that $w(T) = \frac{1}{2} \|T\|$, so that $n(\mathcal{A}) = \frac{1}{2}$. (The condition $T^*H$ orthogonal to $TH$ in [2] is equivalent to $T^2 = 0$.)

2. **Order characterizations.** We recall that the usual order on $\mathcal{S}$ is defined by

$$A \succeq B \Leftrightarrow \langle (A-B)x, x \rangle \geq 0 \quad (x \in H).$$

Let $T, S$ be as in the proof of Theorem 2. Let

$$P = \begin{pmatrix} SS^* & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & (SS^*)^*S \\ S^*(SS^*)^* & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & S^*S \end{pmatrix},$$

so that $P, Q, R \in \mathcal{A}$. We make frequent use of the following lemma.

**Lemma 4.** Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$. Then $\alpha P + \beta Q + \gamma R \succeq 0$ if and only if $\alpha \gamma - \beta^2 \geq 0$.

**Proof.** For $x \in H_1, y \in H_2$ we have

$$\langle (\alpha P + \beta Q + \gamma R)(x+y), x+y \rangle = \|\beta \gamma^{-\frac{1}{2}}(SS^*)^*x + \gamma^\frac{1}{2}Sy \|^2 + \gamma^{-1}(\alpha \gamma - \beta^2) \|S^*x\|^2.$$  

Since $(TH)^{-} = H_1$, for any $x \in H_1$ there exist $y_n \in H_2$ such that $\gamma^{-\frac{1}{2}}Sy_n \rightarrow -\beta \gamma^{-\frac{1}{2}}(SS^*)^*x$. The result follows.

$\mathcal{S}$ is said to be **lattice ordered** if, for each $U \in \mathcal{S}$, there exists $U^+ \geq 0$ such that $U^+ \geq U$ and $U^+ \leq V$ for any $V$ such that $V \geq 0$ and $V \geq U$. $\mathcal{S}$ is said to have the **decomposition property** if, given $A, B, C \in \mathcal{S}$ with $0 \leq A \leq B + C, B \geq 0, C \geq 0$, there exist $A_1, A_2 \in \mathcal{S}$ with $A = A_1 + A_2, 0 \leq A_1 \leq B, 0 \leq A_2 \leq C$.

**Theorem 5.** ([4], [5], [6].) The following statements are equivalent.

(i) $\mathcal{A}$ is commutative.

(ii) $A, B \in \mathcal{A}$, $A \geq B \geq 0 \Rightarrow A^2 \geq B^2$.

(iii) $\mathcal{S}$ is lattice ordered.

(iv) The dual space of $\mathcal{S}$ is lattice ordered.

(v) $\mathcal{S}$ has the decomposition property.

**Proof.** If $\mathcal{A}$ is commutative, the Gelfand–Naimark theorem readily shows that conditions (ii)–(v) hold. Assume that $\mathcal{A}$ is not commutative and let $T$ be as in the proof of Theorem 2.

(ii) $\Rightarrow$ (i). With the above notation, let $A = 8P + 2R, B = 4P + 2Q + R$. Then $A, B \in \mathcal{A}$ and $A \geq B \geq 0$, by Lemma 4. For $y \in H_2$, we have $\langle (A^2 - B^2)y, y \rangle = -\langle (S^*S)y, y \rangle$, so that $A^2 \geq B^2$.

(iii) $\Rightarrow$ (i). Let $\mathcal{S}$ be lattice ordered and let $U = P - R$. Then $U \in S$ and it is elementary that $U^+ = P$. Let $V = 2P + 2Q + R$, and we have $V \in A, V \geq 0, V \geq U$, but $V \lneq U^+$, by Lemma 4.
(iv) \( \Rightarrow \) (i). Let \( \mathcal{S}' \) be the (real) dual space of \( \mathcal{S} \) with the induced dual order and let \( \mathcal{S}' \) be lattice ordered. Given \( x \in H_1 \) and \( y \in H_2 \), let \( f, g \in \mathcal{S}' \) be defined by

\[
f(V) = \langle V_1 x, x \rangle - \langle V_3 y, y \rangle, \quad g(V) = \langle V_1 x, x \rangle,
\]

where

\[
V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_3^* \end{bmatrix}.
\]

If \( V \geq 0 \), then \( V_1 \geq 0 \) and \( V_3 \geq 0 \). Hence \( f \leq g \) and so \( f^+ \leq g \), since \( g \geq 0 \). Then \( f(P) \leq f^+(P) \leq g(P) \) gives \( f^+(P) = \langle Px, x \rangle \) and \( 0 \leq f^+(R) \leq g(R) \) gives \( f^+(R) = 0 \). Also \( (g-f^+)(P + Q + R) = \mp f^+(Q) \geq 0 \), so that \( f^+(Q) = 0 \). Define \( h \in \mathcal{S}' \) by

\[
h(V) = \langle V(2^t x + y), 2^t x + y \rangle = 2\langle V_1 x, x \rangle + 2^t \text{Re} \langle V_2 y, x \rangle + \langle V_3 y, y \rangle.
\]

Then

\[
(h-f)(V) = \langle V_1 x, x \rangle + 2^t \text{Re} \langle V_2 y, x \rangle + 2\langle V_3 y, y \rangle = \langle V(x+2^t y), x+2^t y \rangle,
\]

which gives \( h-f \geq 0 \). But

\[
(h-f^+)(P + Q + R) = \langle Px, x \rangle + 2^t \text{Re} \langle Q_2 y, x \rangle + \langle Ry, y \rangle = \langle (P + 2^t Q + R)(x+y), x+y \rangle,
\]

and, by Lemma 4, we can choose \( x, y \) so that \( h \geq f^+ \).

(v) \( \Rightarrow \) (i). Let \( A = \frac{1}{2} P, B = P + Q + R, C = 4P + 2Q + R \). Then \( 0 \leq A \leq B + C \), by Lemma 4. Suppose that \( A = A_1 + A_2 \), with \( 0 \leq A_1 \leq B, 0 \leq A_2 \leq C \). Since \( A_1 \leq A \), it is easy to show that \( A_1 \) is of the form

\[
A_1 = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}.
\]

Then, since \( A_1 \leq B \), for \( x \in H_1 \) and \( y \in H_2 \) we have \( \langle X x, x \rangle \leq \langle (P + Q + R)(x+y), x+y \rangle = \| (SS^*)^t x + Sy \|^2 \), from the proof of Lemma 4. Since \( H_1 = (TH)^{-} \), we can choose \( y_n \in H_2 \) so that \( y_n \to -(SS^*)^t x \). This gives \( A_1 = 0 \). Hence \( \frac{1}{2} P = A = A_2 \leq C \), which contradicts Lemma 4.

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