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ON HOMEOMORPHISMS OF A 3-DIMENSIONAL HANDLEBODY

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1. Introduction. By a 3-dimensional *handlebody* V_n *of genus* n, we mean an oriented 3-manifold which is a disk-sum of n copies of $D^2 \times S^1$, where D^2 is the unit disk and S^1 is the boundary ∂D^2; and by a *surface* F_n *of genus* n we mean the oriented boundary surface ∂V_n.

Let $\mathscr{H}(F_n)$ be the group of all orientation preserving homeomorphisms of F_n onto itself, and $\mathscr{D}(F_n)$ the normal subgroup consisting of those homeomorphisms which are isotopic to the identity. Then the *mapping class group* $\mathscr{M}(F_n)$ *of* F_n is defined to be the quotient group $\mathscr{H}(F_n)/\mathscr{D}(F_n)$. By a classical result of Dehn [**7**], later simplified and reproved by Lickorish [**14**], the group $\mathscr{M}(F_n)$ is generated by so-called *Dehn twists*, see Birman [**3**; **4**, Chapter 4]. Now we consider a subgroup, say $\mathscr{H}^*(F_n)$, of $\mathscr{H}(F_n)$ consisting of those homeomorphisms which can be extended to homeomorphisms of V_n onto itself, and a subgroup, say $\mathscr{M}^*(F_n)$, of $\mathscr{M}(F_n)$ consisting of isotopy classes of elements in $\mathscr{H}^*(F_n)$. The purpose of this paper is to determine generators for $\mathscr{M}^*(F_n)$, which responds partially to Problem 4 of Birman [**5**]. The group $\mathscr{M}^*(F_0)$ is trivial, the group $\mathscr{M}^*(F_1)$ has been studied extensively, and generators for $\mathscr{M}^*(F_2)$ were determined by Goeritz [**9**].

After establishing a standard model of V_n and loops on ∂V_n, we note in Section 2 a characterization of $\mathscr{H}^*(F_n)$ given by Griffiths [**10**]. In Section 3 we define some elementary maps, and in Section 4 we prove our main theorem. We shall only be concerned with the combinatorial category, so all homeomorphisms and isotopies are piecewise linear, and all curves are polygonal.

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2. A model for V_n and a characterization of $\mathscr{H}^*(F_n)$.

2.1. For the sake of convenience, we first introduce a model for V_n in the 3-dimensional euclidean space R^3.

Let B^3 be a 3-cell in R^3. On ∂B^3 we take n mutually disjoint 2-cells $C_1^2, \ldots, C_n^2$, and also we take two disjoint 2-cells B_{i0} and B_{i1} in $\mathrm{Int}(C_i^2)$ for $1 \leqq i \leqq n$. Let $h_i : D^2 \times I \rightarrow R^3$, $1 \leqq i \leqq n$, be embeddings with

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$$h_i(D^2 \times \{0\}) = B_{i0}, \quad h_i(D^2 \times \{1\}) = B_{i1},$$

$$B^3 \cap h_i(D^2 \times I) = \partial B^3 \cap h_i(D^2 \times \partial I) = B_{i0} \cup B_{i1}$$

and $h_i(D^2 \times I) \cap h_j(D^2 \times I) = \emptyset$ for $i \neq j$, as shown in Fig. 1. We obtain a handlebody $V_n = B^3 \cup h_1(D^2 \times I) \cup \cdots \cup h_n(D^2 \times I)$ of genus n, and we call $h_i(D^2 \times I)$ the i-th *handle* of V_n. V_n has the orientation induced from that of R^3, and we give orientations to B_{i0}, C_i^2 and so $\partial B_{i0} = b_i$, $\partial C_i^2 = s_i$, $1 \leqq i \leqq n$, as shown in Fig. 1. We take simple oriented loops $a_1, \ldots, a_n$ on ∂V_n, a point p in $\partial B^3 - (C_1^2 \cup \ldots \cup C_n^2)$ and simple oriented arcs $d_1, \ldots, d_n$ on ∂V_n such that $a_i \cap b_i$ consists of one crossing point, $a_i \cap s_i = \emptyset$, $\partial d_i = p \cup (a_i \cap b_i)$, $d_i \cap (s_1 \cup \ldots \cup s_n) = d_i \cap s_i$ consists of one crossing point, $d_i \cap d_j = \partial d_i \cap \partial d_j = p$ for $i \neq j$, as shown in Fig. 1.

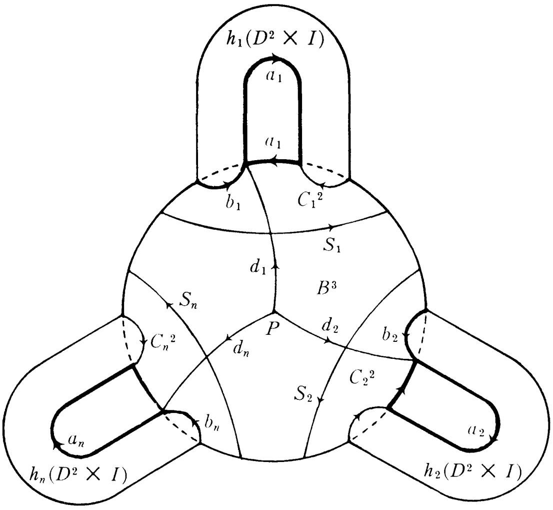

FIGURE 1

To avoid a multiplicity of brackets, we refer to loops rather than to these homotopy or homology classes. Then it is obvious that $\{a_1, \ldots, a_n\}$ and $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ form, respectively, free abelian bases for the first integral homology groups $H_1(V_n; Z)$ and $H_1(F_n; Z)$. We also use a_i, b_i and s_i as p-based loops $d_i a_i d_i^{-1}$, $d_i b_i d_i^{-1}$ and $\tilde{d}_i s_i \tilde{d}_i^{-1}$ unless confusion, where $\tilde{d}_i$ denotes an appropriate subarc of d_i, $1 \leqq i \leqq n$. Then, the fundamental group $\pi_1(V_n, p)$ is freely generated by $\{a_1, \ldots, a_n\}$, and $\pi_1(F_n, p)$ is generated by $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ subject to the single relation

$$\prod_{i=1}^{n} b_i^{-1} a_i^{-1} b_i a_i \simeq 1 \quad \text{rel } p \text{ on } F_n.$$

It holds that $s_i \simeq a_i^{-1} b_i^{-1} a_i b_i$ rel p on F_n, $1 \leqq i \leqq n$.

2.2. By Nielsen [**16**] and Mangler [**15**], $\mathscr{M}(F_n)$ can also be characterized algebraically as the group of classes (mod Inn $\pi_1(F_n, p)$) of automorphisms

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of $\pi_1(F_n, p)$ induced by free substitutions on the generators a_i, b_i which map $\Pi_{i=1}^n b_i^{-1}a_i^{-1}b_ia_i$ to its conjugate, see Birman [**3**, §1] and Birman-Hilden [**6**, §1].

It will be noticed that for any $\psi \in \mathscr{H}(F_n)$, there exists an $\eta \in \mathscr{D}(F_n)$ with $\eta\psi(p) = p$.

Let $\{x_1, \cdots, x_r\}^\nu$ be the smallest normal subgroup of $\pi_1(F_n, p)$ containing the elements $x_1, \cdots, x_r$ of $\pi_1(F_n, p)$.

It will be noted that:

2.3. PROPOSITION. (Griffiths [**11**, Theorem 7.2]) *Let $\iota : F_n \to V_n$ be the natural inclusion, and $K = \ker (\iota_\# : \pi_1(F_n, p) \to \pi_1(V_n, p))$. Then $K = \{b_1, \cdots, b_n\}^\nu$.*

Now we have the following characterization of $\mathscr{H}^*(F_n)$.

2.4. PROPOSITION. (Griffiths [**10**, Theorem 10.1]) *Let $\psi : (F_n, p) \to (F_n, p)$ be an orientation preserving homeomorphism. Then, $\psi \in \mathscr{H}^*(F_n)$ if and only if $\psi_\#(K) \subset K$.*

In [**5**, (14)], Birman defined two conjugate subgroups

$$\mathscr{A} = \{[\psi] \in \mathscr{M}(F_n) | \psi_\#(\{a_1, \cdots, a_n\}^\nu) \subset \{a_1, \cdots, a_n\}^\nu\}$$

and

$$\mathscr{B} = \{[\psi] \in \mathscr{M}(F_n) | \psi_\#(\{b_1, \cdots, b_n\}^\nu) \subset \{b_1, \cdots, b_n\}^\nu\}$$

of $\mathscr{M}(F_n)$. So the group $\mathscr{B}$ is exactly $\mathscr{M}^*(F_n)$.

3. Elementary homeomorphisms of a handlebody. Throughout Sections 3 and 4, we will make free use of fundamental results of curves on a surface which are given by Baer [**1**] and Epstein [**8**]. In this section, we construct various homeomorphisms of V_n onto itself. By $\mathscr{H}(V_n)$ we denote the group of all orientation preserving homeomorphisms of $V_n \to V_n$, and for $\psi \in \mathscr{H}(V_n)$ we denote the restriction $\psi|_{F_n}$ by $\dot{\psi} \in \mathscr{H}^*(F_n)$.

3.1. *Cyclic translation of handles* (cf. Griffiths [**10**, §4]). First we introduce a simple homeomorphism which is the same map as σ in Griffiths [**10**]. Let ρ be the rotation of V_n (and so R^3) on itself, about the vertical axis joining p and the center of the 3-cell B^3, and through $2\pi/n$ radians in the clockwise direction. Of course, $\rho \in \mathscr{H}(V_n)$, and it induces the automorphism

$$\dot{\rho}_\# : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_i \to a_{i+1} & (1 \leqq i \leqq n), \\ b_i \to b_{i+1} & (1 \leqq i \leqq n), \end{cases}$$

where the indices are taken as modulo n.

3.2. *Twisting a knob* (Goeritz [**9**, p. 251], Griffiths [**10**, §5]). Since the loops $s_1, \cdots, s_n$ are contractible in B^3, we have mutually disjoint properly embedded 2-cells $C_1'^2, \cdots, C_n'^2$ in B^3 with $\partial C_i'^2 = s_i$. $C_i'^2$ cuts off a handlebody, say K_i, of genus 1 which contains the i-th handle $h_i(D^2 \times I)$; we call K_i the i-th *knob* of V_n. We take a 2-cell $C^2 \subset K_1$ which is parallel to $C_1'^2$ in V_n. Let $f: I \times D^2 \to V_n$ be an embedding with $f(\{0\} \times D^2) = C_1'^2, f(\{1\} \times D^2) = C^2$ and $f(I \times D^2) \cap F_n = f(I \times \partial D^2)$. We twist the knob K_1 about the line $f(I \times \{0\})$ through π radians keeping $f(\{0\} \times D^2) = C_1'^2$ fixed. Now we have a map $\omega_1 \in \mathscr{H}(V_n)$ with $\omega_1|_{V_n - K_1}$ = identity (see Fig. 2).

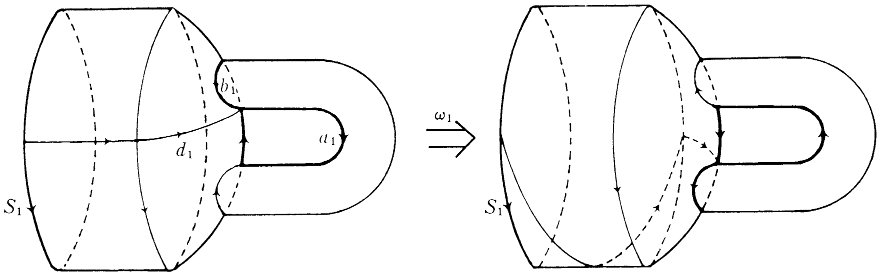

FIGURE 2

The induced automorphism is given by:

$$\dot{\omega}_{1\#} : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_1 \to a_1^{-1} s_1^{-1}, & a_j \to a_j \quad (2 \leqq j \leqq n), \\ b_1 \to a_1^{-1} b_1^{-1} a_1, & b_j \to b_j \quad (2 \leqq j \leqq n). \end{cases}$$

For every i, $1 \leqq i \leqq n$, we define $\omega_i \in \mathscr{H}(V_n)$ by the composites

$$\omega_i = \rho^{i-1} \omega_1 \rho^{-(i-1)}.$$

3.3. *Twisting a handle* (Dehn [**7**], Lickorish [**14**], Birman [**3**; **4**]). The following maps τ_i, $1 \leqq i \leqq n$, are the same maps as "C-homeomorphisms" using b_i in Lickorish [**14**], or "Dehn twists" about b_i in Birman [**3**; **4**]. That is, $\tau_1 \in \mathscr{H}(V_n)$ is defined in the following way. Cut V_n along B_{10}, twist the new free end $h_1(D^2 \times \{0\})$ through 2π, and glue together again. We obtain the induced automorphism as follows:

$$\dot{\tau}_{1\#} : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_1 \to a_1 b_1^{-1}, & a_j \to a_j \quad (2 \leqq j \leqq n), \\ b_i \to b_i \quad (1 \leqq i \leqq n). \end{cases}$$

We also define $\tau_i \in \mathscr{H}(V_n)$, $1 \leqq i \leqq n$, by:

$$\tau_i = \rho^{i-1} \tau_1 \rho^{-(i-1)}.$$

3.4. *Interchanging two knobs* (Griffiths [**10**, pp. 198–201]). The following map $\rho_{12} \in \mathscr{H}(V_n)$ is the same map as ψ in Griffiths [**10**, §6]. In the notation in Section 2, we take a simple arc e on ∂B^3 such that e spans s_1 and s_2 and $e \cap (s_1 \cup \cdots \cup s_n \cup d_1 \cup \cdots \cup d_n) = e \cap (s_1 \cup s_2) = \partial e$. Let D be the regular neighborhood of $C_1^2 \cup e \cup C_2^2$ on ∂B^3. By twisting $C_1^2 \cup e \cup C_2^2$ in D through π radians in the clockwise direction we have a homeomorphism $\rho_{12}' : D \to D$ such that $\rho_{12}'(C_1^2) = C_2^2$, $\rho_{12}'(C_2^2) = C_1^2$, $\rho_{12}'(e) = e$ and $\rho_{12}'|_{\partial D}$ = identity (see Fig. 3).

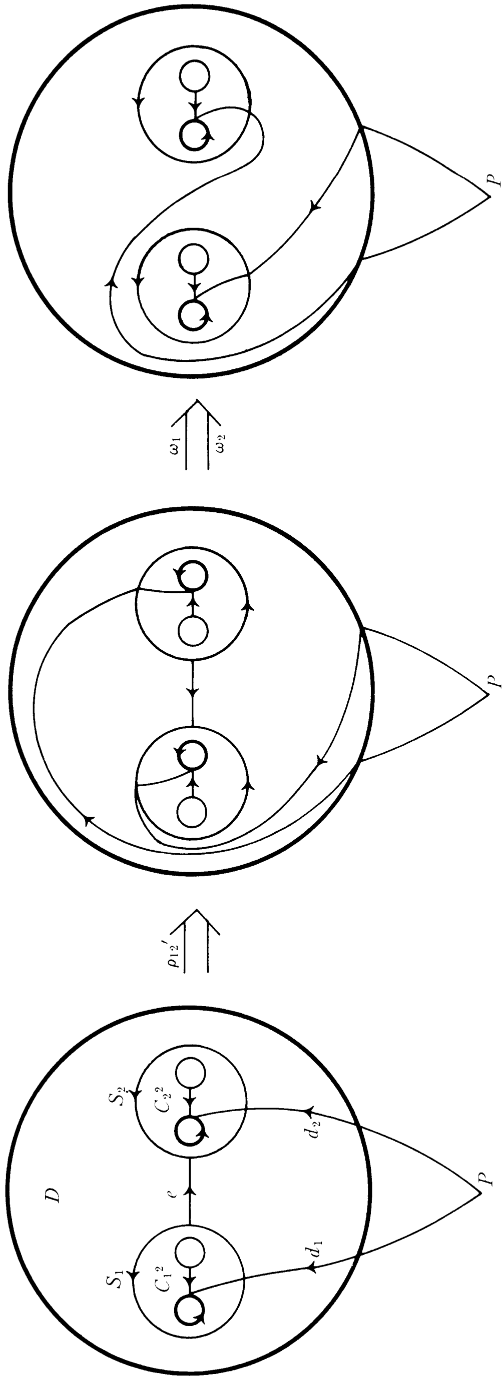

FIGURE 3

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Thus the homeomorphism ρ_{12}' is easily extended to a homeomorphism of V_n onto itself, so that $\rho_{12}'(K_1) = K_2$, $\rho_{12}'(K_2) = K_1$ and $\rho_{12}'|_{V_n - K} =$ identity, where K is the regular neighborhood of $K_1 \cup e \cup K_2$ in V_n. We apply the maps ω_1 and ω_2, and we obtain the map $\rho_{12} \in \mathscr{H}(V_n)$ as

$$\rho_{12} = \omega_2 \omega_1 \rho_{12}'.$$

The induced automorphism is given by:

$$\dot{\rho}_{12\#} : \pi_1(F_n, p) \rightarrow \pi_1(F_n, p) : \begin{cases} a_1 \rightarrow s_1^{-1} a_2 s_1, \quad a_2 \rightarrow a_1, \\ a_j \rightarrow a_j \quad (3 \leqq j \leqq n), \\ b_1 \rightarrow s_1^{-1} b_2 s_1, \quad b_2 \rightarrow b_1, \\ b_j \rightarrow b_j \quad (3 \leqq j \leqq n). \end{cases}$$

We define maps in $\mathscr{H}(V_n)$ by:

$$\rho_{i,i+1} = \rho^{i-1} \rho_{12} \rho^{-(i-1)} \quad (1 \leqq i \leqq n),$$
$$\rho_{1,1+r} = (\rho_{12}^{-1} \cdots \rho_{r-1,r}^{-1}) \rho_{r,r+1} (\rho_{r-1,r} \cdots \rho_{12}),$$
$$\rho_{i,i+r} = \rho^{i-1} \rho_{1,1+r} \rho^{-(i-1)} \quad (1 \leqq i \leqq n, 1 \leqq r \leqq n-1),$$

where the indices are taken as modulo n. It will be noticed that $\rho_{i,i+r}$ is obtained by the same way as that of ρ_{12} using a simple arc $e_{i,i+r}$ on ∂B^3 such that $e_{i,i+r}$ spans s_i and s_{i+r} and $e_{i,i+r} \cap (s_1 \cup \cdots \cup s_n \cup d_1 \cup \cdots \cup d_n) = e_{i,i+r} \cap (s_i \cup s_{i+r}) = \partial e_{i,i+r}$.

3.5. *Spin and sliding* (cf. Birman [**4**, Chapter 4]). Let $V_n{}^i$ be a handlebody of genus $n-1$ obtained from the V_n by removing the i-th handle $h_i(D^2 \times I)$. Let z_{i0} and z_{i1} be the centers of B_{i0} and B_{i1}, respectively. Suppose that c is any simple oriented loop on $\partial V_n{}^i$ with $z_{i0} \notin c$, $z_{i1} \in c$. Let N be a cylindrical neighborhood of c on $\partial V_n{}^i$, parametrized by (y, θ), with $-1 \leqq y \leqq 1$, $0 \leqq \theta \leqq 2\pi$, where c is described by $y = 0$, and $z_{i1} = (0, 0)$. We use the map "*spin of* z_{i1} *about* c" given by Birman [**4**, p. 158] except for obvious modifications. An orientation preserving homeomorphism $\dot{\sigma}_{cz_{i1}} : \partial V_n{}^i \rightarrow \partial V_n{}^i$, which will be called a *spin of* z_{i1} *about* c, is defined by the rule that if a point is in N, then its image is given by:

$$\dot{\sigma}_{cz_{i1}}(y, \theta) = (y, \theta + 2\pi(2y - 1)) \quad \text{if } 1/2 \leqq y \leqq 1,$$
$$\dot{\sigma}_{cz_{i1}}(y, \theta) = (y, \theta - 2\pi(2y + 1)) \quad \text{if } -1 \leqq y \leqq -1/2,$$
$$\dot{\sigma}_{cz_{i1}}(y, \theta) = (y, \theta) \quad \text{if } -1/2 \leqq y \leqq 1/2,$$

while all points of $\partial V_n{}^i - N$ are left fixed, see Fig. 4.

It is easy to see that the $\dot{\sigma}_{cz_{i1}}$ is extended to an orientation preserving homeomorphism of $V_n{}^i \rightarrow V_n{}^i$; which may be denoted by $\sigma_{cz_{i1}}$ and still called a *spin of* z_{i1} *about* c. Without loss of generality we may assume that $B_{i1} \subset N$ with $-1/2 \leqq y \leqq 1/2$, and $B_{i0} \cap N = \emptyset$. So we can extend the $\sigma_{cz_{i1}}$ to a map $\sigma_{cB_{i1}} \in H(V_n)$ with $\sigma_{cB_{i1}}|_{h_i(D^2 \times I)} =$ identity; and we will call it a *sliding of* B_{i1} *about* c. Replacing z_{i1} and B_{i1} with z_{i0} and B_{i0}, we obtain a *spin* $\sigma_{cz_{i0}}$ *of* z_{i0} *about* c and a *sliding* $\sigma_{cB_{i0}}$ *of* B_{i0} *about* c.

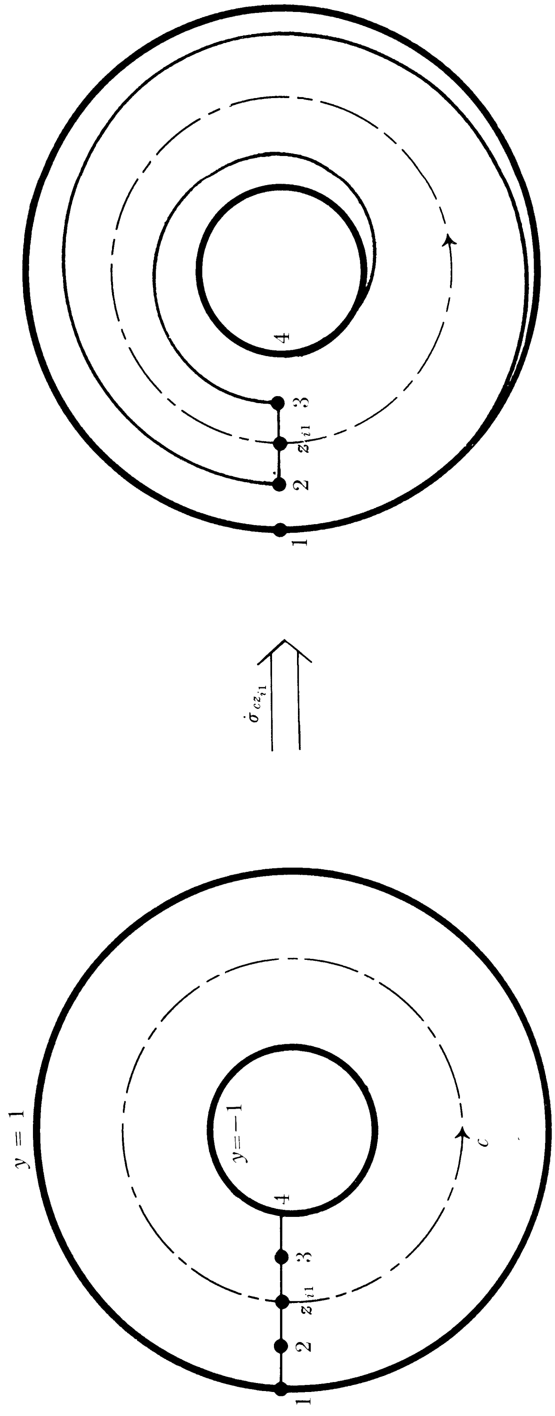

FIGURE 4

We use frequently a simple oriented loop c on $\partial V_n{}^i$ with $z_{i0} \in c$ and $z_{i1} \notin c$ (respectively, $z_{i0} \notin c$ and $z_{i1} \in c$). For brevity, such a simple loop will be called a z_{i0}-*loop* (respectively, a z_{i1}-*loop*). The followings are immediate consequences of the definition of a sliding.

3.6. LEMMA. (1) *Let c be a z_{i1}-loop such that $c \simeq 1 \operatorname{rel} z_{i1}$ on $\partial V_n{}^i - z_{i0}$. Let C^2 be a 2-cell on $\partial V_n{}^i$ with $\partial C^2 = c$ (as sets), and we assume that C^2 has the orientation induced from that of $\partial V_n{}^i$. Then $\sigma_{cB_{i1}}$ is isotopic to $\tau_i{}^{-1}$ or τ_i according as the orientation of c does or does not agree with the orientation of ∂C^2.*

(2) *Let c_1 and c_2 be z_{i1}-loops. Suppose $c_1 \simeq c_2 \operatorname{rel} z_{i1}$ on $\partial V_n{}^i - z_{i0}$; then $\sigma_{c_1B_{i1}}$ is isotopic to $\sigma_{c_2B_{i1}}$ modulo τ_i.*

3.7. LEMMA. *Let $c_0, c_1, \cdots, c_m$ be z_{i1}-loops. Suppose that $c_0 \simeq c_1 \cdots c_m \operatorname{rel} z_{i1}$ on $\partial V_n{}^i - z_{i0}$; then $\sigma_{c_0B_{i1}}$ is isotopic to the product $\sigma_{c_mB_{i1}} \cdots \sigma_{c_1B_{i1}}$ modulo τ_i.*

From now on, we will select some special loops on $\partial V_n{}^i$ and define some special slidings.

3.8 *Slidings* θ. Let α be a z_{11}-loop such that

$$\alpha \cap (a_2 \cup b_2 \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \alpha \cap b_2$$

consists of one crossing point and $\alpha \simeq a_2$ on $\partial V_n{}^1 - z_{10}$. Now we will denote by θ_{12} the composition map $\sigma_{\alpha B_{11}}\tau_1{}^{-1}$. Then the induced automorphism is given by:

$$\dot{\theta}_{12\#} : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_1 \to a_1(b_2{}^{-1}a_2{}^{-1}b_2), \\ a_j \to a_j \quad (j \neq 1), \\ b_2 \to a_2b_2(a_1{}^{-1}b_1a_1)(b_2{}^{-1}a_2{}^{-1}b_2), \\ b_j \to b_j \quad (j \neq 2). \end{cases}$$

We define $\theta_{ij}, \theta_{12}{}^*, \theta_{ij}{}^* \in \mathscr{H}(V_n)$ by:

$$\begin{aligned}
\theta_{1,1+r} &= \rho_{2,1+r}\theta_{12}\rho_{2,1+r}{}^{-1}, \quad 1 \leqq r \leqq n-1, \\
\theta_{i,i+r} &= \rho^{i-1}\theta_{1,1+r}\rho^{-(i-1)}, \quad 1 \leqq r \leqq n-1, 1 \leqq i \leqq n, \\
\theta_{12}{}^* &= \omega_1{}^{-1}\theta_{12}\omega_1, \\
\theta_{1,1+r}{}^* &= \rho_{2,1+r}\theta_{12}{}^*\rho_{2,1+r}{}^{-1}, \quad 1 \leqq r \leqq n-1, \\
\theta_{i,i+r}{}^* &= \rho^{i-1}\theta_{1,1+r}{}^*\rho^{-(i-1)}, \quad 1 \leqq r \leqq n-1, 1 \leqq i \leqq n,
\end{aligned}$$

where ρ_{jj} = identity, and the indices are taken as modulo n.

It should be noted that:

(1) $\theta_{12}{}^*$ is a sliding $\sigma_{\alpha B_{10}}$ of B_{10} about α modulo τ_1, where α is a z_{10}-loop such that $\alpha \cap (a_2 \cup b_2 \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \alpha \cap b_2$ consists of one crossing point and $\alpha \simeq a_2$ on $\partial V_n{}^1 - z_{11}$.

(2) θ_{ij} (respectively, $\theta_{ij}{}^*$) is a sliding $\sigma_{\alpha B_{i1}}$ of B_{i1} (respectively, $\sigma_{\alpha B_{i0}}$ of B_{i0}) about α modulo τ_i, where α is a z_{i1}-loop (respectively, a z_{i0}-loop) such that

$$\alpha \cap (a_1 \cup b_1 \cup \cdots \cup a_{i-1} \cup b_{i-1} \cup a_{i+1} \cup b_{i+1} \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \alpha \cap b_j$$

consists of one crossing point and $\alpha \simeq a_j$ on $\partial V_n{}^i - z_{i0}$ (respectively, $\partial V_n{}^i - z_{i1}$).

3.9. *Slidings* ξ. Let β be a z_{11}-loop such that

$$\beta \cap (a_2 \cup b_2 \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \beta \cap a_2$$

consists of one crossing point and $\beta \simeq b_2$ on $\partial V_n{}^1 - z_{10}$. We also denote by ξ_{12} the composition map $\sigma_{\beta B_{11}} \tau_1{}^{-1}$. Then, the induced automorphism is given by:

$$\xi_{12\#} : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_1 \to b_1 a_1 b_2{}^{-1} s_2 (a_1{}^{-1} b_1{}^{-1} a_1), \\ a_2 \to a_2 b_2 (a_1{}^{-1} b_1{}^{-1} a_1) b_2{}^{-1}, \\ a_j \to a_j \quad (j \neq 1, 2), \\ b_i \to b_i \quad (1 \leqq i \leqq n). \end{cases}$$

We also define ξ_{ij}, $\xi_{12}{}^*$, $\xi_{ij}{}^* \in \mathscr{H}(V_n)$ by:

$$\begin{aligned} \xi_{1,1+r} &= \rho_{2,1+r} \xi_{12} \rho_{2,1+r}{}^{-1}, \quad 1 \leqq r \leqq n-1, \\ \xi_{i,i+r} &= \rho^{i-1} \xi_{1,1+r} \rho^{-(i-1)}, \quad 1 \leqq r \leqq n-1, 1 \leqq i \leqq n, \\ \xi_{12}{}^* &= \omega_1{}^{-1} \xi_{12} \omega_1, \\ \xi_{1,1+r}{}^* &= \rho_{2,1+r} \xi_{12}{}^* \rho_{2,1+r}{}^{-1}, \quad 1 \leqq r \leqq n-1, \\ \xi_{i,i+r}{}^* &= \rho^{i-1} \xi_{1,1+r}{}^* \rho^{-(i-1)}, \quad 1 \leqq r \leqq n-1, 1 \leqq i \leqq n, \end{aligned}$$

where ρ_{jj} = identity, and the indices are taken as modulo n.

It will be noticed that:

(1) $\xi_{12}{}^*$ is a sliding $\sigma_{\beta B_{10}}$ of B_{10} about β modulo τ_1, where β is a z_{10}-loop such that $\beta \cap (a_2 \cup b_2 \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \beta \cap a_2$ consists of one crossing point and $\beta \simeq b_2$ on $\partial V_n{}^1 - z_{11}$.

(2) ξ_{ij} (respectively, $\xi_{ij}{}^*$) is a sliding $\sigma_{\beta B_{i1}}$ of B_{i1} (respectively, $\sigma_{\beta B_{i0}}$ of B_{i0}) about β modulo τ_i, where β is a z_{i1}-loop (respectively, a z_{i0}-loop) such that

$$\beta \cap (a_1 \cup b_1 \cup \cdots \cup a_{i-1} \cup b_{i-1} \cup a_{i+1} \cup b_{i+1} \cup \cdots \cup a_n \cup b_n \cup d_1 \cup \cdots \cup d_n) = \beta \cap a_j$$

consists of one crossing point and $\beta \simeq b_j$ on $\partial V_n{}^i - z_{i0}$ (respectively, $\partial V_n{}^i - z_{i1}$).

3.10. LEMMA. *Let c be any z_{i1}-loop. Then, the sliding $\sigma_{cB_{i1}}$ of B_{i1} about c is isotopic to a power product of θ_{ij}'s, ξ_{ij}'s and τ_i's, where $j \neq i$, $1 \leqq j \leqq n$. This remains valid if $\sigma_{cB_{i0}}$, $\theta_{ij}{}^*$, $\xi_{ij}{}^*$ are substituted for $\sigma_{cB_{i1}}$, θ_{ij}, ξ_{ij}, respectively.*

Proof. From the definitions in 3.3, 3.5, 3.8 and 3.9, it suffices to show the case $i = 1$ of $\sigma_{cB_{11}}$. We choose a system of z_{11}-loops $\{\alpha_2, \cdots, \alpha_n, \beta_2, \cdots, \beta_n\}$ on $\partial V_n{}^1$ such that $\alpha_j \cap \alpha_k = z_{11} (j \neq k)$, $\beta_j \cap \beta_k = z_{11} (j \neq k)$, $\alpha_j \cap \beta_h = z_{11} (1 \leqq j, h \leqq n)$ and α_j and $\beta_j (2 \leqq j \leqq n)$, satisfy the conditions of α and β in 3.8(2) and 3.9(2) with $i = 1$, respectively. We know that the set $\{\alpha_2, \cdots, \alpha_n, \beta_2, \cdots, \beta_n\}$ forms a free basis of $\pi_1(\partial V_n{}^1 - z_{10}, z_{11})$; a free group of rank $2n - 2$. So, c is homotopic (rel z_{11} on $\partial V_n{}^1 - z_{10}$) to a power product of $\alpha_2, \cdots, \alpha_n, \beta_2, \cdots, \beta_n$. By virtue of Lemma 3.6, Lemma 3.7 and the remarks 3.8(2) and 3.9(2), we conclude the lemma.

3.11. COROLLARY. *Any sliding $\sigma_{cB_{i1}}$ of B_{i1} and any sliding $\sigma_{cB_{i0}}$ of B_{i0}, $1 \leqq i \leqq n$, are isotopic to power products of ρ, ρ_{12}, ω_1, τ_1, θ_{12} and ξ_{12}.*

3.12. For future reference, we introduce maps $\mu_i \in \mathscr{H}(F_n)$, $1 \leqq i \leqq n$, with $\mu_i \notin \mathscr{H}^*(F_n)$. With K_1 as in 3.2, let $U_1 = \partial K_1 \cap \partial V_n$; U_1 is a compact oriented 2-manifold of genus 1 with $\partial U_1 = s_1$. Cutting U_1 along the simple loops a_1 and b_1, we have an annulus U_1', and we twist the new boundary of U_1' through $\pi/2$ radians keeping s_1 fixed. Now we have a map $\mu_1 \in \mathscr{H}(F_n)$ with $\mu_1|_{F_n - U_1}$ = identity, $\mu_1(a_1) = b_1$, $\mu_1(b_1) = -a_1$. The induced automorphism is given by:

$$\mu_{1\#} : \pi_1(F_n, p) \to \pi_1(F_n, p) : \begin{cases} a_1 \to a_1^{-1}b_1a_1, & a_j \to a_j \quad (j \neq 1), \\ b_1 \to a_1^{-1}, & b_j \to b_j \quad (j \neq 1). \end{cases}$$

For every i, $1 \leqq i \leqq n$, we define $\mu_i \in \mathscr{H}(F_n)$ by:

$$\mu_i = \rho^{i-1}\mu_1\rho^{-(i-1)}.$$

4. Generators for $\mathscr{M}^*(F_n)$.

In this section, we will establish the following:

4.1. THEOREM. *The group $\mathscr{M}^*(F_n)$ is generated by $[\dot\rho]$, $[\dot\rho_{12}]$, $[\dot\omega_1]$, $[\dot\tau_1]$, $[\dot\theta_{12}]$ and $[\dot\xi_{12}]$. In particular, $\mathscr{M}^*(F_0) \cong 0$, $\mathscr{M}^*(F_1)$ is generated by $[\dot\omega_1]$ and $[\dot\tau_1]$, and $\mathscr{M}^*(F_2)$ by $[\dot\rho]$, $[\dot\omega_1]$, $[\dot\tau_1]$, $[\dot\theta_{12}]$ and $[\dot\xi_{12}]$.*

The proof will be given by an induction on genus n, utilizing Birman's result in [**4**] and Corollary 3.11.

4.2. *The cases $n = 0$ and $n = 1$.* In the case $n = 0$, it is well-known that $\mathscr{M}(F_0) \cong \mathscr{M}^*(F_0) \cong 0$; recall that F_0 is a 2-sphere. In the case $n = 1$, it is also well-known that $\mathscr{M}(F_1) \cong \mathrm{Sp}\ (2, Z)$, the group of 2×2 integral matrices with determinant 1; see, for example, Birman [**3**, p. 58]. $\mathscr{M}^*(F_1)$ is isomorphic to a subgroup of Sp $(2, Z)$ consisting of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c = 0$. Such a subgroup is generated by two matrices $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and these are representation matrices of $\dot\tau_{1\#}$ and $\dot\omega_{1\#}$, respectively, as automorphisms of the free Z-module $\pi_1(F_1, p) \cong H_1(F_1; Z)$ with the basis $\{a_1, b_1\}$. The topological proof is omitted here.

4.3. *Proof of* 4.1; *First step*. From now on, we assume that $n \geqq 2$. For brevity, let G denote the set $\{\rho, \rho_{12}, \omega_1, \tau_1, \theta_{12}, \xi_{12}\}$, and let $\mathscr{D}(V_n)$ denote the subgroup of $\mathscr{H}(V_n)$ whose elements are itosopic to the identity, and $\mathscr{S}(n)$ a subgroup of $\mathscr{H}(V_n)$ which is generated by all slidings, and so by θ_{ij}, θ_{ij}^*, ξ_{ij}, ξ_{ij}^* and τ_i, $1 \leqq i \neq j \leqq n$, by Lemmas 3.6 and 3.10. From the definitions of maps in §3 and Corollary 3.11, it suffices to show that $\mathscr{M}^*(F_n)$ is generated by $[\dot\rho]$, $[\dot\rho_{ij}]$, $[\dot\omega_i]$ and the isotopy classes of slidings in $\mathscr{S}(n)$.

Let $\psi \in \mathscr{H}^*(F_n)$ and $\psi : V_n \to V_n$ an extension homeomorphism. We claim:

4.4. LEMMA. *There exists an element $\psi_0 \in \mathscr{H}(V_n)$ that is power product of elements in G and $\mathscr{D}(V_n)$ such that $\psi_0\psi|_{B_{n0}}$ = identity.*

Proof. Let ∇ denote the union $B_{10} \cup B_{11} \cup \cdots \cup B_{n0} \cup B_{n1}$. Since V_n is irreducible, there exists an $\eta_1 \in \mathscr{D}(V_n)$ such that $\eta_1\psi(B_{n0}) \cap \nabla$ consists of a finite number of simple arcs; we denote $\eta_1\psi$ by ψ_1. Let $l \subset \psi_1(B_{n0}) \cap \nabla$ be an innermost arc on $\psi_1(B_{n0})$, and let $\Delta \subset \psi_1(B_{n0})$ be the 2-cell cut off by l with $\operatorname{Int} \Delta \cap \nabla = \emptyset$. We assume $l \subset \psi_1(B_{n0}) \cap B_{k0}$, and $m = \partial\Delta - l$.

If $m \subset h_k(\partial D^2 \times I)$, there is $\eta_0 \in \mathscr{D}(V_n)$ with $\eta_0\psi_1(B_{n0}) \cap \nabla \subset (\psi_1(B_{n0}) \cap \nabla) - l$. So we may assume that $m \cap h_k(\partial D^2 \times I) = \emptyset$. The simple loop $l \cup m$ divides ∂B^3 into two 2-cells, say Σ_1 and Σ_2, and we assume that $\Sigma_2 \supset B_{k1}$. If $B_{i0} \subset \Sigma_1$ (respectively, $B_{i1} \subset \Sigma_1$), $i \neq k$, we can choose a z_{i0}– (respectively, a z_{i1}–) loop c such that $c \cap \partial B_{k0}$ consists of one crossing point; and we apply $\sigma_{cB_{i0}}$ of B_{i0} (respectively, $\sigma_{cB_{i1}}$ of B_{i1}) about c. Now it is easy to obtain an element $\eta_2 \in \mathscr{D}(V_n)$ such that

$$\eta_2\sigma_{cB_{i\epsilon}}(\Sigma_1) \subset \Sigma_1, \quad \eta_2\sigma_{cB_{i\epsilon}}(\Sigma_1) \cap B_{i\epsilon} = \emptyset, \quad \epsilon = 0 \text{ or } 1.$$

We denote $\eta_2\sigma_{cB_{i\epsilon}}\psi_1$ by ψ_2. It will be noticed that:

(*) If $B_{i\epsilon} \cap \psi_1(B_{n0}) \neq \emptyset$, then some new intersections occur in $\psi_2(B_{n0}) \cap B_{k0}$.

Repeating the procedure, we can assume that $\eta_2\sigma_{cB_{i\epsilon}}(l \cup m) = l \cup \eta_2\sigma_{cB_{i\epsilon}}(m)$ bounds a 2-cell $\eta_2\sigma_{cB_{i\epsilon}}(\Sigma_1)$ on ∂B^3 with $\eta_2\sigma_{cB_{i\epsilon}}(\Sigma_1) \cap \nabla = \Sigma_1 \cap B_{k0}$. Now, there exists $\eta_3 \in \mathscr{D}(V_n)$ with

$$\eta_3\psi_2(B_{n0}) \cap \nabla \subset (\psi_1(B_{n0}) \cap \nabla) - l;$$

it will be noticed that the new intersections given in (*) are also removed.

By the repetition of the procedure, we can conclude that there exist $\eta \in \mathscr{D}(V_n)$ and $\sigma \in \mathscr{S}(n)$ with $\eta\sigma\psi(B_{n0}) \cap \nabla = \emptyset$; let ψ_3 denote $\eta\sigma\psi$.

There are two cases to be considered.

Case 1: There exists a handle, say $h_k(D^2 \times I)$, with $\psi_3(B_{n0}) \subset h_k(D^2 \times I)$. Since $\partial B_{n0} \not\simeq 1$ on F_n, there exists $\eta_4 \in \mathscr{D}(V_n)$ with $\eta_4\psi_3(B_{n0}) = B_{k0}$. Then $\rho^{n-k}\eta_4\psi_3(B_{n0}) = B_{n0}$; let ψ_4 denote $\rho^{n-k}\eta_4\psi_3$. If the orientation of $\psi_4(B_{n0})$ does not agree with that of B_{n0}, we apply ω_n and an appropriate $\eta_5 \in \mathscr{D}(V_n)$, so that $\eta_5\omega_n\psi_4(B_{n0}) = B_{n0}$ and $\eta_5\omega_n\psi_4(B_{n0})$ has the same orientation as that of B_{n0}.

Let $\psi_5 = \psi_4$ or $\eta_5\omega_n\psi_4$ according as the orientation of $\psi_4(B_{n0})$ does or does not agree with that of B_{n0}. Since $\psi_5|_{B_{n0}}$ is orientation preserving, there exists $\eta_6 \in \mathscr{D}(V_n)$ with $\eta_6\psi_5|_{B_{n0}}$ = identity, this completes the proof of 4.4 in Case 1.

Case 2: $\psi_3(B_{n0}) \subset B^3$. Now the simple loop $\psi_3(\partial B_{n0}) \subset \partial B^3 - \nabla$ bounds two 2-cells Σ_1' and Σ_2' on ∂B^3. Since ∂B_{n0} is not homologous to zero on F_n, there exists a handle, say $h_k(D^2 \times I)$, of V_n with $B_{k0} \subset \Sigma_1'$ and $B_{k1} \subset \Sigma_2'$.

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If there is a handle $h_j(D^2 \times I), j \neq k$, with $B_{j0} \subset \Sigma_1'$ (respectively $B_{j1} \subset \Sigma_1'$), we can choose a z_{j0}– (respectively, z_{j1}–) loop d such that $d \cap \partial B_{k0}$ consists of one crossing point, and we apply $\sigma_{dB_{j0}}$ of B_{j0} (respectively. $\sigma_{dB_{j1}}$ of B_{j1}). Then there exists $\eta_7 \in \mathscr{D}(V_n)$ such that $\eta_7\sigma_{dB_{j\epsilon}}(\Sigma_1') \subset \Sigma_1'$, $\eta_7\sigma_{dB_{j\epsilon}}(\Sigma_1') \cap B_{j\epsilon} = \emptyset$, $\epsilon = 0$ or 1; let $\psi_6 = \eta_7\sigma_{dB_{j\epsilon}}\psi_3$. Repeating the procedure, we can now assume that $\psi_6(\partial B_{n0})$ bounds a 2-cell Σ_1' on ∂B^3 with $\Sigma_1' \cap \nabla = \Sigma_1' \cap B_{k0} = B_{k0}$; and we have $\eta_8 \in \mathscr{D}(V_n)$ with $\eta_8\psi_6(B_{n0}) = B_{k0}$. In the same way as that of Case 1, we also conclude 4.4 in Case 2.

4.5. *Proof of* 4.1; *Second step*: By virtue of Lemma 4.4, to prove Theorem 4.1, we may assume that $\psi|_{B_{n0}} =$ identity. From the definition of handles, we may assume that $\psi|_{h_n(D^2\times I)} =$ identity with appropriate isotopy. Let

$$\psi' = \psi|_{V_n{}^n} : V_n{}^n \rightarrow V_n{}^n;$$

ψ' is an orientation preserving homeomorphism with $\psi'|_{B_{n0}\cup B_{n1}} =$ identity; and we regard $V_n{}^n$ as V_{n-1}. It will be noticed that ψ will be isotopic to the identity if and only if ψ' is isotopic to the identity.

Now, induction of genus n is in order. The group $\mathscr{M}^*(F_1)$ is generated by the isotopy classes of the elements of G. We assume, inductively, that $\mathscr{M}^*(F_{n-1})$ is generated by the isotopy classes of the elements of G.

Let $\mathscr{H}^*(F_{n-1}; z_{n0} \cup z_{n1})$ be the group of all orientation preserving homeomorphisms $\psi \in \mathscr{H}^*(F_{n-1})$ with $\psi(z_{n0} \cup z_{n1}) = z_{n0} \cup z_{n1}$, and let $\mathscr{M}^*(F_{n-1}; z_{n0} \cup z_{n1})$ be the group of all isotopy classes of elements in $\mathscr{H}^*(F_{n-1}; z_{n0} \cup z_{n1})$ with respect to isotopies keeping $z_{n0} \cup z_{n1}$ fixed. We can state our version of a special case of Birman's result.

4.6. LEMMA. (Birman [**4**, Theorem 4.2, Theorem 4.3 and pp. 158–160]). *Let* $j_* : \mathscr{M}^*(F_{n-1}; z_{n0} \cup z_{n1}) \rightarrow \mathscr{M}^*(F_{n-1})$ *be the homomorphism induced by the natural inclusion* $j : \mathscr{H}^*(F_{n-1}; z_{n0} \cup z_{n1}) \rightarrow \mathscr{H}^*(F_{n-1})$. *Then,*

$$\mathscr{M}^*(F_{n-1}; z_{n0} \cup z_{n1})$$

is generated by ker j_* *and the lifts of the generators of* $\mathscr{M}^*(F_{n-1})$ *to* $\mathscr{M}^*(F_{n-1}; z_{n0} \cup z_{n1})$. *Moreover,* ker j_* *is generated by* $\dot{\omega}_n' = \dot{\omega}_n|_{F_{n-1}}$ *and spins of* z_{n0} *and* z_{n1} *about appropriate loops.*

We proceed with our proof. From the definitions of maps in Section 3, we conclude that:

4.7. *By slight modifications, if necessary, we may assume that every element of* G *keeps* $z_{n0} \cup z_{n1}$ *fixed as a homeomorphism of* $V_{n-1} \rightarrow V_{n-1}$.

Thus, by Lemma 4.6, ψ' is isotopic to a power product of $\dot{\omega}_n'$, spins of z_{n0} and z_{n1} and $\dot{\rho}$, $\dot{\rho}_{12}$, $\dot{\omega}_1$, $\dot{\tau}_1$, $\dot{\theta}_{12}$ and $\dot{\xi}_{12}$; and so ψ' is isotopic to a power product of $\omega_n' = \omega_n|_{V_{n-1}}$, spins of z_{n0} and z_{n1} and elements of G (as homeomorphisms of

$V_{n-1} \to V_{n-1}$). By the definitions of spin and sliding given in 3.5, we conclude that ψ is isotopic to a power product of ω_n, slidings of B_{n0} and B_{n1} and elements of G. By the definition of ω_n and Corollary 3.11, ψ is isotopic to a power product of elements of G, this completes the proof of 4.1.

4.8. PROPOSITION. *The group* $\mathscr{M}(F_n)$ *is generated by* $[\dot{\rho}]$, $[\dot{\tau}_1]$, $[\dot{\theta}_{12}]$ *and* $[\mu_1]$.

Proof. By Lickorish [**14**] (cf. Birman [**3**; **4**]), $\mathscr{M}(F_n)$ is generated by Dehn twists about simple loops $a_1, \cdots, a_n, b_1, \cdots, b_n$ and $\gamma_1, \cdots, \gamma_{n-1}$ on F_n, where γ_j is contained in $\partial B^3 - (B_{10} \cup B_{11} \cup \cdots \cup B_{n0} \cup B_{n1})$ which bounds a 2-cell $\Gamma_j \subset \partial B^3$ with $\Gamma_j \cap (B_{10} \cup B_{11} \cup \cdots \cup B_{n0} \cup B_{n1}) = B_{j1} \cup B_{j+1,0}$, $1 \leqq j \leqq n-1$. Recall that the Dehn twist about b_i is the same to $\dot{\tau}_i$, see 3.3. Since μ_i maps a_i onto b_i, and $\dot{\theta}_{j,j+1}$ maps γ_j onto b_{j+1}, Dehn twists about a_i and γ_j are isotopic to power products of $\dot{\tau}_i$, μ_i and $\dot{\tau}_{j+1}$, $\dot{\theta}_{j,j+1}$, respectively. From the definitions of maps $\dot{\tau}_i$, μ_i, $\dot{\theta}_{j,j+1}$, we conclude 4.8.

4.9. *Remark.* It is easy to check that our maps are topological realizations of generators for the Siegel's modular group Sp $(2n, Z)$ given by Hua-Reiner [**12**] and Klingen [**13**]; cf. Birman [**2**].

REFERENCES

1. R. Baer, *Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen*, J. reine angew. Math., *159* (1928), 101–111.

2. J. S. Birman, *On Siegel's modular group*, Math. Ann. *191* (1971), 59–68.

3. ——— *Mapping class groups of surfaces: A survey*, Ann. of Math. Studies *79*, Discontinuous Groups and Riemann Surfaces, ed. Greenberg (Princeton Univ. Press, Princeton, N.J., 1974), 57–71.

4. ——— *Braids, links and mapping class groups*, Ann. of Math. Studies *82* (Princeton Univ. Press, Princeton N.J., 1974).

5. ——— *Poincaré conjecture and the homeotopy group of a closed orientable 2-manifold*, J. Aust. Math. Soc. *17* (1974), 214–221.

6. J. S. Birman and H. M. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, Ann. of Math. Studies *66*, Advances on the Theory of Riemann Surfaces, ed. Ahlfors et al. (Princeton Univ. Press, Princeton, N.J., 1972), 81–115.

7. M. Dehn, *Die Gruppe der Abbildungsklassen*, Acta Math. *69* (1938), 135–206.

8. D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. *115* (1966), 83–107.

9. L. Goeritz, *Die Abbildungen der Brezelflachen und der Vollbrezel vom Geschlecht 2*, Abh. Math. Sem. Univ. Hamburg, *9* (1933), 244–259.

10. H. B. Griffiths, *Automorphisms of a 3-dimensional handlebody*, Abh. Math. Sem. Univ. Hamburg *26* (1964), 191–210.

11. ——— *Some elementary topology of 3-dimensional handlebodies*, Comm. Pure and Appl. Math. *17* (1964), 317–334.

12. L. K. Hua and I. Reiner, *On the generators of the symplectic modular group*, Trans. Amer. Math. Soc. *65* (1949), 415–426.

13. H. Klingen, *Charakterisierung der Siegelschen Modulgruppe durch ein endliches System definierender Relationen*, Math. Ann. *144* (1961), 64–82.

14. W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Camb. Phil. Soc. *60* (1964), 769–778. Also Corrigendum, *62* (1966), 679–681.

15. W. Mangler, *Die klassen von topologischen Abbildungen einer geschlossenen Flache auf sich,* Math. Z. *44* (1939), 541–554.

16. J. Nielsen, *Untersuchen zur Topologie der geschlossenen Zweiseitigen Flächen I*, Acta Math. *50* (1927), 184–358. Also, *III*, Acta Math. *58* (1932), 87–167.

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