2

Conformal field theory

Conformal invariance of two-dimensional massless scalar field theory was shown in the previous chapter to associate with the infinite algebra of conserved charges, the Virasoro algebra. In this chapter we describe the basic building blocks of any two-dimensional conformal field theory (CFT). The notions of primary and descendant operators will be introduced and the structure of the Hilbert space of states will be described. We will discuss and classify certain classes of unitary CFTs. Crossing symmetry, duality and bootstrap equations will be defined and applied to computing correlators of CFTs. We then discuss the Verlinde formula which relates the fusion rules and the S transformation. We will end up with two examples of CFTs that demonstrate all of the concepts that have been introduced before. The first one is the theory of a Majorana fermion and the second is the $m = 3$ unitary minimal model, which is shown to be the continuum limit of the two-dimensional Ising model.

Conformal field theory in two dimensions is covered by many review articles and books. The former include [109] which we use intensively in this chapter, also [25], [13], [59], [233] and many others.

Among the books that discuss 2d CFT is [140] and books on string theories [113], [154], [174], [138], [237], [142], [30].

The most complete book on the topic is [77].

The basics of conformal field theory were stated in the seminal paper by Belavin, Polyakov and Zamolodchikov [33]. This includes the introduction of primary fields, the behavior of the energy-momentum tensor and the central charge. Conformal Ward identity and the use of OPEs appears in [93], [95] and [94].

2.1 Conformal symmetry in two dimensions

The theory of the free massless scalar field in two dimensions was shown to be invariant under the holomorphic and anti-holomorphic coordinate transformations

$$z \to z' = f(z); \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}). \tag{2.1}$$

Under such a transformation the metric transforms as

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} \to \mathrm{d}z'\mathrm{d}\bar{z}' = \frac{\partial z'}{\partial z}\frac{\partial \bar{z}'}{\partial \bar{z}}\mathrm{d}z\mathrm{d}\bar{z}. \tag{2.2}$$

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At this point we can understand why we referred to these transformations as conformal transformations. In general in d space-time dimensions the conformal group is the subgroup of coordinate transformations that leaves the metric invariant up to a scale, namely,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.3}$$

It is obvious from (2.2) that the 2d conformal transformations (2.1) indeed produce such a variation of the metric. An important property of conformal transformations in any dimension is that they preserve the angle $\frac{\vec{A}\cdot\vec{B}}{\sqrt{A^2 B^2}}$ between two vectors $\vec{A}$ and $\vec{B}$.

Starting from flat space, the general infinitesimal coordinate transformations $x^\mu \to x^\mu + \epsilon^\mu(x)$ induces a change of the metric $\mathrm{d}s^2 \to \mathrm{d}s^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\mathrm{d}x^\mu \mathrm{d}x^\nu$, so that the condition for conformal transformations reads,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon) g_{\mu\nu}, \tag{2.4}$$

where $g_{\mu\nu}$ is $\eta_{\mu\nu}$ or $\delta_{\mu\nu}$ for a Minkowskian signature, or Euclidean signature, respectively.

It is thus obvious that for two-dimensional Euclidean space-time $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$ are the unique solutions of (2.4), which reduces to the Cauchy–Riemann equation $\partial_1 \epsilon_1 = \partial_2 \epsilon_2$ and $\partial_1 \epsilon_2 = -\partial_2 \epsilon_1$.

We would like now to put aside scalar field theory and explore the general properties of conformal field theories in two dimensions. Any theory with a vanishing trace of the energy-momentum tensor $T^\mu_\mu = 0$, or in complex coordinates $T_{z\bar{z}} = 0$, has necessarily an independent holomorphically (and anti-holomorphically) conserved energy-momentum tensor components, namely,

$$\bar{\partial} T \equiv \bar{\partial} T_{zz} = 0 \quad \partial \bar{T} \equiv \partial T_{\bar{z}\bar{z}} = 0. \tag{2.5}$$

This follows trivially from the usual conservation law $\bar{\partial} T_{zz} + \partial T_{z\bar{z}} = 0$, and its complex conjugation. It is also clear that in fact there are infinitely many conserved currents, since $g(z)T(z)$ for any analytic function $g(z)$ is also a holomorphically conserved current (we sometimes call any conserved tensor "current").

We show in the following section that indeed the energy-momentum tensor $T(z)$ and $\bar{T}(\bar{z})$ generate the conformal transformations given in (2.1).

2.2 Primary fields

Conformal invariance constrains the OPEs of the theory. In particular, since T is holomorphic, the OPE of $T(z)$ with a general operator can be expanded in terms of a Laurent expansion in integer powers of z. The singular part of the OPE takes the form,

$$T(z)\tilde{\mathcal{O}}(w, \bar{w}) = \sum_{n=0}^{\infty} \frac{1}{(z-\omega)^{n+1}} \tilde{\mathcal{O}}^{(n)}(w, \bar{w}), \tag{2.6}$$

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where the sum is usually finite, and the operators $\tilde{\mathcal{O}}^{(n)}(w,\bar{w})$ have to be determined. Using radial quantization as in Section 1.7 and the OPE above, we get for the transformation generated by $T(z)$,

$$\delta_\epsilon \tilde{\mathcal{O}}(w,\bar{w}) = \sum_n \frac{1}{n!}\left[(\partial^n \epsilon)\tilde{\mathcal{O}}^{(n)}(w,\bar{w})\right]. \tag{2.7}$$

We now consider operators that transform under conformal transformation in a way that generalizes the transformation of the metric, (2.2),

$$\mathcal{O}(z,\bar{z}) \to \mathcal{O}'(z'\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}(z'\bar{z}'). \tag{2.8}$$

An operator with such conformal transformations is a *primary field* or a *tensor operator* with *conformal weights* $(h,\bar{h})$, which are sometimes referred to as the holomorphic and anti-holomorphic conformal dimensions.[1] The sum of the weights $h+\bar{h}$ is the *total dimension* that determines the behavior under scaling, whereas $h-\bar{h}$ is the *spin* that controls the behavior under rotations. The infinitesimal transformations that correspond to (2.8) are,

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O}(z,\bar{z}) = \left[(h\partial\epsilon + \epsilon\partial) + (\bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial})\right]\mathcal{O}(z\bar{z}). \tag{2.9}$$

This form of transformation implies that the singular part of the OPE of T and $\mathcal{O}(w,\bar{w})$ reduces to,

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h}{(z-\omega)^2}\mathcal{O}(w,\bar{w}) + \frac{1}{(z-\omega)}\partial\mathcal{O}(w,\bar{w}). \tag{2.10}$$

Applying these notions to the free scalar field we find that $\partial X(z)$ has $(1,0)$ weights, $\bar{\partial}\bar{X}(\bar{z})$ has $(0,1)$ and the weights of $: \mathrm{e}^{i\alpha X(z,\bar{z})} :$ are $(\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$.

In Chapter 1 the notion of OPE was discussed in the context of scalar field theory. The generalization to any CFT is straightforward. Normalize the operators with fixed conformal weights as,

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\rangle = \delta_{ij}\frac{1}{(z-w)^{2h_i}}\frac{1}{(\bar{z}-\bar{w})^{2\bar{h}_i}}, \tag{2.11}$$

then, for a complete set, the OPE of any pair of such operators is, to leading singularity,

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) \sim \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\mathcal{O}_k(w,\bar{w}), \tag{2.12}$$

where C_{ijk} are the *product coefficients* of the theory.

[1] The notion of conformal primary field and its descendants was introduced in [33] and further discussed in [236].

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2.3 Conformal properties of the energy-momentum tensor

For the free massless scalar field we found that the OPE of $T(z)T(w)$ is not of the form shown as (2.6), due to the anomaly term as in (1.71). The form of $T(z)T(w)$ OPE for any CFT is rather,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}, \tag{2.13}$$

where c is the *central charge* (or the Virasoro anomaly), a constant that characterizes the theory. The second term represents the dimensions and the third the property of translations under T. For theories with positive semi-definite Hilbert space $c \geq 0$, as follows from,

$$< T(z)T(w) >= \frac{c/2}{(z-w)^4}.$$

This type of OPE implies the following infinitesimal transformation of T:

$$\delta_{\epsilon(z)} T(z) = \frac{c}{12}\partial^3\epsilon(z) + 2(\partial\epsilon(z))T(z) + \epsilon(z)\partial T(z). \tag{2.14}$$

The corresponding finite transformation $T(z) \to T'(z')$ takes the form,

$$T'(z') = (\partial z')^2 T(z) + \frac{c}{12}\{z', z\}, \tag{2.15}$$

where $\{z', z\}$ is the Schwarzian derivative,

$$\{f, z\} = \frac{2\partial^3 f \partial f - 3\partial^2 f \partial^2 f}{2\partial f \partial f}. \tag{2.16}$$

To derive (2.16), we first note that by applying a second transformation $f \to \omega$ we get,

$$\{w, z\} = (\partial_z f)^2 \{w, f\} + \{f, z\}. \tag{2.17}$$

Then, we take $\omega = f + \delta f$, thus obtaining a functional equation,

$$\delta f \frac{\delta}{\delta f}\{f, z\} = (\partial_z f)^2 \frac{\partial^3 \delta f}{\partial^3 f}. \tag{2.18}$$

Expressing the right-hand side as derivatives with respect to z,

$$\frac{1}{f'}(\delta f)^{'''} - \frac{3f^{''}}{(f')^2}(\delta f)^{''} + \left[\frac{3(f^{''})^2}{(f')^3} - \frac{f^{'''}}{(f')^2}\right](\delta f)^{'},$$

we can integrate the equation to get (2.16). The first term suggests integrating to $f^{'''}/f^{'}$, the variation of which gives $1/f^{'}(\delta f)^{'''} - f^{'''}/(f^{'})^2(\delta f)^{'}$, while the second term suggests $-3(f^{''})^2/2(f^{'})^2$, the variation of which gives $-3f^{''}/(f^{'})^2(\delta f)^{''} + 3(f^{''})^2/(f^{'})^3(\delta f)^{'}$.

For the massless scalar case T can be written as $T(z) = -\frac{1}{2} : J(z)J(z) :$, as we saw in (1.5). In fact, as will be discussed in Chapter 3, there is a large class of theories that share this so-called *Sugawara form*. For this type of theory the proof that the finite transformation is of the form of (2.15) is as follows. Recall

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that as a primary field of weights $(1,0)$, $J(z) \to \frac{\partial z'}{\partial z} J(z')$. If we write $T(z) = -\frac{1}{2}\lim_{z\to w}(J(z)J(w) + \frac{1}{(z-w)^2})$ and substitute the transformation of the currents we end up after some lengthy but straightforward calculation with (2.15).

2.4 Virasoro algebra for CFT

Let us use the Laurent expansion of T for CFT, following (1.60),

$$T = \sum_{n=-\infty}^{\infty} L_n z^{-(n+2)} \quad \bar{T} = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{-(n+2)}}, \tag{2.19}$$

so that,

$$L_n = \frac{1}{2\pi i} \oint \mathrm{d}z z^{n+1} T(z). \tag{2.20}$$

The expansion is chosen such that L_n has scale dimension n under $z \to z/a$, namely, $L_n \to a^n L_n$.

The Virasoro algebra[2] can now be derived using the OPE of $T(z)T(w)$ given in (2.13),

$$[L_n, L_m] = \left(\frac{1}{2\pi i}\right)^2 \oint \mathrm{d}z \oint \mathrm{d}w [z^{n+1}w^{m+1} - z^{m+1}w^{n+1}] T(z)T(w). \tag{2.21}$$

The double integral is performed by fixing w and transforming the difference of the two $\oint \mathrm{d}z$ integrals into one integral around w,

$$\begin{aligned}
[L_n, L_m] &= \left(\frac{1}{2\pi i}\right)^2 \oint \mathrm{d}z \oint \mathrm{d}w \left[z^{n+1}w^{m+1} - z^{m+1}w^{n+1}\right] \\
&\quad \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}\right] \\
&= \left(\frac{1}{2\pi i}\right) \oint \mathrm{d}w \left[c/12(n^3 - n)w^{n+m-1}\right. \\
&\quad \left. + \left[2(n+1) - (n+m+2)\right] w^{n+m+1} T(w)\right] \\
&= \frac{c}{12}(n^3 - n)\delta(n+m) + (n-m)L_{n+m}.
\end{aligned} \tag{2.22}$$

Performing identical steps for $\bar{L}_n$ we get that $\bar{L}_n$ obeys the same infinite algebra, with some central charge $\bar{c}$, and that $[L_n, \bar{L}_m] = 0$.

Any CFT is a representation of the Virasoro algebra characterized by c and $\bar{c}$. It is straightforward to identify the following properties of the algebra:

- The generators $(L_{\pm 1}, L_0)$ span an $SL(2, \mathcal{R})$ algebra,

$$[L_{+1}, L_{-1}] = 2L_0 \quad [L_0, L_{\pm 1}] = \mp L_{\pm} \tag{2.23}$$

[2] The first use of the Virasoro algebra was by M. Virasoro in the context of the dual resonance model [212]. Its application to two-dimensional CFT was presented in [33].

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Table 2.1. *The conformal family*

Level	Weight	Fields
0	h	ϕ
1	h+1	$L_{-1}\phi$
2	h+2	$L_{-2}\phi, L_{-1}^2\phi$
3	h+3	$L_{-3}\phi, L_{-2}L_{-1}\phi, L_{-1}^3\phi$
$\vdots$	$\vdots$	$\vdots$
N	h+N	$P(N)$ fields

- For $n > 0$, L_{-n} is a raising operator and L_n is a lowering one, since $[L_0, L_n] = -nL_n$ so that if $|\psi>$ is an eigenstate of L_0, $L_0|\psi> = h|\psi>$ then $L_0|L_n\psi> = (h-n)|L_n\psi>$.

2.5 Descendant operators

From every primary operator $\phi(z,\bar{z})$ one can construct an infinite tower of Virasoro descendant operators,

$$(L_{-n}\phi(w,\bar{w})) = \frac{1}{2\pi i}\oint \mathrm{d}z \frac{1}{z^{n-1}} T(z)\phi(w,\bar{w}). \tag{2.24}$$

A distinguished descendant operator is the energy momentum tensor $T(z)$ since,

$$L_{-2}\mathbf{1} = \frac{1}{2\pi i}\oint \frac{\mathrm{d}z}{z} T(z)\mathbf{1} = T(0). \tag{2.25}$$

The set containing the primary field $\phi(z,\bar{z})$ and all its descendant operators is called a *conformal family* and it is denoted by $[\phi]$. A conformal family is a tower of operators where each layer is characterized by its level as shown in Table 2.1, where $P(N)$ is the number of partitions of N into positive integer parts, which can be written in terms of the generating function $\prod_{n=1}\frac{1}{(1-q^n)} = \sum_{N=0}^{\infty} P(N)q^N$.

We can now use the conformal family to rewrite the expression of the OPE (2.12) of two primary fields,

$$\begin{aligned}\phi_i(z,\bar{z})\phi_j(w,\bar{w}) \\ = \sum_{k\{l\bar{l}\}} C_{ijk}^{\{l\bar{l}\}} (z-w)^{h_k - h_i - h_j + \sum_n l_n} (\bar{z}-\bar{w})^{\bar{h}_k - \bar{h}_i - \bar{h}_j + \sum_n \bar{l}_n} \phi_k^{l\bar{l}}(w,\bar{w}),\end{aligned} \tag{2.26}$$

where we denote by $\phi_k^{l\bar{l}}(w,\bar{w})$ the descendants $L_{-l_1}\dots L_{-l_n}\bar{L}_{-\bar{l}_1}\dots\bar{L}_{-\bar{l}_n}\phi_k(w,\bar{w})$ with the normalization given in (2.11). The product coefficients $C_{ijk}^{\{l\bar{l}\}}$ are given in terms of those of (2.12) C_{ijk} as,

$$C_{ijk}^{\{l\bar{l}\}} = C_{ijk}\beta_{ij}^{k\{l\}}\bar{\beta}_{ij}^{k\{\bar{l}\}}, \tag{2.27}$$

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where $\beta_{ij}^{k\{l\}}$ are determined by conformal invariance and are functions of c and h_i, h_j, h_k, and similarly for $\bar{\beta}_{ij}^{k\{\bar{l}\}}$. This follows from a detailed analysis that we do not show here.

The OPEs of any pair of descendant fields can also be deduced from (2.12) which implies in fact that all the information about the OPE is encoded in the product coefficients C_{ijk}. Moreover since the structure of (2.26) holds for all the primaries and their descendants, one can write the so-called *fusion algebra* for conformal, families, which takes the form,

$$[\phi_i][\phi_j] = \sum_k N_{ij}^k [\phi_k]. \tag{2.28}$$

2.6 Hilbert space of states

Our next task is to construct the Hilbert space of states. First we define the ground state $|0\rangle$ by,

$$L_n|0\rangle = 0 \quad n \geq 0. \tag{2.29}$$

The next step in this program is to build the highest weight states (hws). Consider the state generated from the vacuum by a primary field $\phi(z)$ of dimension h,

$$|h\rangle = \phi(0)|0\rangle\ . \tag{2.30}$$

It is easy to check that for $n > 0$, $[L_n, \phi(0)] = 0$ since,

$$[L_n, \phi(w)] = \frac{1}{2\pi i}\oint \mathrm{d}z z^{n+1} T(z)\phi(w) = h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w). \tag{2.31}$$

Hence the *highest weight state* $|h\rangle$ obeys

$$L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0 \ \ n > 0. \tag{2.32}$$

Expanding the primary field $\phi(z)$ in a Laurent series $\sum_n \phi_n z^{(n-h)}$, one can write the highest weight state symbolically as $\phi_h|0\rangle$.

Descendant states are generated by applying the descendant operators $L_{-n}\phi$ on the vacuum or alternatively by applying L_{-n} on highest weight states, namely,

$$L_{-n}|h\rangle = L_{-n}\phi(0)|0\rangle = (L_{-n}\phi)|0\rangle\ . \tag{2.33}$$

It is thus clear that the highest weight states, or equivalently the primary operators, play a major role in constructing representations of the Virasoro algebra. In fact one can show that every representation is characterized by a primary operator. Consider an eigenstate of L_0, $L_0|\psi\rangle = h_\psi|\psi\rangle$. Now act on it with the lowering operator L_n with $n > 0$. The L_0 eigenvalue of the new state $L_n|\psi\rangle$ is $h_\psi - n$. Since we require that the Hamiltonian is bounded from below, L_0 has to be also bounded. This implies that after repeating the lowering process one finally hits a state that is annihilated by L_n for every $n > 0$ and hence an hws.

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It is thus clear that any state in a positive Hilbert space is a linear combination of hws, and their descendants. The representation given in Table 2.1 is referred to as the *Verma module*. Denoting it by $\mathcal{V}(c,h)$ and its analogous representation for the anti-holomorphic Virasoro algebra by $\bar{\mathcal{V}}(\bar{c},\bar{h})$, the Hilbert space of the theory is a direct sum of the products $\mathcal{V}(c,h)\otimes\bar{\mathcal{V}}(\bar{c},\bar{h})$, namely,

$$\mathcal{H}=\sum_{h,\bar{h}}\mathcal{V}(c,h)\otimes\bar{\mathcal{V}}(\bar{c},\bar{h}). \tag{2.34}$$

The Verma module may be *reducible* in the sense that there is a submodule that is by itself a Verma module. Such a submodule whose states transform amongst themselves under any conformal transformation, is built from a $|h_{\text{null}}\rangle$. The latter is both an hws., namely $L_n|h_{\text{null}}\rangle=0$ for $n>0$, as well as a descendant. Such a state is called *null state* or *null vector*, motivated by what follows. It generates its own Verma module which is included in the parent module. It is orthogonal to the whole Verma module as well as to itself $\langle h_{\text{null}}|h_{\text{null}}\rangle=0$, since $\langle h_{\text{null}}|L_{-k_1}\ldots L_{-k_n}|h\rangle=\langle h|L_{k_n}\ldots L_{k_1}|h_{\text{null}}\rangle^*=0$, and in particular it has a zero norm $\langle h_{\text{null}}|h_{\text{null}}\rangle=0$ and similarly also its descendants. The null state corresponds to a null operator which is simultaneously a primary and a secondary field.

Let us now demonstrate the construction of a null vector. Consider a general linear combination of the states of level 2,

$$L_{-2}|h\rangle+aL_{-1}^2|h\rangle, \tag{2.35}$$

we would like to check whether for certain values of the mixing coefficient a, this state is a null state. If indeed it is $|\text{null}\rangle$, then so is the state $[L_n|\text{null}\rangle]$. In fact it is easy to verify that at level 2, one has to check these consistency conditions only for L_1 and L_2. Now using the Virasoro algebra we find that,

$$\begin{aligned}[][L_1,L_{-2}]\,|h\rangle+a\left[L_1,L_{-1}^2\right]|h\rangle&=(3+2a(2h+1))L_{-1}|h\rangle,\\ [L_2,L_{-2}]|h\rangle+a\left[L_2,L_{-1}^2\right]|h\rangle&=\left(4h+\frac{c}{2}+6ah\right)|h\rangle\,.\end{aligned} \tag{2.36}$$

It is thus clear that for the following values of a and c,

$$a=-\frac{3}{2(2h+1)}\qquad c=\frac{2h}{2h+1}(5-8h), \tag{2.37}$$

the linear combination state (2.35) is a null state. In the unitary case we have h and c positive (see next section). Hence in this example $h<\frac{5}{8}$.

An irreducible representation of the Virasoro algebra can be constructed from a Verma module that contains a null vector by a quotient procedure, taking out of the Verma module the null module. In the next section we discuss this construction.

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2.7 Unitary CFT and Kac determinant

Unitarity is obviously lost if there are negative norm states in the Verma module. Hence, our task is to derive the conditions for having a negative norm state. In the basis of the Verma module,

$$L_{-k_1} \dots L_{-k_i} |h\rangle \equiv |s\rangle \quad (1 \leq k_1 \leq \dots \leq k_i), \tag{2.38}$$

the matrix of inner products $\mathbf{I}_{ss'} = \langle s|s'\rangle$ is block diagonal with blocks $\mathbf{I}^{(N)}$ for states at level $N(\sum_i k_i = N)$. For a given Verma module the elements of $\mathbf{I}$ are functions of (h, c). It is easy to realize that unitarity dictates $c > 0$ and $h > 0$. This follows from $\langle h|L_n L_{-n}|h\rangle = [2nh + 1/12cn(n^2 - 1)] \langle h|h\rangle$, which is positive for $n = 1$ only if $h > 0$ and for large enough n only for $c > 0$. To determine the full set of constraints for unitarity let us analyze further the properties of $\mathbf{I}$. A general state $|\hat{s}\rangle = \sum_k c_k |s\rangle$ has a norm $\langle \hat{s}|\hat{s}\rangle = \hat{c}^\dagger \mathbf{I} \hat{c}$, with $\hat{c}$ the vector of the c_k. Now since $\mathbf{I}$ is hermitian it can be diagonalized by a unitary matrix U so that the norm can be written as $\langle \hat{s}|\hat{s}\rangle = \sum_k l_k |t_k|^2$ where $t = U\hat{c}$ and l_k are the eigenvalues of $\mathbf{I}$, which are real. It is thus clear that there are negative norm states if and only if $\mathbf{I}$ has negative eigenvalues. A vanishing eigenvalue indicates a null vector which means a reducible Verma module.

For the low lying levels these matrices take the following form:

$$\begin{aligned}
\mathbf{I}^{(0)} &= 1 \\
\mathbf{I}^{(1)} &= 2h \\
\mathbf{I}^{(2)} &= \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + c/2 \end{pmatrix}.
\end{aligned} \tag{2.39}$$

The derivation of the various elements is straightforwad, for instance,

$$\begin{aligned}
\mathbf{I}^{(2)}_{11} &= \langle h|L_1 L_1 L_{-1} L_{-1}|h\rangle = \langle h|L_1 L_{-1} L_1 L_{-1}|h\rangle + 2 \langle h|L_1 L_0 L_{-1}|h\rangle \\
&= 4 \langle h|L_1 L_{-1} L_0|h\rangle + 2 \langle h|L_1 L_{-1}|h\rangle = 8h^2 + 4h.
\end{aligned} \tag{2.40}$$

The determinant of $\mathbf{I}^{(2)}$ is given by

$$\det[\mathbf{I}^{(2)}] = 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}), \tag{2.41}$$

where $h_{1,1} = 0$ and $h_{1,2}, h_{2,1}$ are $(1/16)[(5 - c) \pm \sqrt{(1-c)(25-c)}]$. The trace of $\mathbf{I}^{(2)}$ is $\mathrm{Tr}[\mathbf{I}^{(2)}] = 8h(h+1) + c/2$. Since the trace and the determinant are the sum and product of the two eigenvalues, unitarity is lost if either the trace or the determinant is negative.

The determinant for $\mathbf{I}^{(N)}$ at general level N, which is referred to as the *Kac determinant,*[3] has the form

$$\det[\mathbf{I}^{(N)}] = \alpha_N \prod_{pq \leq N} [h - h_{p,q}(c)]^{P(N-pq)}, \tag{2.42}$$

[3] The proof of the Kac determinant is detailed in [89], [206] and [95].

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where α_N are constants independent of (c, h) and $h_{p,q}(c)$ can be expressed in terms of $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$ as,

$$h_{p,q}(c) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}. \tag{2.43}$$

Note that we can choose either the plus or the minus sign in the expression for m, as their interchange is like interchanging p with q, which does not change the determinant. Note also that $h_{p,q}$ is invariant under $p \to m - p, q \to m + 1 - q$. Let us also mention that for $N = 2$ the result is identical to (2.41).

In the (h, c) plane the determinant vanishes along the curves $h = h_{p,q}(c)$ which are therefore called the *vanishing curves*. If the determinant (2.42) is negative it means that there is an odd number of negative eigenvalues and hence the corresponding Virasoro representation is not unitary. If the determinant is vanishing or positive one needs to further analyze the determinant as follows:

- For $c > 1$ and $h > 0$ it is straightforward to show that the determinant does not vanish.

  In the domain $1 < c < 25$ the value for m has an imaginary part. Thus $h_{p,q}$ are complex for $p \neq q$, and as they come in complex conjugate pairs the product of the appropriate two factors in the determinant is positive. For $p = q$ the value of $h_{p,q}$ is negative. Thus the determinant is positive in that domain.

  For $c > 25$ the $h_{p,q}$ are negative.

  For large h the matrix is dominated by its diagonal elements.

  Since these elements are positive, the eigenvalues for large h are all positive. Now since the determinant never vanishes in the region considered $(h > 0,\ c > 1)$ all the eigenvalues have to be positive on the entire region.

  Note that in $\mathbf{I}^{(2)}$ the off-diagonal element is larger at large h than the 22 element, but still the determinant is dominated at large h by the diagonal elements, and thus also the eigenvalues, as a 2×2 matrix.
- For $c = 1$ we have $h_{p,q} = (p - q)^2/4$, and so the determinant is never negative. However, it vanishes when $h = n^2/4$ for some integer n.
- For $0 < c < 1,\ h > 0$ a closer look at the determinant is required. We draw $h_{p,q}(c)$ in Fig. 2.1.

  By expanding the curves around $c = 1$ one can show that any point in the region can be connected to the right of $c = 1$ by crossing a single vanishing curve. The vanishing of the determinant is due to one eigenvalue that reverses its sign which implies that there are negative norm states at any point in the region that are not on the vanishing curve. In fact it turns out that there are additional negative norm states at points along the vanishing curve except at

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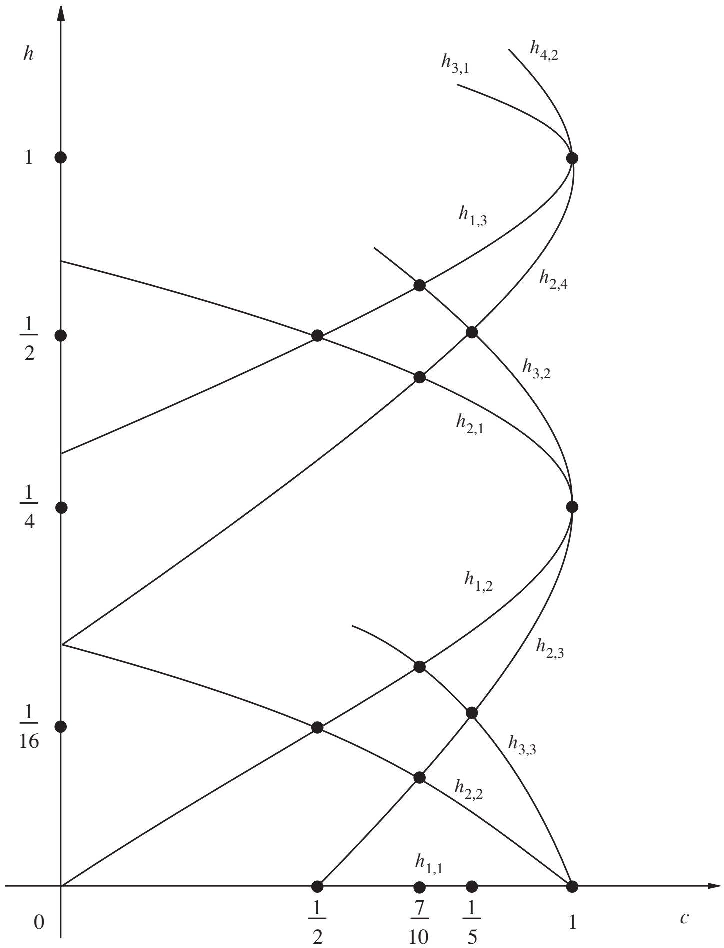

Fig. 2.1. $h_{p,q}(c)$ as a function of c for various values of (p, q).

certain points where they intersect. On these points the central charge c is a solution of $m = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$ for the cases of m an integer from 3 up,

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots \tag{2.44}$$

For each such *unitary minimal model*[4] there are $m(m-1)/2$ primary fields with h given by (2.43) where p, q are integers satisfying $1 \leq p \leq m-1,\ 1 \leq q \leq p$. The simplest of those models is the Ising model, given in the section $m = 3,\ c = 1/2$ with $h_{1,1} = 0,\ h_{2,1} = 1/2,\ h_{2,2} = 1/16$. It will be described in Section 2.13.

[4] The minimal models were presented in [33] and discussed in [95].

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2.8 Characters

The structure of the Verma module, and in particular the degeneracy of states at each level, is captured in the generating function $\chi_{(c,h)}(\tau)$, the *character* of the Verma module, defined by,

$$\chi_{(c,h)}(\tau) = \mathrm{Tr}[q^{L_0 - \frac{c}{24}}] = \sum_{n=0}^{\infty} \dim(h+n) q^{h+n-\frac{c}{24}}, \tag{2.45}$$

where $q \equiv \mathrm{e}^{2\pi i\tau}$, τ is a complex number, and $\dim(n+h)$ is the number of linearly independent states of the module at level n. The latter is equal to $P(n)$ the partitions of n in the generic case, but may be smaller when there are null states. For $|q| < 1$, namely, τ in the upper half plane, the series is uniformly convergent, since $|q| < 1$ is the domain of convergence of the inverse of the Euler function $\varphi(q)$ defined by,

$$\frac{1}{\varphi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} P(n) q^n. \tag{2.46}$$

In terms of this function the character of a generic Verma module is given by,

$$\chi_{(c,h)}(\tau) = \frac{q^{h-\frac{c}{24}}}{\varphi(q)}. \tag{2.47}$$

The character can be expressed also in terms of the Dedekind $\eta(\tau)$ function,

$$\eta(\tau) \equiv q^{\frac{1}{24}} \varphi(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \tag{2.48}$$

in the form

$$\chi_{(c,h)}(\tau) = \frac{q^{h+\frac{1-c}{24}}}{\eta(\tau)}. \tag{2.49}$$

To get the character of a minimal model one has to determine the irreducible Verma module using the quotient procedure discussed in the previous section. We do not give the derivation here, just the final result, which is,

$$\chi(c(p,p'), h_{rs}(p,p')) = \frac{q^{h-\frac{c}{24}}}{\varphi(q)} = \sum_{n\in\mathcal{Z}} \left[q^{\frac{(2pp'n+pr-p's)^2}{4pp'}} - q^{\frac{(2pp'n+pr+p's)^2}{4pp'}} \right], \tag{2.50}$$

where

$$c(p,p') = 1 - 6\frac{(p-p')^2}{pp'}, \tag{2.51}$$

and

$$h_{rs}(p,p') = \frac{(pr-p's)^2 - (p-p')^2}{4pp'}. \tag{2.52}$$

Note that these are the non-unitary minimal models, except for the cases $p - p' = \pm 1$, which coincide with the cases of the previous section with the identification of $\mathrm{p} = \mathrm{m}$ or $\mathrm{p}' = \mathrm{m}$.

2.9 Correlators and the conformal Ward identity

Now that the Hilbert space of states has been analyzed we would like to determine the correlation functions of all possible operators of a given CFT. Naturally, we first investigate correlators of primary fields and then those also involving descendents.

A very useful tool for determining correlators are the symmetries of the system. In the present case we obviously implement conformal invariance. In particular we first determine the consequences of the $SL(2,C)$ *Ward identities*. Recall that the vacuum is annihilated by $L_{0,\pm 1}$ and $\bar{L}_{0,\pm 1}$, and hence is invariant under $SL(2,C)$, namely, $U|0\rangle = |0\rangle$ for $U \in SL(2,C)$. It thus follows that,

$$<0|U^{-1}\phi_1(z_1,\bar{z}_1)U \dots U^{-1}\phi_n(z_n,\bar{z}_n)U|0> = <0|\phi_1(z_1,\bar{z}_1)\dots\phi - n(z_n,\bar{z}_n)|0> . \tag{2.53}$$

Recall that by definition a primary field of dimension h transforms under an $SL(2,C)$ transformation $z \to f(z) = \frac{az+b}{cz+d}$ (with $ad - bc = 1$), as,

$$U^{-1}\phi(z,\bar{z})U = (\partial f(z))^h \phi(f(z),\bar{z}). \tag{2.54}$$

Let us mention that $SL(2,C)$ invariance holds for CFT in any dimension.

The invariance of the vacuum implies, in infinitesimal form,

$$<0|[L_k,\phi_1(z_1,\bar{z}_1)]\dots\phi_n(z_n,\bar{z}_n)|0> + \dots <0|\phi_1(z_1,\bar{z}_1)\dots[L_k,\phi_n(z_n,\bar{z}_n)]|0> = 0, \tag{2.55}$$

for $k = 0, \pm 1$. Using $[L_k, \phi(z,\bar{z})] = h(k+1)z^k\phi(z,\bar{z}) + z^{k+1}\partial\phi(z,\bar{z})$ we get Ward identities in terms of differential equations:

$$\begin{aligned}
k = -1: &\quad \sum_i \partial_i <0|\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)|0> = 0\\
k = \ \ 0: &\quad \sum_i (z_i\partial_i + h_i) <0|\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)|0> = 0\\
k = +1: &\quad \sum_i (z_i^2\partial_i + 2z_ih_i) <0|\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)|0> = 0.
\end{aligned} \tag{2.56}$$

These Ward identities are associated with the invariance under translations, dilations and special conformal transformations. Applying these equations to the two point function one finds that,

$$G_2(z_1,\bar{z}_1,z_2,\bar{z}_2) \equiv <0|\phi_1(z_1,\bar{z}_1)\phi_1(z_2,\bar{z}_2)|0> = \frac{c_2}{(z_1 - z_2)^{2h_1}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}_1}}, \tag{2.57}$$

where c_2 is a constant, to be put to 1 in the normalization (2.11). Note also that when taking two different fields ϕ_1 and ϕ_2, $SL(2,C)$ implies that $h_1 = h_2$ is necessary for a non-zero two-point function.

In a similar manner the three-point function is given by,

$$G_3(z_i,\bar{z}_i) = c_{123}\left(\frac{1}{z_{12}^{h_1+h_2-h_3}\,z_{13}^{h_1+h_3-h_2}\,z_{23}^{h_2+h_3-h_1}}\right)(z \to \bar{z}, h \to \bar{h}), \tag{2.58}$$

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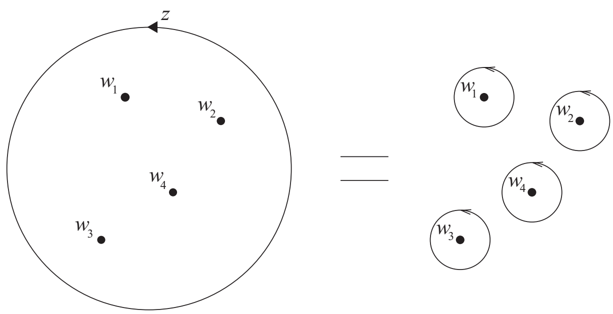

Fig. 2.2. Integration along C that bounds all the operators.

where $z_{ij} = z_i - z_j$ and c_{123} is the correpsonding product coefficient defined in (2.12). Using the $SL(2C)$ invariance one can set the points z_1, z_2, z_3 at $\infty, 1, 0$, respectively so that the constant c_{123} is determined from the corresponding correlator via $\lim_{z_1, \bar{z}_1 \to \infty} \left[z_1^{2h} \bar{z}_1^{2\bar{h}} G_3 \right] = c_{123}$. For G_n with $n > 3$ the global conformal transformations do not fully determine the correlator. For instance the four-point function G_4 can be written using these transformations as,

$$G_4(z_i, \bar{z}_i) = f(\mathcal{Z}, \bar{\mathcal{Z}}) \left[\left(\prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \right) \left(z \to \bar{z}, h \to \bar{h} \right) \right], \tag{2.59}$$

where $h = \sum_{i=1}^{4} h_i$ and the cross ratio $\mathcal{Z}$ is defined as $\mathcal{Z} = \frac{z_{12} z_{34}}{z_{13} z_{24}}$, which is an $SL(2, C)$ invariant.

For a general n-point function, denoting the power of z_{ij} by $-h_{ij}$, we get, $h_{ij} = \left[\frac{2}{n-2}(h_i + h_j) - \frac{2}{(n-1)(n-2)} h \right]$ for $n \geq 3$.

So far we have implemented the global Ward identities. To get the *local Ward identity* one performs a conformal transformation of an n-point function of primary fields G_n. This is achieved by integrating $\epsilon(z) T(z)$ along a contour C which bounds a region that includes all the operators (see Fig. 2.2)

Now using analyticity one can deform the contour into a sum of countours each of which encircles one operator. The result of the integral is therefore,

$$\begin{aligned} &\left\langle \oint \frac{\mathrm{d}z}{2\pi i} \epsilon(z) T(z) \phi(w_1, \bar{w}_1) \ldots \phi(w_n, \bar{w}_n) \right\rangle \\ &\quad = \sum_{i=1}^{n} \left\langle \phi(w_1, \bar{w}_1) \ldots \oint \frac{\mathrm{d}z}{2\pi i} \epsilon(z) T(z) \phi(w_i, \bar{w}_i) \ldots \phi(w_n, \bar{w}_n) \right\rangle \\ &\quad = \sum_{i=1}^{n} \langle \phi(w_1, \bar{w}_1) \ldots \delta_\epsilon \phi(w_i, \bar{w}_i) \ldots \phi(w_n, \bar{w}_n) \rangle . \end{aligned} \tag{2.60}$$

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Using (2.9) we substitute now for $\delta_\epsilon \phi(w_i, \bar{w}_i) = \epsilon(w_i)\partial + h\partial\epsilon(w_i)\phi(w_i, \bar{w}_i)$. Since this holds for arbitrary ϵ we can get a local form of the Ward identity,

$$\langle T(z)\phi(w_1, \bar{w}_1) \dots \phi(w_n, \bar{w}_n)\rangle = \sum_{i=1}^{n} \left(\frac{h_i}{(z - w_i)^2} + \frac{1}{(z - w_i)} \frac{\partial}{\partial_{w_i}} \right) \langle \phi(w_1, \bar{w}_1) \dots \phi(w_n, \bar{w}_n)\rangle \,, \tag{2.61}$$

similar to the transition from (2.9) to (2.10). It is thus clear that the correlation function above is a meromorphic function of z with singularities at the positions of the operators.

A useful tool for computing correlators is the use of null vectors. Rather than discussing this for a general null vector we demonstrate this procedure on a level two null vector. Recall that in models with a primary of weight h such that $c = \frac{2h}{2h+1}(5 - 8h)$ there is a null vector at level two of the form $(L_{-2} + aL_{-1}^2)\Phi^{(h)} = 0$ where $a = -\frac{3}{2(2h+1)}$. As $L_{-1}\phi^{(h)}(z) = \partial\phi^{(h)}(z)$ one can trade $L_{-2}\phi^{(h)}(z)$ with $-a\partial^2\phi^{(h)}(z)$. Now $L_{-2}\phi^{(h)}(w)$ is given by,

$$L_{-2}\phi^{(h)}(w) = \lim_{z\to w} \left[T(z)\phi^{(h)}(w) - \frac{h\phi^{(h)}(w)}{(z - w)^2} - \frac{\partial_w\phi^{(h)}(w)}{(z - w)} \right]. \tag{2.62}$$

Substituting this into (2.61) one finds the following differential equation,

$$-a\partial_{w_1}^2 \langle \phi(w_1, \bar{w}_1) \dots \phi(w_n, \bar{w}_n)\rangle = \sum_{i\neq 1}^{n} \left(\frac{h_i}{(w_1 - w_i)^2} + \frac{1}{(w_1 - w_i)} \frac{\partial}{\partial_{w_i}} \right) \langle \phi(w_1, \bar{w}_1) \dots \phi(w_n, \bar{w}_n)\rangle \,. \tag{2.63}$$

This exact differential equation will enable us to compute the four-point function for the Ising model as we discuss in Section (2.13).

Next we would like to deduce the implications of the associativity on correlation functions of primaries and descendant operators.

2.10 Crossing symmetry, duality and bootstrap

The complete package of information that specifies a CFT is its Virasoro anomaly c, the set of primary fields $\phi_i(z, \bar{z})$, with their weights $(h_i, \bar{h}_i)$ and the operator product coefficients C_{ijk}. Hence, to determine all consistent CFTs one has to find all the allowed sets of such data. The latter have to comply with the constraints that follow from confomal symmetry as well as with the associativity of the operator algebra. To study the implications of associativity it is useful to consider the four-point function,

$$\langle \phi_i(w_1, \bar{w}_1)\phi_j(w_2, \bar{w}_2)\phi_k(w_3, \bar{w}_3)\phi_l(w_4, \bar{w}_4)\rangle \,. \tag{2.64}$$

The idea is to compare the computation of this correlator using the OPE of ϕ_i and ϕ_j and of ϕ_k and ϕ_l, with those of ϕ_i and ϕ_k and of ϕ_j and ϕ_l, namely calculation where $(z_1 \to z_2), (z_3 \to z_4)$ versus one in which $(z_1 \to z_3), (z_2 \to z_4)$.

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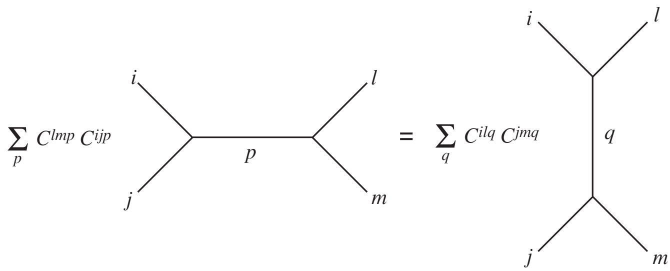

Fig. 2.3. Crossing symmetry.

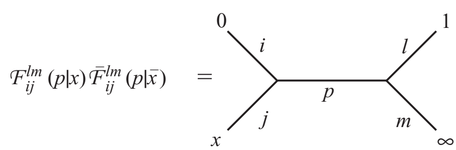

Fig. 2.4. Single channel amplitude.

The requirement that the two ways of computing coincide, referred to as *crossing symmetry*, is expressed in Fig. 2.3.[5]

Using conformal transformations, we can relate the diagram on the left-hand side of Fig. 2.3 to the diagram drawn in Fig. 2.4, which corresponds to the sum of the contributions of intermediate states belonging to the conformal family $[\phi_p]$ with the four-point function of operators located at $(w_1, w_2, w_3, w_4) = (0, z, 1, \infty)$. Note that in such a situation, z is actually also the cross ratio $\mathcal{Z}$. We denote this amplitude by the *conformal block* $\mathcal{F}^{kl}_{ij}(m|z)\bar{\mathcal{F}}^{kl}_{ij}(m|\bar{z})$ which depends on the Virasoro anomaly of the theory and the dimensions of all the operators involved. In terms of conformal blocks the crossing symmetry condition takes the form,

$$\sum_m C_{ijm} C_{klm} \mathcal{F}^{kl}_{ij}(m|z)\bar{\mathcal{F}}^{kl}_{ij}(m|\bar{z})$$
$$= \sum_n C_{ijn} C_{kln} \mathcal{F}^{jl}_{ik}(n|1-z)\bar{\mathcal{F}}^{jl}_{ik}(n|1-\bar{z}). \qquad (2.65)$$

For a given set of conformal blocks (2.65) is a set of equations that determine the C_{ijk} and the weights. The general set of solutions of these equations is not known, but for a particular class of theories like the minimal models these equations can be solved.

[5] Crossing symmetry, duality and bootstrap was discussed in [33].

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2.11 Verlinde's formula

The fusion rules (2.28), namely,

$$[\phi_i][\phi_j] = \sum_k N_{ij}^k [\phi_k],$$

constitute a commutative associative algebra. The commutativity implies that $N_{ij}^k = N_{ji}^k$ and the associativity means that,

$$\sum_k N_{ij}^k N_{kl}^m = \sum_k N_{ik}^m N_{jl}^k. \tag{2.66}$$

Using matrix notation in which $N_{ij}^k = (N_i)_j^k$ the associativity translates into the commutativity of the matrices, namely $N_i N_l = N_l N_i$. Thus, the matrices N_i are also members of an associative commutative algebra. Hence they can be diagonalized simultaneously to form a one-dimensional representation. This implies that there is a common matrix $\tilde{S}$,

$$N_{ij}^k = \sum_{lm} \tilde{S}_j^l \lambda_i^{(l)} \delta_l^m (\tilde{S}^{-1})_m^k = \sum_l \tilde{S}_j^l \lambda_i^{(l)} (\tilde{S}^{-1})_l^k, \tag{2.67}$$

where we denote the eigenvalues of N_i by $\lambda_i^{(l)}$. If j is the vacuum state $j = 0$ then $N_{i0}^k = \delta_i^k$, if all the representations labeled by i are irreducible. We now multiply from the right by $\tilde{S}_k^n$ to get,

$$\begin{aligned} N_{i0}^k \tilde{S}_k^n &= \sum_l \tilde{S}_0^l \lambda_i^{(l)} (\tilde{S}^{-1})_l^k \tilde{S}_k^n \\ \tilde{S}_i^n &= \sum_l \tilde{S}_0^l \lambda_i^{(l)} \delta_l^n = \tilde{S}_0^n \lambda_i^{(n)}, \end{aligned} \tag{2.68}$$

which means that $\lambda_i^{(n)} = \frac{\tilde{S}_i^n}{\tilde{S}_0^n}$ and therefore,

$$N_{ij}^k = \sum_l \frac{\tilde{S}_j^l \tilde{S}_i^l (\tilde{S}^{-1})_l^k}{\tilde{S}_0^l}. \tag{2.69}$$

Now, for the reader who knows about the τ parameter and the characters (discussed in Section 2.8), we recall that under the S-transformation $\tau \to -\frac{1}{\tau}$ the characters of a given CFT transform as,

$$\chi_j(-\frac{1}{\tau}) = \sum_k S_j^k \chi_k(\tau). \tag{2.70}$$

Verlinde's formula[6] states that the matrix $\tilde{S}$ above is identical to the S-transformation matrix,

$$S = \tilde{S}. \tag{2.71}$$

This is a remarkable relation.

[6] The Verlinde formula was introduced in the seminal paper [210].

2.12 Free Majorana fermions – an example of a CFT

The theory of free massless fermions in two dimensions is an example of a 2d conformal theory of the utmost importance. In this section we describe this theory in detail following the steps taken in the general analysis of conformal field theories. The well-known Dirac action of a massless free fermion in two Euclidean dimensions is,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 x \bar{\Psi}\ \partial\!\!\!/ \Psi. \tag{2.72}$$

Expressing the Dirac fermion in terms of *chiral (or Weyl)* fermions, a left ψ and a right $\tilde{\psi}$, with $\Psi \equiv (\psi, \tilde{\psi})$, and using the fact that in two dimensions one can take $\gamma^0 = \sigma^2$ and $\gamma^1 = \sigma^1$, we rewrite the action as,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z (\psi^\dagger \bar{\partial}\psi + \tilde{\psi}^\dagger \partial \tilde{\psi}). \tag{2.73}$$

We remind the reader that $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. The equations of motion are,

$$\bar{\partial}\psi = 0 \quad \partial\tilde{\psi} = 0 \ \rightarrow\ \psi = \psi(z) \quad \tilde{\psi} = \tilde{\psi}(\bar{z}). \tag{2.74}$$

In analogy to the symmetries of the scalar field it is straighforward to realize that the action is invariant under left holomorphic chiral and right anti-holomorphic transformations,

$$\psi \rightarrow \psi' = \mathrm{e}^{i\alpha(z)}\psi \quad \tilde{\psi} \rightarrow \tilde{\psi}' = \mathrm{e}^{i\tilde{\alpha}(\bar{z})}\tilde{\psi}. \tag{2.75}$$

The corresponding "affine current algebra" currents, given by,

$$J = i\psi^\dagger\psi \quad \bar{J} = i\tilde{\psi}^\dagger\tilde{\psi}, \tag{2.76}$$

are holomorphically and anti-holomorphically conserved.

In addition the theory is obviously invariant under conformal transformations $z \rightarrow f(z),\ \bar{z} \rightarrow \bar{f}(\bar{z})$.

Dirac (or Weyl) fermions can be further decomposed into Majorana (or Weyl–Majorana) fermions as $\Psi = \frac{1}{\sqrt{2}}(\Psi_1 + i\Psi_2)$ (or $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$). Substituting these, the action reads,

$$S = \frac{1}{8\pi}\int \mathrm{d}^2 z \left(\sum_{i=1}^{2} \psi_i\bar{\partial}\psi_i + \tilde{\psi}_i\partial\tilde{\psi}_i\right). \tag{2.77}$$

From this point on we discuss the theory of single Majorana fermions, namely χ and $\tilde{\chi}$ are a left and a right Weyl–Majorana fermion with the action,

$$= \frac{1}{8\pi}\int \mathrm{d}^2 z (\chi\bar{\partial}\chi + \tilde{\chi}\partial\tilde{\chi}). \tag{2.78}$$

The equations of motion are still as in (2.74) so that χ is a holomorphic function and $\tilde{\chi}$ is an anti-holomorphic one (their extensions, as they are real on the real line).

Before spelling out the conformal structure of the theory we pause for a moment with the complex coordinate formulation and discuss canonical quantization in two dimensions with a Minkowski signature. The conjugate momentum to χ is $\pi_\chi = \frac{\partial \mathcal{L}}{\partial_0 \chi} = \frac{1}{2}\chi$, and as we are dealing with a real field $\{\pi_\chi, \chi\}$ has a factor $\frac{1}{2}$ multiplying a delta function,which gives,

$$\{\chi(x_1, x_0), \chi(y_0, y_1)\}|_{x_0=y_0} = \delta(x_1 - y_1). \tag{2.79}$$

Combining a pair of two Majorana fermions (each consisting of two Weyl–Majorana) into a Dirac fermion, one finds for the latter the usual anti-commutation relations, namely,

$$\{\Psi^\dagger(x_1, x_0), \Psi(y_1, y_0)\}|_{x_0=y_0} = \delta(x_1 - y_1) \quad \{\Psi(x_1, x_0), \Psi(y_1, y_0)\}|_{x_0=y_0} = 0. \tag{2.80}$$

The Noether currents associated with conformal transformations, namely the components of the energy-momentum tensor, are given by,

$$T(z) = -\frac{1}{2} :\chi\partial\chi: \quad \bar{T}(\bar{z}) = -\frac{1}{2} :\tilde{\chi}\bar{\partial}\tilde{\chi}:, \tag{2.81}$$

where $:\chi\chi:$ the normal ordered product, stands for the product with the subtraction of its OPE. The latter is given by,

$$\chi(z)\chi(w) = \frac{1}{z-w} \quad \tilde{\chi}(z)\tilde{\chi}(w) = \frac{1}{\bar{z}-\bar{w}}. \tag{2.82}$$

Using this basic OPE in $T(z)\chi(w)$ one finds,

$$T(z)\chi(w) = \frac{1}{2}\frac{\chi(w)}{(z-w)^2} + \frac{\partial\chi(w)}{z-w}, \tag{2.83}$$

which implies that χ is a *primary* field of conformal dimensions of $(\frac{1}{2}, 0)$, and similarly $\tilde{\chi}$ with $(0, \frac{1}{2})$. The Virasoro anomaly, which comes as usual from $T(z)T(w)$, is $c = \frac{1}{2}$, and $\bar{c} = \frac{1}{2}$ from $\bar{T}(z)\bar{T}(w)$.

Recall that the energy-momentum tensor of the scalar field (1.27) takes the form of a bilinear of the "current algebra" currents (1.24). We want to examine now if such a construction can be applied also for the fermionic fields. Since for Weyl–Majorana fermions there are no such currents it is left only to check for the $T(z)$ of Weyl fermions. Let us note first the OPE of the currents and the Weyl fermions that read,

$$J(z)\psi(w) = -i\frac{\psi(z)}{(z-w)} \quad J(z)\psi^\dagger(w) = i\frac{\psi^\dagger(z)}{(z-w)}, \tag{2.84}$$

where $J = i:\psi^\dagger\psi:$ with our conventions.

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Using this OPE one finds,

$$\begin{aligned}
T(z) &= -\frac{1}{2} : J(z)J(z) := -\frac{1}{2}\lim_{z\to w}\left[J(z)J(w) + \frac{1}{(z-w)^2}\right] \\
&= -\frac{1}{2}\lim_{z\to w}\lim_{\epsilon\to 0}\left[iJ(z)\left(\psi^\dagger(w-\epsilon)\psi(w+\epsilon) + \frac{1}{2\epsilon}\right) + \frac{1}{(z-w)^2}\right] \\
&= -\frac{1}{2}\lim_{z\to w}\lim_{\epsilon\to 0}\left[-\frac{\psi^\dagger(z)\psi(w+\epsilon)}{[z-(w-\epsilon)]} + \frac{\psi^\dagger(w-\epsilon)\psi(z)}{[z-(w+\epsilon)]} + \frac{1}{(z-w)^2}\right] \\
&= -\frac{1}{2}\left[\psi^\dagger\partial\psi - \partial\psi^\dagger\psi\right]. \qquad (2.85)
\end{aligned}$$

This construction of the energy-momentum tensor in terms of a normal ordered product of two currents, which is known as the *Sugawara construction*, will play a key role in the discussion in Chapter 4.

The mode expansion of the Weyl–Majorana fermion takes the form,

$$\psi = \sum_{r\in\mathcal{Z}+\nu}\frac{\psi_r}{z^{r+\frac{1}{2}}} \qquad \psi_r = \frac{1}{2\pi i}\oint \mathrm{d}z z^{r-\frac{1}{2}}\psi(z), \qquad (2.86)$$

with $z = \mathrm{e}^{-iw}$, ν is related to the boundary conditions as

$$\psi(w+2\pi) = \mathrm{e}^{2\pi i\nu}\psi(w), \qquad (2.87)$$

so that there are two types of fermions:

Ramond fermions $\nu = \frac{1}{2}$ $\quad\leftrightarrow$ periodic boundary condition
Neveu–Schwarz fermions $\nu = 0$ $\quad\leftrightarrow$ anti-periodic boundary condition.

The anti-commutation relations of the fermionic modes follow straightforwardly upon using (2.86) and the OPEs (2.82), namely,

$$\{\psi_r, \psi_s\} = \delta_{r+s}. \qquad (2.88)$$

The form (2.82) holds for the periodic case. For the anti-periodic case there is an extra factor of $\frac{1}{2}(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}})$, which tends to 1 as $z \to w$.

The canonical quantization conditions in terms of real space-time coordinates take the following form,

$$\{\psi(x^1), \psi(y^1)\}|_{x^0=y^0} = \frac{1}{2}\delta(x^1 - y^1), \qquad (2.89)$$

since ψ is the conjugate momentum of itself. Combining two Majorana fermions into a Dirac one yields the following anti-commutation relations for the Dirac fermions,

$$\{\Psi^\dagger(x^1), \Psi(y^1)\}|_{x^0=y^0} = \delta(x^1 - y^1) \quad \{\Psi(x^1), \Psi(y^1)\}|_{x^0=y^0} = 0, \qquad (2.90)$$

so that now $\Psi^\dagger$ is the conjugate momentum of Ψ.

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2.13 The Ising model – the $m = 3$ unitary minimal model

The first unitary minimal model with $m = 3$ has $c = 1/2$, just like the Majorana fermion discussed above. We now analyze the primaries of this model, their fusion rules and their correlators. Comparing the latter with correlation functions of the Ising model,[7] we show that in fact the $m = 3$ unitary minimal model is the continuum limit of the Ising model. Recall that the set of primaries of the $m = 3$ model are characterized by the following conformal weights:

$$h_{1,1} = 0 \quad h_{2,1} = \frac{1}{2} \quad h_{2,2} = \frac{1}{16}, \tag{2.91}$$

which determine the two-point functions,

$$\begin{aligned} \left\langle \Phi^{(1/2)}(z,\bar{z})\Phi^{(1/2)}(w,\bar{w}) \right\rangle &= \frac{1}{|z-w|^2} \\ \left\langle \Phi^{(1/16)}(z,\bar{z})\Phi^{(1/16)}(w,\bar{w}) \right\rangle &= \frac{1}{|z-w|^{1/4}}, \end{aligned} \tag{2.92}$$

where $\Phi^{(h)}(z,\bar{z}) = \phi^{(h)}(z)\bar{\phi}^{(h)}(\bar{z})$. It turns out that at the critical point of the Ising model, the two-point function of the spin operator σ at a lattice point n and at the origin behaves like $< \sigma_n \sigma_0 > \sim \frac{1}{|n|^{1/4}}$. Thus in the continuum, it has the same "critical exponent" as that of $\Phi^{(1/16)}$, and similarly the Greens function of the energy density falls like $< \epsilon_n \epsilon_0 > \sim \frac{1}{|n|^2}$, namely like the two-point function of $\Phi^{(1/2)}$.

There are additional properties of the $m = 3$ unitary model that can be shown to match those of the Ising model. Here we demonstrate this with the determination of the four-point function of $\Phi^{(1/16)}$, namely,

$$\left\langle \Phi^{(1/16)}(z_1,\bar{z}_1)\dots\Phi^{(1/16)}(z_4,\bar{z}_4) \right\rangle. \tag{2.93}$$

From equation (2.63) we have that

$$\left[\frac{4}{3}\partial_{w_1}^2 - \sum_{i\neq 1}^{4}\left(\frac{1/16}{(w_1-w_i)^2} + \frac{1}{(w_1-w_i)}\frac{\partial}{\partial_{w_i}}\right)\right]\langle \phi(w_1,\bar{w}_1)\dots\phi(w_4,\bar{w}_4)\rangle = 0, \tag{2.94}$$

where ϕ denotes $\Phi^{(1/16)}$.

Using the global Ward identities we express G_4 as in (2.59),

$$G_4(z_1,\bar{z}_1\dots z_4,\bar{z}_4) = \tilde{f}(\mathcal{Z},\bar{\mathcal{Z}})\left[(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{-1/24}(C.C.)\right], \tag{2.95}$$

where $\mathcal{Z} = \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Using also $\frac{z_{14}z_{23}}{z_{13}z_{24}} = 1 - \mathcal{Z}$, we can rewrite as,

$$G_4(z_1,\bar{z}_1\dots z_4,\bar{z}_4) = \tilde{f}(\mathcal{Z},\bar{\mathcal{Z}})\left[(z_{13}z_{24})^{-1/8}\left[\mathcal{Z}(1-\mathcal{Z})\right]^{-1/24}(C.C.)\right]. \tag{2.96}$$

[7] The two-dimensional Ising model has a long history. It was discussed in [137]. The relation to Majorana fermions was discussed in [187].

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Anticipating the result of the Ising model, we actually write,

$$G_4(z_1, \bar{z}_1 \dots z_4, \bar{z}_4) = f(\mathcal{Z}, \bar{\mathcal{Z}}). \left[[z_{13} z_{24} \mathcal{Z} \, (1 - \mathcal{Z})]^{-1/8} \, (C.C.) \right]. \tag{2.97}$$

If we now substitute this ansatz into (2.94) we find the following differential equation for f,

$$\left[\mathcal{Z}(1 - \mathcal{Z})\partial^2 + (1/2 - \mathcal{Z})\partial + 1/16 \right] f(\mathcal{Z}, \bar{\mathcal{Z}}) = 0. \tag{2.98}$$

A similar equation holds for $\bar{\mathcal{Z}}$. The solutions of this differential equation are $f_{1,2}(\mathcal{Z}) = \left(1 \pm \sqrt{1 - \mathcal{Z}}\right)^{1/2}$ and so finally the four-point function takes the form,

$$G_4(z_1, \bar{z}_1 \dots z_4, \bar{z}_4) = \left| \left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{34}} \right) \right|^{1/4} \left(|f_1(\mathcal{Z})|^2 + |f_2(\mathcal{Z})|^2 \right), \tag{2.99}$$

where the unique combination is dictated by the requirement for a single value. This is identical to the result found in the Ising model for G_4.

Note also that $f(1 - \mathcal{Z}, 1 - \bar{\mathcal{Z}})$ is a solution, a result of the symmetry under the interchange of z_1 with z_3.

Note also that although $\Phi^{(1/2)}$ is a free fermion, $\Phi^{(1/16)}$ cannot be constructed in a local way from the fermion.

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