Extension of Frenet's Formulae to a Curve in Flat Space of n Dimensions.

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1. A curve in n dimensions may be taken as the limit to which a polygonal figure $ABCDE \ldots$ tends, when the sides AB, BC, CD, etc., all diminish towards the limit zero.

The curvature at A is $\lim\limits_{AB \to 0} \dfrac{\sin A\widehat{B}C}{AB}$.

The tortuosity at A is $\lim\limits_{AB \to 0} \dfrac{\sin A\overline{BC}D}{AB}$,

where $A\overline{BC}D$ denotes the dihedral angle between the planes ABC and BCD.

The p^{th} flexure at A is $\lim\limits_{AB \to 0} \dfrac{\sin A\,\overline{BC \ldots K}\,L}{AB}$,

where $A\,\overline{BC \ldots K}\,L$ denotes the angle between the flat p-dimensional spaces $ABC \ldots K$ and $BCD \ldots KL$.

Thus the curvature is the first flexure,

the tortuosity is the second flexure,

and the higher flexures exist only when the curve is not confined to the three-dimensional space.

These flexures may be denoted by $\phi_1, \phi_2, \phi_3 \ldots \phi_p$ respectively.

2. Using the notation $S_q(A_0, A_1, A_2 \ldots A_q)$ to denote the flat q-dimensional space in which the points $A_0, A_1, \ldots A_q$ are contained, then in the case of a polygonal n-dimensional figure $A_0 A_1 A_2 A_3 \ldots A_n$.

$S_1(A_0 A_1)$ is the osculating line or tangent at $A_0 A_1$

$S_2(A_0 A_1 A_2)$ is the osculating plane at $A_0 A_1 A_2$

$S_3(A_0 A_1 A_2 A_3)$ is the osculating S_3 at $A_0 A_1 A_2 A_3$

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$S_p(A_0 A_1 A_2 \ldots A_p)$ is the osculating S_p at $A_0 A_1 A_2 \ldots A_p$.

And the limits toward which those spaces tend when the points $A_0, A_1 \dots A_p$ approach coincidence, are the osculating spaces at A of the curve of which the polygonal figure is the limit.

The principal normal to the curve at A is that which is perpendicular to the tangent, but lies in the osculating plane.

The binormal at A is that which is perpendicular to the osculating plane, and lies in the osculating S_3.

The p^{th} normal is that which is perpendicular to the osculating S_p, and lies in the osculating S_{p+1}. Thus the straight line through A_0, lying in S_{p+1} $(A_0 A_1 \dots A_{p+1})$ and perpendicular to $S_p (A_0 A_1 \dots A_p)$, may be called the p^{th} normal to the polygonal figure $A_0 A_1 A_2 \dots$ at the point A_0, and this straight line has for its limit the p^{th} normal at A_0 of the curve which is the limit of the polygon.

To define the *sense* of the p^{th} normal at A_0 we may choose its positive direction to be that which makes an *acute* angle with the $(p-1)^{\text{th}}$ normal at A_1 (the tangent to be counted in this connection as the 0^{th} normal) so that the 1^{st}, or principal, normal will be directed towards the centre of curvature.

The angle between $A_0 A_1$ and $A_1 A_2$ we denote by θ_1; and, generally, by θ_p we denote the angle between the osculating S_p's at A_0 and A_1. Thus $\text{Lim}\,(\theta_p \div A_0 A_1) = \phi_p$.

Let $A_0 N_1, A_0 N_2 \dots A_0 N_p \dots$ be the 1^{st}, 2^{nd}, $\dots$ $p^{\text{th}} \dots$ normals to the polygon at A_0.

Let $A_1 M_1, A_1 M_2, \dots A_1 M_p \dots$ be the 1^{st}, 2^{nd}, $\dots p^{\text{th}} \dots$ normals to the polygon at A_1, so that $A_1 M_p$ lies in S_{p+1} $(A_1 A_2 \dots A_{p+2})$ and is perpendicular to $S'_R (A_1 A_2 \dots A_{p+1})$.

Let each of these normals be of unit length, and let

$$A_1 P_1 \equiv A_0 N_1,\ A_1 P_2 \equiv A_0 N_2, \text{ etc.,}$$

where $\equiv$ asserts equality in magnitude and sameness in direction.

Now let $A_1 P_1$ rotate through θ_1 in the plane $A_0 A_1 A_2$, in direction *away from* $A_1 A_2$ to the position $A_1 P_1'$. Then since the same rotation would bring $A_1 A_0$ into line with $A_1 A_2$, $A_1 P_1'$ will be perpendicular to $A_1 A_2$. Let it then rotate through θ_2 to $A_1 P_1''$, towards $A_1 P_2$ in the plane which is perpendicular to $A_1 A_2$ and which lies in $S_3(A_0 A_1 A_2 A_3)$. Then $A_1 P_1''$ is in the plane $A_1 A_2 A_3$ and is perpendicular to $A_1 A_2$. Hence $A_1 P_1''$ coincides with $A_1 M_1$.

Similarly, let $A_1 P_p$ rotate through θ_p about $S_{p-1}(A_1 A_2 \ldots A_p)$, away from $A_1 M_{p-1}$ to a position $A_1 P_p'$ in $S_{p+1}(A_0 A_1 \ldots A_{p+1})$. Since the same rotation would bring $S_p(A_0 A_1 \ldots A_p)$ into coincidence with $S_p(A_1 A_2 \ldots A_{p+1})$, $A_1 P_p$ being perpendicular to the former S_p, $A_1 P_p'$ will be perpendicular to the latter.

But $A_1 P_{p+1}$ is perpendicular to $S_{p+1}(A_0 A_1 A_2 \ldots A_{p+1})$ and therefore to $S_p(A_1 A_2 \ldots A_{p+1})$ as is also $A_1 M_p$, and all three of these lines lie in $S_{p+2}(A_0 A_1 \ldots A_{p+2})$. Hence they lie in the same plane perpendicular to $S_p(A_1 A_2 \ldots A_{p+1})$, $A_1 P_{p+1}$ being perpendicular to $A_1 P_p'$. Now a rotation of amount θ_{p+1} about $S_p(A_1 A_2 \ldots A_{p+1})$ brings $S_{p+1}(A_0 A_1 \ldots A_{p+1})$ into coincidence with $S_{p+1}(A_1 A_2 A_3 \ldots A_{p+2})$.

Hence it will bring $A_1 P_p'$ which lies in the former S_{p+1} to a position $A_1 P_p''$ which will lie in the latter, and, being perpendicular to $S_p(A_1 A_2 \ldots A_{p+1})$, must coincide with $A_1 M_p$.

Thus $A_1 P_p$ can be brought into coincidence with $A_1 M_p$ by two rotations, one of amount θ_p, *away* from $A_1 M_{p-1}$, and the other of amount θ_{p+1}, towards $A_1 P_{p+1}$.

Hence if we suppose the sides of the Polygonal figure to be small, so that we may reject quantities of the second order, we have $P_p P_p' = \theta_p$, its direction being that of $M_p A_1$, and $P_p' P_p'' = \theta_{p+1}$, its direction being that of $A_1 P_{p+1}$.

Thus taking $l_0, m_0, n_0, o_0, p_0 \ldots$ to denote the direction cosines of $A_0 A_1$,

and taking $l_1, m_1, n_1, p_1 \ldots$ to denote the direction cosines of $A_0 N_1$ or $A_1 P_1$,

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we have $\delta l_p =$ projection of $P_p M_p$ on $0x$

$$= -l_{p-1}\theta_p + l_{p+1}\theta_{p+1}.$$

Taking δs to denote $A_0 A_1$, we deduce

$$\frac{\delta l_p}{\delta s} = - l_{p-1} \frac{\theta_p}{\delta s} + l_{p+1} \frac{\theta_{p+1}}{\delta s}.$$

Going to the limit for $\delta s \rightarrow 0$, we get the formula

$$\frac{d l_p}{d s} = - l_{p-1} \phi_p + l_{p+1} \phi_{p+1} .$$

Similarly we find $\dfrac{dm_p}{d s} = - m_{p-1} \phi_p + m_{p+1} \phi_{p+1}$, etc.

These formulae include those of Frenet as particular cases, when the curve is confined to three dimensions, so that ϕ_3 and the higher flexures are all zero