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ON MINIMAX AND RELATED MODULES

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ABSTRACT. A module M is called a *minimax module*, if it has a finitely generated submodule U such that M/U is Artinian. This paper investigates minimax modules and some generalized classes over commutative Noetherian rings. One of our main results is: *M is minimax iff every decomposition of a homomorphic image of M is finite.*

From this we deduce that:

- All couniform modules are minimax.
- All modules of *finite codimension* are minimax.
- Essential covers of minimax modules are minimax. With the aid of these corollaries we completely determine the structure of couniform modules and modules of finite codimension.

We then examine the following variants of the minimax property:

- replace U "finitely generated" by U "coatomic" (i.e. every proper submodule of U is contained in a maximal submodule);
- replace M/U "Artinian" by M/U "semi-Artinian" (i.e. every proper submodule of M/U contains a minimal submodule).

Introduction. Let R, like all rings in this paper, be a commutative Noetherian ring. As in [14], an R-module M is called *minimax module* if it has a finitely generated submodule N such that M/N is Artinian, i.e. if M is an extension of a finitely generated module by an Artinian module.

The main aim of this paper is to characterize this property by finiteness conditions for decompositions of factor modules of M.

In the first section, we introduce strongly faithful modules (see below). With their help we are able to establish

COROLLARY 1.6. *Let M be a semi-Artinian, non-Artinian R-module. Then M has a factor module which possesses an infinite decomposition.*

This result is used in § 2 to prove the following characterization of minimax modules:

THEOREM 2.1. *For an R-module M the following statements are equivalent:*

(i) M is a minimax module.

(ii) Every semi-Artinian factor module of M is Artinian.

(iii) Every factor module of M has ACC for direct summands.

(iv) Every decomposition of a factor module of M is finite.

With the aid of this theorem we are able to show that

– all couniform modules are minimax modules;

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– all modules of "finite codimension" are minimax modules;
– essential covers of minimax modules are minimax modules.

In the third paragraph we investigate some classes of generalized minimax modules. While Corollary 1.6 shows there is a "very large" factor module in every semi-Artinian, non-Artinian module, Proposition 3.1 constructs a "very large" factor module for certain non semi-Artinian modules. From this we can deduce

THEOREM 3.4. *For an R-module M the following are equivalent:*
(i) Every radical factor module of M is a minimax module.
(ii) Every decomposition of a radical factor module of M is finite.
(iii) Every semi-Artinian factor module of M is the sum of a coatomic and an Artinian module.
(iv) M is an extension of a coatomic module by an Artinian module.

In the case of a local ring $(R, \mathfrak{m})$, Theorem 3.3 shows additional equivalent statements. The most interesting one is

M is the sum of a minimax module and a discrete module.

Finally, Theorem 3.8 characterizes those radical R-modules which are locally minimax modules:

THEOREM 3.8. *Let R be arbitrary and M a radical R-module. Then the following statements are equivalent:*
(i) $M_{\mathfrak{m}}$ *is an* $R_{\mathfrak{m}}$*-minimax module for all maximal ideals* $\mathfrak{m} \subset R$.
(ii) M is an extension of a coatomic module by a semi-Artinian, locally Artinian module.

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0. **Definitions.** Let R be an arbitrary Noetherian ring. By Ω we denote the set of all maximal ideals of R. Let $\mathfrak{a} \subset R$. Then we write $M[\mathfrak{a}] = \{x \in M \mid \mathfrak{a}x = 0\}$.

An R-module M is called *radical* if it has no maximal submodules, i.e. $\mathrm{Rad}(M) = M$. By $P(M)$ we denote the sum of the radical submodules of M. $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, M is called *reduced.*

An R-module is called *semi-Artinian* if every proper submodule contains a minimal submodule. For any module M we denote by $L(M)$ the sum of all Artinian submodules of M. $L(M)$ is the largest semi-Artinian submodule of M. $L(M)$ always has a decomposition $L(M) = \bigoplus_{\mathfrak{m}\in\Omega} L_{\mathfrak{m}}(M)$, where $L_{\mathfrak{m}}(M) = \sum_{n=1}^{\infty} M[\mathfrak{m}^n]$ is called the $\mathfrak{m}$*-primary component* of $L(M)$. If $L(M) = 0$, M is called *socle-free.*

Let M be an R-module. The *Goldie dimension* of M (we write $\dim(M)$) can be defined in the following way (cf. [4, Definition 6]): $\dim(M) = n$ iff M has an essential submodule B that is a direct sum of n uniform modules; $\dim(M) = \infty$ iff M contains a submodule which has an infinite decomposition.

1. **Strongly faithful modules.** We call an R-module M *strongly faithful,* if rM Artinian implies $r = 0$ for any $r \in R$. Obviously every strongly faithful module is faithful. If M has a strongly faithful submodule or a strongly faithful factor module, M itself is strongly faithful. If R is a domain and M is a divisible R-module, then M is strongly faithful iff M is not Artinian.

PROPOSITION 1.1. *Let R be a domain, not a field, and let M be a faithful R-module with $L(M)$ reduced. Then M is strongly faithful.*

PROOF. Let $s \in R$ with sM Artinian. Then $sM \subset L(M)$, thus sM is of finite length. Consequently there exists a non-zero $t \in R$ with $tsM = 0$, so by the hypothesis we have $ts = 0$, $s = 0$.

For every R-module M the set

$$\mathrm{Art}_R(M) = \{\, r \in R \mid rM \text{ is Artinian}\}$$

is an ideal in R. We denote $\mathfrak{a} = \mathrm{Art}_R(M)$, and since R is Noetherian $\mathfrak{a}M$ is Artinian and $M/\mathfrak{a}M$ is a strongly faithful $R/\mathfrak{a}$-module.

For any ideal $\mathfrak{b} \subset R$ there exists a module N with $\mathrm{Art}_R(N) = \mathfrak{b}$; choose e.g. $N = (R/\mathfrak{b})^{(\mathbf{N})}$.

Furthermore, we define for M a set of ideals

$$\mathcal{S}(M) = \{\, \mathfrak{a} \subsetneq R \mid M/\mathfrak{a}M \text{ is a strongly faithful } R/\mathfrak{a}\text{-module}\}.$$

If M is not Artinian, then $\mathrm{Art}_R(M) \subsetneq R$, thus $\mathrm{Art}_R(M) \in \mathcal{S}(M) \neq \emptyset$. In this case obviously $\mathrm{Art}_R(M)$ is the smallest element of $\mathcal{S}(M)$.

EXAMPLES.

(a) If $(R, \mathfrak{m})$ is local and $M = \coprod_{i=1}^{\infty} R/\mathfrak{m}^i$, then $\mathcal{S}(M) = \{\, \mathfrak{a} \subset R \mid \mathfrak{a} \neq R\}$, since $M/\mathfrak{a}M \cong \coprod_{i=1}^{\infty} R/(\mathfrak{a} + \mathfrak{m}^i)$ is a strongly faithful $R/\mathfrak{a}$-module for every $\mathfrak{a} \neq R$.

(b) If R is a domain and M is a divisible and non-Artinian R-module, then $\mathcal{S}(M) = \{0\}$.

Our aim is to find, for an R-module M, a factor module which is strongly faithful over an integral, non-Artinian factor ring of R. Therefore we are looking for non-maximal prime ideals in $\mathcal{S}(M)$.

PROPOSITION 1.2. *Let $M \neq 0$ be a strongly faithful R-module, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal prime ideals of R. Then $M/\mathfrak{p}_iM$ is a strongly faithful $R/\mathfrak{p}_i$-module for every $1 \leq i \leq n$.*

PROOF. Assume, the assertion fails. Then we can assume $M/\mathfrak{p}_1M$ to be not strongly faithful over $R/\mathfrak{p}_1$, i.e. there is a $t_1 \notin \mathfrak{p}_1$ with $(t_1M + \mathfrak{p}_1M)/\mathfrak{p}_1M$ Artinian. Choose, for every $2 \leq i \leq n$, $s_i \in \mathfrak{p}_i \setminus \mathfrak{p}_1$ and put $t = t_1s_2 \ldots s_n$ (in the case $n = 1$ we put $t = t_1$). Of course $t \in \mathfrak{p}_i \setminus \mathfrak{p}_1$ for all $2 \leq i \leq n$, hence $\mathfrak{p}_1t \subset \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. Consequently there is $e \geq 1$ with $(\mathfrak{p}_1t)^e = 0$.

The set of natural numbers $\mathcal{M} = \{i \geq 1 \mid \mathrm{Ann}_R(\mathfrak{p}_1^i) \cap \{t, t^2, \ldots\} \neq \emptyset\}$ is not empty (because $t^e \in \mathrm{Ann}_R(\mathfrak{p}_1^e)$) and has therefore a minimal element m. Note that $1 \leq m \leq e$. There exists $k \geq 1$ with $t^k \mathfrak{p}_1^m = 0$, and consequently $(t^k M + \mathfrak{p}_1 M)/\, \mathfrak{p}_1 M \subset (t_1 M + \mathfrak{p}_1 M)/\, \mathfrak{p}_1 M$ is Artinian according to the assumption. If we multiply numerator and denominator by the ideal $t^k \mathfrak{p}_1^{m-1}$, we see that $(t^{2k}\mathfrak{p}_1^{m-1}M + t^k\mathfrak{p}_1^m M)/\, t^k\mathfrak{p}_1^m M$ and hence $t^{2k}\mathfrak{p}_1^{m-1}M$ are also Artinian. Since M is strongly faithful over R, $t^{2k}\mathfrak{p}_1^{m-1} = 0$. As, by construction, t cannot be nilpotent, we have $m - 1 \geq 1$ and therefore $m - 1 \in \mathcal{M}$, in contradiction to the minimality of m.

COROLLARY 1.3. *Let M be an R-module. If $\mathfrak{a} \in \mathcal{S}(M)$, then all minimal prime divisors of $\mathfrak{a}$ are contained in $\mathcal{S}(M)$. The maximal elements of $\mathcal{S}(M)$ are prime ideals. If* $\dim\big(R/\,\mathrm{Art}(M)\big) \geq 1$, *$\mathcal{S}(M)$ contains a non-maximal prime ideal.*

To prove the following theorems we make use of Matlis duality (cf. [5, Theorem 4.2] and [6, Theorem 4.6]). If $(R, \mathfrak{m})$ is a local ring and M is an R-module, we denote by E the injective envelope of $R/\,\mathfrak{m}$ and set $M^\circ = \mathrm{Hom}_R(M, E)$.

As in [13, § 2], we define $\mathrm{Coass}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p}$ is the annihilator of an Artinian factor module of $M\}$. By [14, Lemma 3.1] $\mathrm{Coass}(M) = \mathrm{Ass}(M^\circ)$.

Our next aim is to prove that a semi-Artinian, strongly faithful module over a local, complete domain $(R, \mathfrak{m})$ has a "very large" factor module which is isomorphic to $E^{(\mathbb{N})}$.

THEOREM 1.4. *Let $(R, \mathfrak{m})$ be a complete, local domain, let M be a semi-Artinian R-module. The following statements are equivalent:*

(i) M is strongly faithful.

(ii) There is an epimorphism $M \longrightarrow E^{(\mathbb{N})}$.

PROOF. (ii)$\Rightarrow$(i): $E^{(\mathbb{N})}$ is divisible and not Artinian, hence strongly faithful. Then clearly M is strongly faithful, too.

(i) $\Rightarrow$(ii): If R is a field, the vector space M is strongly faithful iff it has infinite dimension, i.e. if there is an epimorphism $M \longrightarrow R^{(\mathbb{N})} \cong E^{(\mathbb{N})}$.

If R is not a field, we construct by induction finitely generated submodules $V_1, V_2, \ldots$ of M with :

(a) the sum of the V_i is direct and

(b) $\mathrm{Ann}_R(V_i) \subset \mathfrak{m}^i$ for all $i \in \mathbb{N}$.

Choose $0 \neq x_1 \in M$ and put $V_1 = Rx_1$. If $V_1, \ldots, V_n$ are chosen such that (a) and (b) are valid we put $V = V_1 \oplus \cdots \oplus V_n$. Let W be a maximal element of the set $\{X \subset M \mid X \cap V = 0\}$. Now $M/\,W$ is Artinian because V is Artinian and the monomorphism $V \longrightarrow M/\,W$ is essential, so that W is strongly faithful, too. According to [1, Theorem 1.1], W has a countably generated submodule W' which is still faithful. Since R is complete, we may apply Chevalley's Theorem [11, VIII § 5, Theorem 13] to choose a finitely generated submodule V_{n+1} of W' with $\mathrm{Ann}_R(V_{n+1}) \subset \mathfrak{m}^{n+1}$. Thus $V_1, \ldots, V_{n+1}$ satisfy the conditions (a) and (b).

Next we will divide the set of the V_j into infinitely many, infinite pairwise disjoint sets. We achieve this by indicating with prime powers. Let $\mathcal{P}$ be the set of prime numbers. For any $j \in \mathbb{N}$ let $U_j := \bigoplus_{p \in \mathcal{P}} V_{p^j}$. U_j is faithful for all $j \geq 1$. By Proposition 1.1 U_j is even strongly faithful for all $j \geq 1$.

If we apply the technique described above to every U_j in place of M, we get $U_j \supset X_j = \bigoplus_{i=1}^{\infty} Y_{i,j}$ with strongly faithful $Y_{i,j}$ for all $i,j \in \mathbb{N}$. We now claim $0 \in \mathrm{Coass}(X_j)$ for all $j \geq 1$, i.e. X_j° not torsionfull for all $j \geq 1$. Now $X_j^{\circ} \cong \prod_{i=1}^{\infty} Y_{i,j}^{\circ}$, where every factor is faithful. As the local ring R always has finite Krull dimension we can apply [3, Corollary 5.6] and see: X_j° is not a torsion module. Hence we have proved $0 \in \mathrm{Coass}(X_j)$ for all $j \geq 1$. According to the following Proposition 1.5 ((i)⇒(ii)), we therefore have epimorphisms $X_j \longrightarrow E$ for all $j \geq 1$. Together they yield an epimorphism $\bigoplus_{j=1}^{\infty} X_j \longrightarrow E^{(\mathbb{N})}$. Since $E^{(\mathbb{N})}$ is injective, this mapping can be extended to an epimorphism $M \longrightarrow E^{(\mathbb{N})}$.

PROPOSITION 1.5. *Let $(R, \mathfrak{m})$ be a complete, local domain. For an R-module M the following conditions are equivalent:*

(i) $0 \in \mathrm{Coass}(M)$.

(ii) There is an epimorphism $M \longrightarrow E$.

If M is semi-Artinian these are also equivalent to:

(iii) M is faithful.

PROOF. (i)⇒(ii): M has a nonzero Artinian, divisible factor module X. The dual module X° is torsion-free and nonzero, and therefore R can be embedded into X°. Since X is Artinian, $X \cong X^{\circ\circ}$ by [5, Corollary 4.3], and we obtain an epimorphism $M \longrightarrow X^{\circ\circ} \longrightarrow R^{\circ} \cong E$.

(ii)⇒(i): E is divisible, so $\{0\} = \mathrm{Coass}(E) \subset \mathrm{Coass}(M)$.

(ii)⇒(iii): We have $\mathrm{Ann}(M) \subset \mathrm{Ann}(E) = 0$.

(iii)⇒(i): Let M be semi-Artinian. If M is strongly faithful, the assertion follows from Theorem 1.4. Otherwise there is $0 \neq r \in R$ with rM Artinian. Now rM is faithful, too, and $\mathrm{Coass}(rM)$ is finite. We conclude from [13, Beispiel 3]: $0 = \sqrt{\big(\mathrm{Ann}(rM)\big)} = \bigcap \mathrm{Coass}(rM)$, thus $0 \in \mathrm{Coass}(rM) \subset \mathrm{Coass}(M)$.

REMARK. We would like to emphasize the following analogy between Theorem 1.4 and Proposition 1.5: Let $(R, \mathfrak{m})$ be a complete, local domain and M a semi-Artinian R-module. Then

(1) M is faithful iff there is an epimorphism $M \longrightarrow E$;

(2) M is strongly faithful iff there is an epimorphism $M \longrightarrow E^{(\mathbb{N})}$.

COROLLARY 1.6. *Let R be an arbitrary Noetherian ring and M a semi-Artinian, non-Artinian R-module. Then M has a factor module which possesses an infinite decomposition.*

PROOF. If infinitely many primary components of M are nonzero, M itself has the desired property. Otherwise there is a component $L_{\mathfrak{m}}(M)$ of M which is not Artinian. We may assume $M = L_{\mathfrak{m}}(M)$. As the $\hat{R}_{\mathfrak{m}}$-module M and the R-module M have the same submodules we may assume R to be local and complete. Since M is not Artinian we have $\mathrm{Art}(M) \subsetneq R$. Let $\mathfrak{p}$ be a minimal prime divisor of $\mathrm{Art}(M)$. By Corollary 1.3 $M/\mathfrak{p}M$

is a strongly faithful $R/\mathfrak{p}$-module. According to Theorem 1.4 there is an epimorphism $M/\mathfrak{p}M \longrightarrow E[\mathfrak{p}]^{(\mathbb{N})}$, giving a factor module of M with the desired property.

2. **A characterization of minimax modules.** With the help of Corollary 1.6 we now are able to prove the following characterization of minimax modules. Note that each of the equivalent statements (ii)–(iv) seems to be weaker than its predecessor.

THEOREM 2.1. *For an R-module M the following statements are equivalent:*
(i) M is a minimax module.
(ii) Every semi-Artinian factor module of M is Artinian.
(iii) Every factor module of M has ACC for direct summands.
(iv) Every decomposition of a factor module of M is finite.

PROOF. (i)⇒(ii): Clear

(ii)⇒(iii): Let X be a factor module of M. Then X has a semi-Artinian factor module X' with the same Goldie-dimension. By *(ii)* X' is Artinian, and hence X is of finite Goldie-dimension. Consequently X has ACC for direct summands.

(iii)⇒(iv): Clear.

(iv)⇒(i): Suppose that M is not a minimax module. Then according to [15, Anhang], M has an image M' of infinite Goldie-dimension. Hence there are $M' \supset B = \bigoplus_{i\in I} U_i$ with U_i nonzero cyclic and I infinite. Choose maximal submodules $Y_i \subsetneq U_i$ for all $i \in I$. Then $A = \bigoplus_{i\in I} Y_i \subset B$ and clearly B/A is semi-Artinian but not Artinian. Let C/A be a maximal submodule of M'/A with respect to $C/A \cap B/A = 0$. Then M'/C is semi-Artinian but not Artinian. But then, by Corollary 1.6, M has a factor module with an infinite decomposition, which contradicts (iv).

An extreme case of (iv) is the case of a module whose factor modules all are indecomposable. These modules are called *couniform.* We conclude from Theorem 2.1:

COROLLARY 2.2. *Every couniform module is a minimax module.*

REMARK. With the aid of Corollary 2.2 we are able to describe the structure of couniform modules completely. These results, as well as a description of the structure of "complemented modules," are presented in [9].

Another application of Theorem 2.1 leads to a fairly explicit description of modules having finite "codimension."

We recall the concept of *coindependent sets* of submodules. Let M be an R-module. As in [10], a set $\mathcal{M}$ of submodules of M is called coindependent if, for pairwise different $U_1, \ldots, U_n \in \mathcal{M}$, $U_1 + \bigcap_{i=2}^{n} U_i = M$ holds (this property is called "meet-independent" in [4] and "kodirekt" in [14]). The empty set and any set with only one element are coindependent. In R the set Ω of all maximal ideals is coindependent.

THEOREM 2.3. *For an R-module M the following statements are equivalent:*
(i) Every coindependent set of submodules of M is finite.
(ii) M is a minimax module and $R/\mathfrak{p}$ *is semi-local for all* $\mathfrak{p} \in \mathrm{Ass}(M)$.
(iii) M is a minimax module and $M/\mathrm{Rad}(M)$ *is semi-simple.*

PROOF. (ii)⇒(i): According to [14, Satz 3.6] we have to show that M is an essential cover of an Artinian module. Let U be a nonzero finitely generated submodule of M with M/U Artinian. Let $\mathfrak{a} = \sqrt{\left(\mathrm{Ann}(U)\right)} = \cap \mathrm{Ass}(U)$. According to the hypothesis there lie only finitely many maximal ideals over $\mathfrak{a}$, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ $(n \geq 1)$. U is divisible by all other maximal ideals, hence $\mathrm{Rad}(U) = \mathfrak{m}_1 \cdots \mathfrak{m}_n U$. Therefore, by Nakayama's lemma, M is an essential cover of the Artinian module $M/\mathfrak{m}_1 \cdots \mathfrak{m}_n U$.

(i)⇒(iii): According to [14, Satz 3.6] M fulfills the condition (*iv*) of Theorem 2.1 and therefore is a minimax module. By the same reference M is an essential cover of an Artinian module and hence $M/\mathrm{Rad}(M)$ is semi-simple.

(iii)⇒(ii): $M/\mathrm{Rad}(M)$ is Artinian as a semi-simple minimax module. Let U be a finitely generated submodule of M with M/U Artinian. Now $U/U \cap \mathrm{Rad}(M) \cong U + \mathrm{Rad}(M)/\mathrm{Rad}(M)$ is Artinian and $U \cap \mathrm{Rad}(M)$ is small in M as a finitely generated submodule of $\mathrm{Rad}(M)$. Therefore M is an essential cover of an Artinian module A. For almost all $\mathfrak{m} \in \Omega$ one has $A_{\mathfrak{m}} = 0$, and hence by [12, Lemma 4.1] $M_{\mathfrak{m}} = 0$. Therefore, for every $\mathfrak{p} \in \mathrm{Ass}(M)$, $(R/\mathfrak{p})_{\mathfrak{m}} = 0$, hence $\mathfrak{p} \not\subset \mathfrak{m}$ for almost all $\mathfrak{m} \in \Omega$.

REMARKS. (1) As property (ii) is inherited by submodules, so is property (i).

(2) Property (i) is a dualization of the following characterization of finite Goldie-dimension: "Every independent set of submodules of M is finite". The papers [4], [7] and [10] show that property (i) indeed leads to a well behaved concept of "codimension".

COROLLARY 2.4. *Every essential cover of a minimax module is a minimax module.*

PROOF. We verify (ii) in Theorem 2.1: Let $V \subset U$ with M/V semi-Artinian. There is a small submodule U of M such that M/U is a minimax module. Then $U + V/V$ is small in M/V and $M/U + V$ is minimax and semi-Artinian, hence Artinian. So M/V is an essential cover of an Artinian module. By Theorem 2.3 (i)⇒(ii), M/V is Artinian as desired.

3. **Generalized minimax-modules.**

The following proposition is a sharper version of [15, Anhang, Satz]. With its help we will be able to characterize modules which are extensions of coatomic by Artinian modules.

PROPOSITION 3.1. *Let $(R, \mathfrak{m})$ be local.*

(a) Let $n \geq 1$ and let M be an R-module which is not semi-Artinian. Then there are finitely generated submodules $S \subset U$ of M with U socle-free, U/S Artinian, but $\mathfrak{m}^n(U/S) \neq 0$. We have a monomorphism $L(M) \times U/S \longrightarrow L(M/S)$.

(b) Assume that M/A is not semi-Artinian for any finitely generated submodule A of M. Then M has a factor module $X \neq 0$, such that $\mathfrak{m}^k X$ has infinite Goldie-dimension for all $k \geq 1$.

PROOF. (a) M has a submodule $U \cong R/\mathfrak{p}$ for some $\mathfrak{p} \neq \mathfrak{m}$. Define $S = \mathfrak{m}^{n+1}U$. Then clearly U/S is Artinian and $\mathfrak{m}^n(U/S) \neq 0$. As $L(M) \cap U = 0$, we have $\left(L(M) + S\right) \cap U = S$, hence $L(M) \times U/S \cong \left(L(M) + S\right)/S \oplus U/S \subset L(M/S)$, which gives the desired monomorphism.

(b) According to the hypothesis M is not semi-Artinian. So, by (a), we may choose finitely generated submodules $S_1 \subset U_1 \subset M$ with $U_1 \cap L(M) = 0$, U_1/S_1 Artinian, but $\mathfrak{m}(U_1/S_1) \neq 0$. Further we have a monomorphism $U_1/S_1 \longrightarrow L(M/S_1)$.

Since S_1 is finitely generated, according to the hypothesis M/S_1 is not semi-Artinian, so by (a) we have finitely generated submodules $S_2/S_1 \subset U_2/S_1 \subset M/S_1$ with $U_2/S_1 \cap L(M/S_1) = 0$, U_2/S_2 Artinian, but $\mathfrak{m}^2(U_2/S_2) \neq 0$. Further we have a monomorphism $U_1/S_1 \times U_2/S_2 \longrightarrow L(M/S_1) \times U_2/S_2 \longrightarrow L(M/S_2)$.

Since S_2 is finitely genenerated, M/S_2 is not semi-Artinian either, so we can continue the construction indicated above. By induction we obtain a sequence of finitely generated modules $0 = S_0 \subset S_1 \subset S_2 \subset \cdots \subset M$, together with finitely generated $U_i \supset S_i$ with U_i/S_{i-1} socle-free, U_i/S_i Artinian, but

$$(*) \qquad \mathfrak{m}^i(U_i/S_i) \neq 0.$$

We have monomorphisms $U_1/S_1 \times U_2/S_2 \times \cdots \times U_i/S_i \longrightarrow L(M/S_i)$ for all $i \geq 1$.

Now, for all $i \geq 1$, let $S_i'/S_i = L(M/S_i)$ and $T = \sum_{i=1}^{\infty} S_i$. Then $S_{i+1}/S_i \subset U_{i+1}/S_i$ is socle-free, so $(S_{i+1}/S_i) \cap (S_i'/S_i) = 0$, i.e. $S_{i+1} \cap S_i' = S_i$. Therefore $T \cap S_i' = S_i$ for all $i \geq 1$, since $S_{i+2} \cap S_i' = S_{i+2} \cap S_{i+1}' \cap S_i' = S_{i+1} \cap S_i' = S_i$, etc. Consequently, $T \cap S_i' = (\sum_{j=1}^{\infty} S_j) \cap S_i' = \sum_{j=i+1}^{\infty} (S_j \cap S_i') = S_i$.

Now $(S_i' + T)/T \cong S_i'/(S_i' \cap T) = S_i'/S_i$. Consequently there are monomorphisms $f_i\colon U_1/S_1 \times U_2/S_2 \times \cdots \times U_i/S_i \longrightarrow S_i'/S_i \longrightarrow M/T$ for all $i \geq 1$. Let $B_i \subset M$ be defined by $B_i/T = \mathrm{Im}(f_i)$ for all $i \geq 1$.

According to [2, IV, § 1.2 Proposition 4] there is $T'/T \subset M/T$ with T'/T socle-free and $X := M/T'$ semi-Artinian. Now, for every $i \geq 1$, the canonical mapping $U_1/S_1 \times \cdots \times U_i/S_i \cong B_i/T \longrightarrow M/T' = X$ is injective since $B_i/T \cap T'/T = 0$. Let $k \geq 1$. Then $(*)$ implies $\mathfrak{m}^k(U_i/S_i) \neq 0$ for every $i \geq k$, and consequently $\dim(\mathfrak{m}^k X) > i - k$ for all $i > k$. We conclude $\dim(\mathfrak{m}^k X) = \infty$.

COROLLARY 3.2. *Let $(R, \mathfrak{m})$ be local, M an R-module. The following statements are equivalent:*

(i) There is an $n \geq 1$ such that $\mathfrak{m}^n M$ is a minimax module.

(ii) For every factor module X of M there is an $n \geq 1$ such that $\mathfrak{m}^n X$ is of finite Goldie-dimension.

PROOF. (i)$\Rightarrow$(ii): Clear. (ii)$\Rightarrow$(i): Suppose (i) does not hold. Then, for any finitely generated submodule A of M, M/A is not semi-Artinian. (Otherwise, according to the hypothesis, there is $n \geq 1$ with $\mathfrak{m}^n(M/A) \cong \mathfrak{m}^n M/(\mathfrak{m}^n M \cap A)$ Artinian, hence $\mathfrak{m}^n M$ is a minimax module, which violates our assumption.)

We may now apply Proposition 3.1, which shows that there is a factor module X of M with the property $\dim(\mathfrak{m}^n X) = \infty$ for all $n \geq 1$, a contradiction.

For an arbitrary Noetherian ring R, an R-module M is *coatomic* if it has no non-zero radical factor module. Observe that any coatomic submodule of a radical module M is small since M has no nonzero coatomic images.

If $(R, \mathfrak{m})$ is local, we call an R-module B *discrete* (in the $\mathfrak{m}$-adic topology) if there is an $n \geq 1$ with $\mathfrak{m}^n B = 0$. By [12, Satz A] an R-module M is coatomic iff M is the sum of a finitely generated and a discrete submodule. Observe that if S is an arbitrary ring and M is a semi-Artinian S-module, M is locally discrete iff M is coatomic.

THEOREM 3.3. *Let $(R, \mathfrak{m})$ be local. For an R-module M the following are equivalent:*
(i) Every radical factor module of M is a minimax module.
(ii) Every decomposition of a radical factor module of M is finite.
(iii) For every factor module X of M there is an $n \geq 1$ such that $\mathfrak{m}^n X$ has ACC for direct summands.
(iv) There is an $n \geq 1$ such that $\mathfrak{m}^n M$ is a minimax module.
(v) $M = A + B$ with a minimax module A and a discrete module B.
(vi) M is an extension of a coatomic module by an Artinian module.

PROOF. (iv)$\Rightarrow$(iii) is clear, (iii)$\Rightarrow$(i) is valid by Theorem 2.1 and (i)$\Rightarrow$(ii) is clear again.

(ii) $\Rightarrow$(iv): By Corollary 3.2 it remains to show that for every factor module X of M there is $n \geq 1$ with $\dim(\mathfrak{m}^n X) < \infty$. Suppose this condition is violated. Then we can assume $\dim(\mathfrak{m}^n M) = \infty$ for all $n \geq 1$, and in a *first step* we construct a semi-Artinian factor module X of M which again violates the condition.

Case I: $\dim\big(M/L(M)\big) = \infty$. Let D be a maximal element in the set $\{B \subset M \mid B \cap L(M) = 0\}$. Then $D \longrightarrow M/L(M)$ is an essential monomorphism and hence $\dim(D) = \infty$. Therefore, by [4, Theorem 5] there are $D \supset D' = \bigoplus_{i=1}^{\infty} D_i$, D_i non-zero. Choose $\mathfrak{p}_i \in \mathrm{Ass}(D_i)$ for all $i \geq 1$. Then $\mathfrak{p}_i \neq \mathfrak{m}$ for all $i \geq 1$, and there is a monomorphism $\coprod_{i=1}^{\infty} R/\mathfrak{p}_i \longrightarrow \bigoplus_{i=1}^{\infty} D_i = D' \subset D \subset M$. Let C be the image of this mapping in M. Put $U_i = R/(\mathfrak{p}_i + \mathfrak{m}^i)$ for all $i \geq 1$. Now all U_i are Artinian and $\mathfrak{m}^i U_{i+1} \neq 0$ for all $i \geq 0$, since $U_i \cong \bar{R}/\bar{\mathfrak{m}}^i$ over the non Artinian ring $\bar{R} = R/\mathfrak{p}_i$.

If V is the kernel of the induced epimorphism $C \longrightarrow \coprod_{i=1}^{\infty} U_i$, there is a monomorphism $f \colon \coprod_{i=1}^{\infty} U_i \longrightarrow M/V$. Choose a maximal element W/V in the set $\{B/V \subset M/V \mid \mathrm{Im}(f) \cap B/V = 0\}$. Then there is an essential monomorphism $\coprod_{i=1}^{\infty} U_i \longrightarrow M/W =: X$. Now X is semi-Artinian and $\dim(\mathfrak{m}^n X) \geq \dim\Big(\coprod_{i=1}^{\infty} \mathfrak{m}^n U_i\Big) = \infty$ for all $n \geq 1$.

Case II: $\dim\big(M/L(M)\big) < \infty$. Let again D be a maximal element in the set $\{B \subset M \mid B \cap L(M) = 0\}$. Then $X = M/D$ is semi-Artinian, being an essential extension of $L(M)$. Further, in $\mathfrak{m}^n X = \mathfrak{m}^n M + D/D \cong \mathfrak{m}^n M/\mathfrak{m}^n M \cap D$ the last denominator is of finite Goldie-dimension, since D can be embedded in $M/L(M)$. According to the assumption we conclude $\dim(\mathfrak{m}^n X) = \infty$ for all $n \geq 1$.

In the *second step* we construct a radical factor module of X which violates (ii). Since X is semi-Artinian, we may assume R to be complete. Now $\dim\big(R/\mathrm{Art}(X)\big) \geq 1$, since otherwise we have $\mathfrak{m}^i \subset \mathrm{Art}(X)$ for an $i \geq 1$ and therefore $\mathfrak{m}^i X$ Artinian in contradiction to the construction of X. According to Corollary 1.3 there is a prime ideal $\mathfrak{p}$ with $\mathrm{Art}(X) \subset \mathfrak{p} \subsetneq \mathfrak{m}$ such that $X/\mathfrak{p}X$ is strongly faithful over the complete domain $R/\mathfrak{p}$. Now by Theorem 1.4 we have an epimorphism $X/\mathfrak{p}X \longrightarrow E[\mathfrak{p}]^{(\mathbb{N})}$. The latter module is the desired radical factor module which violates condition (ii).

(iv)⇒(vi): Let $a_1, \ldots, a_k \in R$ such that $\mathfrak{m}^n = (a_1, \ldots, a_k)$. Then the homomorphism $M \longrightarrow (\mathfrak{m}^n M)^k$, $x \longmapsto (a_1 x, \ldots, a_k x)$ has the kernel $M[\mathfrak{m}^n]$, and therefore $M/M[\mathfrak{m}^n]$ is minimax. Consequently there is a finitely generated submodule $N/M[\mathfrak{m}^n]$ of $M/M[\mathfrak{m}^n]$ such that M/N is Artinian, and clearly N is coatomic.

(vi)⇒(v): Let N be a coatomic submodule of M such that M/N is Artinian. By [12, Satz A] there is a discrete submodule B of N such that N/B is finitely generated. Therefore M/B is minimax. There exists a minimal element A of the set $\{X \subset M \mid X+B = M\}$. Since A is an essential cover of M/B, according to Corollary 2.4 A is a minimax module, and of course $M = A + B$.

(v)⇒(iv): Clear.

REMARK. The statements of Theorem 3.3 are obviously equivalent to the following: "Every radical factor module of M has finite Goldie-dimension". In [8], we proved a stronger version of Proposition 3.1, which allows a characterization of modules whose socle-free factor modules are of finite Goldie-dimension [8, Satz 2.5].

In the next theorem, we will generalize some of the equivalences of Theorem 3.3 to modules over arbitrary Noetherian rings.

THEOREM 3.4. *For an R-module M the following statements are equivalent:*

(i) Every radical factor module of M is a minimax module.

(ii) Every decomposition of a radical factor module of M is finite.

(iii) Every semi-Artinian factor module of M is the sum of a coatomic and an Artinian module.

(iv) M is an extension of a coatomic module by an Artinian module.

PROOF. (i)⇔(ii): Clear by Theorem 2.1.

(i)⇒(iii): We may assume M to be semi-Artinian. By Theorem 3.3, for every maximal ideal $\mathfrak{m} \subset R$ there is $L_{\mathfrak{m}}(M) = A(\mathfrak{m}) + B(\mathfrak{m})$ with $A(\mathfrak{m})$ Artinian and $B(\mathfrak{m})$ discrete. We may assume $A(\mathfrak{m})$ to be radical. Now $\coprod_{\mathfrak{m}\in\Omega}\big(L_{\mathfrak{m}}(M)/B(\mathfrak{m})\big)$ is radical, hence Artinian by (i). Therefore we have $A(\mathfrak{m}) = 0$ for almost all $\mathfrak{m} \in \Omega$. Consequently we have a presentation $M = \oplus_{\mathfrak{m}\in\Omega} A(\mathfrak{m}) + \oplus_{\mathfrak{m}\in\Omega} B(\mathfrak{m})$, where the first summand is Artinian and the second one is coatomic.

(iii)⇒(i): We can assume M to be radical. Let X be a semi-Artinian factor module of M. Then by (iii) $X = A + B$ with A Artinian and B coatomic. As X is radical, B is small in X and hence X is Artinian. Therefore, by Theorem 2.1 M is a minimax module.

(iii)⇒(iv): We show in a *first step:*

$$M_{\mathfrak{m}} \text{ is a coatomic } R_{\mathfrak{m}}\text{-module for almost all } \mathfrak{m} \in \Omega.$$

PROOF. Assume that we have a sequence $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ of pairwise different maximal ideals such that $M_{\mathfrak{m}_i}$ is not coatomic. Since $M_{\mathfrak{m}_1}$ again satisfies (iii), by Theorem 3.3 there is a finitely generated submodule U_1 of M with $(M/U_1)_{\mathfrak{m}_1}$ semi-Artinian. Let S_1 be a maximal element in the set $\{V \subset U_1 \mid V \cap L(U_1) = 0\}$. S_1 is socle-free and U_1/S_1 is Artinian, being an essential extension of $L(U_1)$. Consequently, $(M/S_1)_{\mathfrak{m}_1}$ is semi-Artinian and not coatomic.

As above, there is a finitely generated, socle-free submodule $S_2/S_1 \subset M/S_1$ with $(M/S_2)_{\mathfrak{m}_2}$ semi-Artinian and not coatomic.

By induction we obtain a sequence $S_1 \subset S_2 \subset \ldots$ of finitely generated submodules of M such that S_{i+1}/S_i is socle-free and $L_{\mathfrak{m}_i}(M/S_i) \cong (M/S_i)_{\mathfrak{m}_i}$ is not coatomic. Now put $L(M/S_i) = S'_i/S_i$; then $S'_i/S_i \cap S_{i+1}/S_i = 0$. With $T = \sum_{i=1}^{\infty} S_i$ we have (similar to the proof of Proposition 3.1) $S'_i \cap T = S_i$. Consequently $S'_i + T/T \cong S'_i/S_i$, so $L(M/S_i)$ is embeddable in $L(M/T)$. Now $L(M/T)$ is the sum of a coatomic and an Artinian module. Therefore almost all $L_{\mathfrak{m}_i}(M/T)$, and hence almost all $L_{\mathfrak{m}_i}(M/S_i)$, are coatomic. This is a contradiction.

In the *second step* we prove (iv): Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be those maximal ideals for which the localization of M is not coatomic. Since, by Theorem 3.3, $M_{\mathfrak{m}_i}$ is an extension of a coatomic module by an Artinian module, according to [12, Lemma 3.2] there is a coatomic submodule B_i of M with $(M/B_i)_{\mathfrak{m}_i}$ Artinian. Hence $B = \sum_{i=1}^{n} B_i$ is also coatomic, and $(M/B)_{\mathfrak{m}_i}$ is Artinian for all $1 \leq i \leq n$. $(M/B)_{\mathfrak{m}}$ is coatomic for all $\mathfrak{m} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$. Let $\bar{M} = M/B$. Then almost all of the primary components of $L(\bar{M})$ are coatomic, and the others are Artinian. Therefore $L(\bar{M})$ is an extension of a coatomic module by an Artinian module. Put $N = \bar{M}/L(\bar{M})$. Then $N_{\mathfrak{m}}$ is coatomic for all $\mathfrak{m} \in \Omega$, since $N_{\mathfrak{m}_i} = 0$ for all $1 \leq i \leq n$. According to [12, Folgerung zu Lemma 1.1], N is coatomic. Consequently, $\bar{M}$ is an extension of a coatomic module by an Artinian module. Then M has the same property.

(iv)$\Rightarrow$(i): By the hypothesis, M has a coatomic submodule V with M/V Artinian. Now let U be a submodule of M with M/U radical. Then $(V+U)/U$ is coatomic and therefore small in M/U. By Corollary 2.4, M/U is a minimax module, being an essential cover of the Artinian module $M/(U+V)$.

Finally we investigate modules which are locally minimax modules. Our first result will show, for rings of finite Krull-dimension, that these modules are extensions of a coatomic module by a semi-Artinian module. For any ring S, by $\mathcal{H}(S)$ we denote the class of S-modules which are extensions of coatomic by semi-Artinian S-modules. If S is local and $M \in \mathcal{H}(S)$ then $\mathrm{Ass}_S(M)$ is finite, since every coatomic and every semi-Artinian S-module has only finitely many associated prime ideals.

PROPOSITION 3.5. *Let M be an R-module with $M_{\mathfrak{m}} \in \mathcal{H}(R_{\mathfrak{m}})$ for all maximal ideals $\mathfrak{m} \subset R$. Then:*

(a) For all $\mathfrak{p} \in \mathrm{Ass}_R\big(P(M)\big)$ one has $\mathrm{coh}(\mathfrak{p}) \leq 1$.

(b) $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module for all non-maximal ideals $\mathfrak{p} \subset R$.

PROOF. a) For every maximal $\mathfrak{m} \supset \mathfrak{p}$ we have $\mathfrak{p}R_{\mathfrak{m}} \in \mathrm{Ass}_{R_{\mathfrak{m}}}\big((PM)_{\mathfrak{m}}\big) \subset \mathrm{Ass}_{R_{\mathfrak{m}}}\big(P(M_{\mathfrak{m}})\big)$, so by [14, Lemma 1.1.e] $\dim(R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}) \leq 1$. Consequently, $\mathrm{coh}(\mathfrak{p}) = \dim(R/\mathfrak{p}) \leq 1$.

b) Let $\mathfrak{p} \subsetneq \mathfrak{m}$. There is a finitely generated submodule B of M with $(M/B)_{\mathfrak{m}}$ semi-Artinian. Let $\mathfrak{q} \in \mathrm{Ass}_R(M/B)$. Then $\mathfrak{q} \not\subset \mathfrak{p}$, since otherwise $\mathfrak{q} \subset \mathfrak{m}$, $\mathfrak{q}R_{\mathfrak{m}} \in \mathrm{Ass}\big((M/B)_{\mathfrak{m}}\big) \subset \{\mathfrak{m}R_{\mathfrak{m}}\}$, hence $\mathfrak{q} = \mathfrak{p} = \mathfrak{m}$ which is not true. Consequently $\mathrm{Ass}_{R_{\mathfrak{p}}}\big((M/B)_{\mathfrak{p}}\big) = \emptyset$, hence $(M/B)_{\mathfrak{p}} = 0$.

PROPOSITION 3.6. *Let M be a non-zero R-module. Assume there exists $n \in \mathbb{N}$ such that $n = \max\{\operatorname{coh}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$. Then the following statements are equivalent:*

(i) $M \in \mathcal{H}(R)$.

(ii) $M_\mathfrak{m} \in \mathcal{H}(R_\mathfrak{m})$ for all maximal ideals $\mathfrak{m} \subset R$.

PROOF. (i)$\Rightarrow$(ii): Clear.

(ii) $\Rightarrow$(i): We prove this statement by induction on n. If $n = 0$, M is semi-Artinian, hence $M \in \mathcal{H}(R)$.

Now let $n \geq 1$ and the assertion be valid for all predecessors of n. Put $\mathcal{M} = \{\mathfrak{p} \in \operatorname{Ass}(M) \mid \operatorname{coh}(\mathfrak{p}) = n\}$. By [2, IV § 1.2, Proposition 4] there is a submodule N of M with $\operatorname{Ass}(N) = \mathcal{M}$ and $\operatorname{Ass}(M/N) = \operatorname{Ass}(M)\setminus \mathcal{M}$. Then obviously $\operatorname{coh}(\mathfrak{p}) \leq n-1$ for every $\mathfrak{p} \in \operatorname{Ass}(M/N)$. Since (ii) is valid for the factor module M/N, by induction $M/N \in \mathcal{H}(R)$, so it remains to show $N \in \mathcal{H}(R)$. Clearly $N_\mathfrak{m} \in \mathcal{H}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \Omega$. As the elements of $\operatorname{Ass}(N)$ all have coheight n, they are pairwise incomparable.

Now let $\mathfrak{p} \in \operatorname{Ass}(N)$. As before there is $X \subset N$ with $\operatorname{Ass}(X) = \{\mathfrak{p}\}$, $\operatorname{Ass}(N/X) = \operatorname{Ass}(N)\setminus\{\mathfrak{p}\}$. By Proposition 3.5.b $X_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module, so there is a finitely generated submodule Y of X with $Y_\mathfrak{p} = X_\mathfrak{p} = N_\mathfrak{p}$ (for the last equality, observe that no associated prime ideal of N/X lies under $\mathfrak{p}$).

As before, for every $\mathfrak{p} \in \operatorname{Ass}(N)$, we can choose a finitely generated $U(\mathfrak{p}) \subset N$ with $U(\mathfrak{p})_\mathfrak{p} = N_\mathfrak{p}$, $\operatorname{Ass}\big(U(\mathfrak{p})\big) = \{\mathfrak{p}\}$. Define $V = \sum\{U(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(N)\}$. V is coatomic: by [12, Folgerung zu Lemma 1.1] we only have to check this locally. So let $\mathfrak{m} \in \Omega$. Then $V_\mathfrak{m} = \sum\{U(\mathfrak{p})_\mathfrak{m} \mid \mathfrak{p} \in \operatorname{Ass}(N) \text{ and } \mathfrak{p} \subset \mathfrak{m}\}$, and this sum contains only finitely many summands, as $\big|\{\mathfrak{p} \in \operatorname{Ass}(N) \mid \mathfrak{p} \subset \mathfrak{m}\}\big| = \big|\operatorname{Ass}_{R_\mathfrak{m}}(N_\mathfrak{m})\big|$ is finite since $N_\mathfrak{m} \in \mathcal{H}(R_\mathfrak{m})$. Consequently $V_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module.

For each $\mathfrak{q} \in \operatorname{Ass}(N/V)$ we have $\operatorname{coh}(\mathfrak{q}) \leq n-1$, because there is $\mathfrak{p} \in \operatorname{Ass}(N)$ with $\mathfrak{p} \subset \mathfrak{q}$, and since $\big(N/U(\mathfrak{p})\big)_\mathfrak{p} = 0$ implies $(N/V)_\mathfrak{p} = 0$, we conclude $\mathfrak{p} \subsetneq \mathfrak{q}$. By induction, $N/V \in \mathcal{H}(R)$, and consequently $N \in \mathcal{H}(R)$ as desired.

COROLLARY 3.7. *Let R be of finite Krull-dimension and M an R-module such that $M_\mathfrak{m}$ is an $R_\mathfrak{m}$-minimax module for all $\mathfrak{m} \in \Omega$. Then M is an extension of a coatomic module by a semi-Artinian, locally Artinian module.*

THEOREM 3.8. *Let R be arbitrary and M a radical R-module. Then the following statements are equivalent:*

(i) $M_\mathfrak{m}$ is an $R_\mathfrak{m}$-minimax module for all maximal ideals $\mathfrak{m} \subset R$.

(ii) M is an extension of a coatomic module by a semi-Artinian, locally Artinian module.

PROOF. (i)$\Rightarrow$(ii): Because of Proposition 3.5.a, $\operatorname{coh}(\mathfrak{p}) \leq 1$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$, so, by Proposition 3.6, $M \in \mathcal{H}(R)$.

(ii)$\Rightarrow$(i): Every $M_\mathfrak{m}$ is radical and satisfies (ii) as an $R_\mathfrak{m}$-module, so we may assume R to be local and we have to show M minimax. Let V be a coatomic submodule of M with M/V Artinian. Then, as M is radical, V is small in M, and therefore M is a minimax module by Corollary 2.4.

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