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W-GRAPHS AND GYOJA'S W-GRAPH ALGEBRA

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Abstract. Let (W, S) be a finite Coxeter group. Kazhdan and Lusztig introduced the concept of W-graphs, and Gyoja proved that every irreducible representation of the Iwahori–Hecke algebra $H(W, S)$ can be realized as a W-graph. Gyoja defined an auxiliary algebra for this purpose which—to the best of the author's knowledge—was never explicitly mentioned again in the literature after Gyoja's proof (although the underlying ideas were reused). The purpose of this paper is to resurrect this W-graph algebra, and to study its structure and its modules. A new explicit description of it as a quotient of a certain path algebra is given. A general conjecture is proposed which would imply strong restrictions on the structure of W-graphs. This conjecture is then proven for Coxeter groups of type $I_2(m)$, B_3 and A_1–A_4.

§1. Introduction

Let (W, S) be a finite Coxeter group. Kazhdan and Lusztig introduced W-graphs in [7] in an attempt to capture certain combinatorial features of Kazhdan–Lusztig-cells and of the cell representations associated to them. By definition, every cell representation is a W-graph representation. The converse is not true.

Gyoja proved that every irreducible representation (and hence every reducible representation as well) of the Hecke algebra $H(W, S)$ can be realized as a W-graph representation if W is finite (see [4, 2.3.(1)]). In that proof, the Iwahori–Hecke algebra is embedded into a larger algebra, which I denote Ω in this paper, and it is proven that there exists a left inverse of this embedding. The W-graph algebra Ω is constructed in such a way that its modules correspond to W-graphs (up to choice of an appropriate basis). Using any one of these left inverses, every H-module can be considered as an Ω-module, and the result follows.

Gyoja's proof is nonconstructive, as it does not provide a concrete left inverse of the embedding $H \hookrightarrow \Omega$ and does not offer additional information about the W-graphs that were constructed in this fashion or any information

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about general W-graphs. In my thesis [5], I discovered that a careful analysis of Ω reveals a fine structure that gives much more detailed information about W-graphs. An explicit left inverse utilizing Lusztig's asymptotic algebra is also provided in [5, Satz 4.3.2].

The starting point for this analysis is the observation that Ω is a quotient of a path algebra over a quiver which is describable entirely in terms of the Dynkin diagram, a fact that is implicitly contained in Gyoja's paper but was not interpreted in that way. Gyoja's definition [4, 2.5] gives elements of Ω that basically realize the vertex idempotents and the edge elements of a path algebra. (This is made precise in Lemma 7.) The first main result of my paper is to give an explicit set of relations for this quotient (Theorem 13). The relations are inspired by the work of Stembridge [8], where similar equations appear for the edge weights of so-called admissible W-graphs, although they were neither formulated for general W-graphs nor interpreted as relations for an underlying algebra. This set of relations seems to be different from the presentation Gyoja gives in the appendix of his paper.

Once this new presentation of Ω is established, it is applied to breaking down the structure of Ω further. At the moment, this is only done for some small Coxeter groups by a case-by-case analysis, but the proofs are so similar in spirit that I proposed a general conjecture in my thesis whose essence is that Ω should also be a quotient of a generalized path algebra over a different quiver which should have $\mathrm{Irr}(W)$ as its vertex set and should be acyclic. The algebras associated to the vertices should be matrix algebras.

In the cases for which the conjecture is true, it has several important consequences like the following.

- $k\Omega$ is finitely generated as a k-module, where k is a so-called good ring for (W, S); that is, a ring $k \subseteq \mathbb{C}$ with $2\cos(2\pi/m_{st}) \in k$ for all $s, t \in S$ and $p \in k^\times$ for all bad primes p. (See [3, Table 1.4] for a detailed description of what that means for each type of finite Coxeter group.)
- The Jacobson radical $\mathrm{rad}(k\Omega)$ is finitely generated by an explicitly describable finite list of elements and $k\Omega/\mathrm{rad}(k\Omega) \cong \prod_{\lambda\in\mathrm{Irr}(W)} k^{d_\lambda \times d_\lambda}$, where d_λ denotes the degree of the irreducible character λ. This implies that Gyoja's conjecture (cf. [4, 2.18]) holds.
- There is an enumeration $\lambda_1, \ldots, \lambda_n$ of $\mathrm{Irr}(W)$ such that every $k\Omega$-module V has a natural filtration

$$0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n = V,$$

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which realizes the decomposition of V into irreducibles in the sense that V^i/V^{i-1} is isomorphic to a direct sum of irreducibles of isomorphism class λ_i.

Because of the last consequence I named the conjecture the "W-graph decomposition conjecture". The second consequence, and in particular the connection to Gyoja's conjecture, was my original motivation for investigating the W-graph algebra and its fine structure. At the time of writing, the decomposition conjecture has been proven for Coxeter groups of types A_1–A_4, $I_2(m)$ and B_3.

The paper is organized as follows. The first section introduces some notation, recalls the definition of W-graphs (following [3], which is slightly more general than Kazhdan and Lusztig's), the definition of the W-graph algebra (following [4] though with a different notation) and proves some basic lemmas establishing the connection between W-graphs and Ω-modules. Section 3 is devoted to stating and proving an explicit description of Ω in terms of generators and relations which are the basis for all subsequent proofs. Section 4 contains the statement of the decomposition conjecture and a short discussion of its consequences, while Section 5 is devoted to the proofs of the conjecture for small Coxeter groups.

§2. Preliminaries

2.1 Notation

Throughout the paper, fix a finite Coxeter system (W, S). The Iwahori–Hecke algebra $H = H(W, S)$ of (W, S) is the $\mathbb{Z}[v^{\pm 1}]$-algebra (where v is an indeterminate), which is freely generated by $(T_s)_{s\in S}$ subject only to the relations

$$\forall s \in S : T_s^2 = 1 + (v - v^{-1})T_s \qquad \text{and}$$
$$\forall s, t \in S : \Delta_{m_{st}}(T_s, T_t) = 0,$$

where m_{st} denotes the order of $st \in W$ and $\Delta_m(x, y)$ is the mth *braid commutator* of ring elements x and y, which is defined as follows:

$$\Delta_m(x, y) := \underbrace{xyx\ldots}_{m \text{ factors}} - \underbrace{yxy\ldots}_{m \text{ factors}}.$$

In particular, $\Delta_0(x, y) = 0$, $\Delta_1(x, y) = x - y$, $\Delta_2(x, y) = xy - yx$, $\Delta_3(x, y) = xyx - yxy$, and so on.

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Also fix a good ring for (W, S); that is, a ring $k \subseteq \mathbb{C}$ with $2\cos(2\pi/m_{st}) \in k$ for all $s, t \in S$ and $p \in k^\times$ for all so-called bad primes p. (See [3, Table 1.4] for a detailed description of what that means for each type of finite Coxeter group.)

A ring is good if it is big enough for the purposes of representation theory of Coxeter groups. For example, every good field is a splitting field for W.

If A is a k-algebra and k' is a commutative k-algebra, then $k'A$ is used as shorthand for the k'-algebra $k' \otimes_k A$. Similarly, the abbreviation $k'V$ is used for the $k'A$-module $k' \otimes_k V$ if V is an A-module.

2.2 W-graphs

Definition 1. (Cf. [7] and [3]) A W-graph with edge weights in k is a triple $(\mathfrak{C}, \mathcal{I}, m)$ consisting of a finite set $\mathfrak{C}$ of *vertices*, a *vertex labeling* map $\mathcal{I} : \mathfrak{C} \to \{I \mid I \subseteq S\}$ and a family of *edge weight* matrices $m^s \in k^{\mathfrak{C}\times\mathfrak{C}}$ for $s \in S$ (here, $k^{\mathfrak{C}\times\mathfrak{C}}$ denotes the ring of matrices whose rows and columns are indexed with $\mathfrak{C}$ and whose entries are elements of k) such that the following conditions hold.

(1) $\forall x, y \in \mathfrak{C} : m^s_{xy} \neq 0 \implies s \in \mathcal{I}(x) \setminus \mathcal{I}(y)$.
(2) The matrices

$$\omega(T_s)_{xy} := \begin{cases} -v^{-1} \cdot 1_k & \text{if } x = y, s \in \mathcal{I}(x), \\ v \cdot 1_k & \text{if } x = y, s \notin \mathcal{I}(x), \\ m^s_{xy} & \text{otherwise} \end{cases}$$

induce a matrix representation $\omega : k[v^{\pm 1}]H \to k[v^{\pm 1}]^{\mathfrak{C}\times\mathfrak{C}}$.

The associated directed graph is defined as follows. The vertex set is $\mathfrak{C}$ and there is a directed edge $x \leftarrow y$ if and only if $m^s_{xy} \neq 0$ for some $s \in S$. If this is the case, then the value m^s_{xy} is called the *weight* of the edge. The set $I(x)$ is called the *vertex label* of x.

Note that condition 1 and the definition of $\omega(T_s)$ already guarantee $\omega(T_s)^2 = 1 + (v - v^{-1})\omega(T_s)$, so that the only nontrivial requirement in condition 2 is the braid relation $0 = \Delta_{m_{st}}(\omega(T_s), \omega(T_t))$.

The definition seems to allow up to $|I(x) \setminus I(y)|$ different edge weights for a single edge $x \leftarrow y$. We prove later that all values m^s_{xy} with $s \in I(x) \setminus I(y)$ are in fact equal.

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Given a W-graph as above, the matrix representation ω turns the space $k[v^{\pm 1}]^{\mathfrak{C}}$ of column vectors indexed with $\mathfrak{C}$ with entries in $k[v^{\pm 1}]$ into a left module for the Hecke algebra $k[v^{\pm 1}]H$. It is natural to ask whether the converse is true. In situations where the Hecke algebra is split semisimple, the answer is yes, as shown by Gyoja.

THEOREM 2. (Cf. [4]) *Let $K \subseteq \mathbb{C}$ be a splitting field for W. Every irreducible representation of $K(v)H$ can be realized as a W-graph module for some W-graph with edge weights in K.*

2.3 Gyoja's W-graph algebra

DEFINITION 3. Define Ξ as the $\mathbb{Z}$-algebra that is freely generated by e_s, x_s for $s \in S$ with respect to the following relations:

(1) $\forall s \in S : e_s^2 = e_s$;
(2) $\forall s, t \in S : e_s e_t = e_t e_s$;
(3) $\forall s \in S : e_s x_s = x_s,\ x_s e_s = 0$.

Furthermore, define

$$\iota(T_s) := -v^{-1} e_s + v(1 - e_s) + x_s \in \mathbb{Z}[v^{\pm 1}]\Xi$$

for all $s \in S$. The braid commutator $\Delta_{m_{st}}(\iota(T_s), \iota(T_t))$ can be written as $\sum_{\gamma \in \mathbb{Z}} y^\gamma(s,t) v^\gamma$ with uniquely determined elements $y^\gamma(s,t) \in \Xi$.

The *W-graph algebra* Ω is defined as the $\mathbb{Z}$-algebra obtained as the quotient of Ξ modulo the relations $y^\gamma(s,t) = 0$ for all $s, t \in S$ and all $\gamma \in \mathbb{Z}$.

By abuse of notation, the quotient map $\Xi \to \Omega$ is not explicitly mentioned for the remainder of this paper, and symbols like e_s, x_s and $\iota(T_s)$ are therefore used for elements of Ξ as well as the corresponding elements of Ω.

The definition, and in particular the observation $x_s^2 = (e_s x_s)(e_s x_s) = 0$, immediately implies that $T_s \mapsto \iota(T_s)$ defines a homomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras $\iota : H \to \mathbb{Z}[v^{\pm 1}]\Omega$ (which is in fact injective, as we prove in Corollary 10). This observation also appears in Gyoja's paper [4, Remark 2.4.3].

2.4 Morphisms

Giving an algebra by generators and relations means having a universal property for homomorphisms on the resulting algebra. Since the relations for Ω are not explicit enough to be verifiable by explicit calculations, we use the following universal property instead.

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LEMMA 4. *Consider the category of all rings. Then, precomposing with the quotient* $\Xi \to \Omega$ *is a natural isomorphism*

$$\operatorname{Hom}(\Omega, A) \cong \left\{ f : \Xi \to A \;\middle|\; \begin{array}{l} \text{the induced map } \mathbb{Z}[v^{\pm 1}]\Xi \to \mathbb{Z}[v^{\pm 1}]A \\ \text{annihilates } \Delta_{m_{st}}(\iota(T_s), \iota(T_t)) \text{ for all } s, t \in S \end{array} \right\}.$$

Proof. Precomposing with the quotient map certainly is an injective natural transformation $\operatorname{Hom}(\Omega, -) \to \operatorname{Hom}(\Xi, -)$. We prove that its image is exactly the subset of the claim.

Choose $s, t \in S$ and write $\Delta_{m_{st}}(\iota(T_s), \iota(T_t)) = \sum_{\gamma \in \mathbb{Z}} y^\gamma(s,t) v^\gamma$ as before. Thus, for any homomorphism $f : \Xi \to A$, the induced map $\mathbb{Z}[v^{\pm 1}]\Xi \to \mathbb{Z}[v^{\pm 1}]A$ satisfies

$$f(\Delta_{m_{st}}(\iota(T_s), \iota(T_t))) = \sum_{\gamma \in \mathbb{Z}} f(y^\gamma(s,t)) v^\gamma.$$

Because an element $\sum_\gamma a_\gamma v^\gamma \in \mathbb{Z}[v^{\pm 1}]A$ with $a_\gamma \in A$ is zero if and only if $a_\gamma = 0$ for all $\gamma \in \mathbb{Z}$, the map f descends to a well-defined homomorphism $\Omega \to A$ if and only if f annihilates all $y^\gamma(s,t)$ if and only if the induced map annihilates all braid commutators $\Delta_{m_{st}}(\iota(T_s), \iota(T_t))$. $\square$

The following easy corollary establishes symmetries of Ω which are used to simplify the proofs of the decomposition conjecture in the last section of the paper.

COROLLARY 5.

(1) *If* $\alpha : S \to S$ *is a bijection with* $\operatorname{ord}(\alpha(s)\alpha(t)) = \operatorname{ord}(st)$ *(in other words, a graph automorphism of the Dynkin diagram of* (W, S)*), then there is a unique automorphism of* Ω *with* $e_s \mapsto e_{\alpha(s)}$, $x_s \mapsto x_{\alpha(s)}$.
(2) *There is a unique antiautomorphism* δ *of* Ω *with* $e_s \mapsto 1 - e_s$, $x_s \mapsto -x_s$.

2.5 Modules and W-graphs

The following definition appears in Gyoja's paper [4, Definition 2.5], although with different notation.

DEFINITION 6. In Ξ, define the following elements for all $I, J \subseteq S$, $s \in S$:

$$E_I := \left(\prod_{t \in I} e_t\right)\left(\prod_{t \in S \setminus I} (1 - e_t)\right)$$
$$X^s_{IJ} := E_I x_s E_J.$$

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What Gyoja did not mention in his paper is that these elements actually give Ω the structure of a quotient of a path algebra. This is the content of the following lemma.

LEMMA 7. *With the above notation, the following statements are true.*

(1) $E_I E_J = \delta_{IJ} E_I$, $\sum_{I \subseteq S} E_I = 1$ *and* $e_s = \sum_{\substack{I \subseteq S \\ s \in I}} E_I$.

(2) $X^s_{IJ} = 0$ *if* $s \notin I \setminus J$ *and* $x_s = \sum_{\substack{I,J \subseteq S \\ s \in I \setminus J}} X^s_{IJ}$.

(3) Ξ *is isomorphic to the path algebra* $\mathbb{Z}\mathcal{Q}$ *over the quiver* $\mathcal{Q}$ *whose vertex set is the power set of* S *and which has exactly* $|I \setminus J|$ *edges* $I \leftarrow J$ *for every pair of vertices* $I, J \subseteq S$.

Proof. The first equation follows immediately from the definition, $e_s(1 - e_s) = (1 - e_s)e_s = 0$ and the fact that the e_s commute with each other. The decomposition of the identity follows by expanding $1 = \prod_{s \in S}(e_s + (1 - e_s))$, and the expression for e_s follows by applying the decomposition of the identity in $e_s \cdot 1$.

The expression for x_s follows by applying the decomposition of the identity twice in $1 \cdot x_s \cdot 1$.

The path algebra $\mathbb{Z}\mathcal{Q}$ can be described as the algebra freely generated by $\{\tilde{E}_K, \tilde{X}^s_{IJ} | K, I, J \subseteq S, s \in I \setminus J\}$ with respect to the relations

$$\tilde{E}_I \tilde{E}_J = \delta_{IJ} \tilde{E}_I, \qquad \sum_{I \subseteq S} \tilde{E}_I = 1 \qquad \text{and} \qquad \tilde{X}^s_{IJ} = \tilde{E}_I \tilde{X}^s_{IJ} \tilde{E}_J.$$

This implies that $\tilde{E}_I \mapsto E_I$, $\tilde{X}^s_{IJ} \mapsto X^s_{IJ}$ induces a ring homomorphism $\mathbb{Z}\mathcal{Q} \to \Xi$. Going in the other direction, one readily verifies that the unique ring homomorphism $\Xi \to \mathbb{Z}\mathcal{Q}$ with $e_s \mapsto \sum_{\substack{I \subseteq S \\ s \in I}} \tilde{E}_I$ and $x_s \mapsto \sum_{\substack{I,J \subseteq S \\ s \in I \setminus J}} \tilde{X}^s_{IJ}$ is inverse to the first morphism. ∎

REMARK 8. For later use, we observe the following.

(1) The algebra automorphism induced by a graph automorphism α maps $E_I \mapsto E_{\alpha(I)}$ and $X^s_{IJ} \mapsto X^{\alpha(s)}_{\alpha(I)\alpha(J)}$.

(2) The antiautomorphism δ maps $E_I \mapsto E_{I^c}$ and $X^s_{IJ} \mapsto -X^s_{J^c I^c}$, where I^c denotes the complement of I in S.

The following theorem also appears in Gyoja's paper as a remark without proof and establishes the connection between Ω and W-graphs.

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THEOREM 9. (Cf. [4, Remark 2.7]) *Let k be a commutative ring. There is a correspondence between Ω-modules and W-graphs by the choice of a suitable basis. More precisely, the following statements hold.*

(1) *(From W-graphs to Ω-modules)*
Let $(\mathfrak{C}, \mathcal{I}, m)$ be a W-graph with edge weights in k. Define $\omega : k\Omega \to k^{\mathfrak{C}\times\mathfrak{C}}$ by

$$\omega(e_s)_{xy} := \begin{cases} 1 & x = y, s \in \mathcal{I}(x) \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \omega(x_s) := m^s.$$

Then, ω is a well-defined k-algebra homomorphism such that the composition

$$k[v^{\pm 1}]H \xrightarrow{\iota} k[v^{\pm 1}]\Omega \xrightarrow{\omega} k[v^{\pm 1}]^{\mathfrak{C}\times\mathfrak{C}}$$

is exactly the matrix representation of H attached to $(\mathfrak{C}, \mathcal{I}, m)$.

(2) *(From Ω-modules to W-graphs)*
Let V be a $k\Omega$-module with representation $\omega : k\Omega \to \mathrm{End}_k(V)$. Define $V_I := E_I V$ for all $I \subseteq S$.
If V_I is a finitely generated free k-module and $\mathfrak{C}_I \subseteq V_I$ is a k-basis for all $I \subseteq S$, define $(\mathfrak{C}, \mathcal{I}, m)$ as follows: set $\mathfrak{C} := \bigcup_{I\subseteq S} \mathfrak{C}_I$, set $\mathcal{I}(x) := I$ for all $x \in \mathfrak{C}_I$ and define m^s to be the matrix of $\omega(x_s)$ with respect to the basis $\mathfrak{C}$. With these definitions, $(\mathfrak{C}, \mathcal{I}, m)$ is a W-graph and its W-graph module is $k[v^{\pm 1}] \otimes_k V$.

Proof. (1) The matrices $\omega(e_s)$ and $\omega(x_s)$ satisfy the relations of Ξ by definition of W-graphs. We therefore view ω as an algebra homomorphism $\Xi \to k^{\mathfrak{C}\times\mathfrak{C}}$. Because $\omega(\iota(T_s))$ is exactly equal to the matrices $\omega(T_s)$ in the definition of W-graphs, and those matrices satisfy the braid relations, it follows that ω descends to a homomorphism $\Omega \to k^{\mathfrak{C}\times\mathfrak{C}}$ by the universal property.

(2) The second assertion is easily verified. The condition $m^s_{xy} \neq 0 \implies s \in \mathcal{I}(x) \setminus \mathcal{I}(y)$ follows from $X^s_{IJ} \neq 0 \implies s \in I \setminus J$. The matrices occurring in the definition of W-graphs are exactly the matrices $\omega(\iota(T_s))$, and hence satisfy the necessary braid relations because the elements $\iota(T_s) \in \Omega$ satisfy them. ∎

COROLLARY 10. *If W is finite, then the following hold.*

(1) *$\iota : k[v^{\pm 1}]H \to k[v^{\pm 1}]\Omega$ is injective.*
(2) *All E_I are nonzero as elements of $k\Omega$.*

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In particular, H is considered as a subalgebra of the scalar extension $\mathbb{Z}[v^{\pm 1}]\Omega$ for the rest of this paper.

Proof. Consider the Kazhdan–Lusztig-W-graph as defined in [7]. It is a W-graph $(\mathfrak{C}, \mathcal{I}, m)$ with $\mathfrak{C} := W$, $\mathcal{I}(w) := \{s \in S \mid sw < w\}$ and integer edge weights such that the associated W-graph module is the regular H-module. This can be considered as a W-graph with edge weights in k.

The representation $k[v^{\pm 1}]H \xrightarrow{\iota} k[v^{\pm 1}]\Omega \to k[v^{\pm 1}]^{W\times W}$ induced by this W-graph equals the map $k[v^{\pm 1}]H \to \mathrm{End}_{k[v^{\pm 1}]}(k[v^{\pm 1}]H)$, $h \mapsto (x \mapsto hx)$. The latter map is injective, so that $\iota : k[v^{\pm 1}]H \to k[v^{\pm 1}]\Omega$ is injective too.

If W is finite, then all the elements $E_I \in k\Omega$ are nonzero because there are $w \in \mathfrak{C}$ with $\mathcal{I}(w) = I$ (for example, the longest elements of the corresponding parabolic subgroup W_I). $\square$

Remark 11. The finiteness condition is in fact superfluous. A more carefully phrased version of the definition of W-graphs and of Theorem 9 which also includes the infinite-dimensional case makes the same proof work for the first statement. The second statement, however, cannot be proved in the same way because there is an element $w \in W$ with $\mathcal{I}(w) = I$ if and only if W_I is finite, so that this proof does not work for infinite Coxeter groups (contrary to what I believed when I wrote my thesis, which contains the special proof for the general statement). An alternative general proof of the second statement will be contained in my next paper [6].

§3. Ω as a quotient of a path algebra

It is observed in Lemma 7 that Ξ is a path algebra. In this section, we give an explicit set of relations for the quotient $\Xi \to \Omega$ in terms of this path algebra structure. The proof is inspired by equations appearing in Stembridge's paper [8].

We need the following lemma, which is a slight generalization of [8, Proposition 3.1].

Lemma 12. *Define polynomials $\tau_r \in \mathbb{Z}[T]$ by the following recursion:*

$$\tau_{-1} := 0, \qquad \tau_0 := 1, \qquad \tau_r := T\tau_{r-1} - \tau_{r-2}.$$

With this notation the following holds.

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If R is any ring, and $x, y \in R$ are solutions of the equation $T^2 = 1 + \zeta T$ for some fixed $\zeta \in R$, then their braid commutators satisfy

$$\Delta_{r+1}(x, y) = (-1)^r \tau_r(x + y - \zeta) \cdot (x - y).$$

Observe that τ_r is a monic polynomial of degree r for all $r \in \mathbb{N}$. In particular, $\{\tau_0, \ldots, \tau_r\}$ is a $\mathbb{Z}$-basis of $\{f \in \mathbb{Z}[T] \mid \deg(f) \leqslant r\}$. Furthermore, τ_r is an even polynomial for even r and an odd polynomial for odd r; that is, $\tau_r(-T) = (-1)^r \tau_r(T)$. This follows immediately from the recursion.

Proof. The claim for the braid commutator is true for $r = -1$ and $r = 0$. Furthermore, the following holds:

$$\begin{aligned}
(x+y)\Delta_{r+1}(x, y) &= x^2 \underbrace{yx\ldots}_{r} - \underbrace{xyx\ldots}_{r+2} + \underbrace{yxy\ldots}_{r+2} - y^2 \underbrace{xy\ldots}_{r} \\
&= (1+\zeta x) \underbrace{yx\ldots}_{r} - \underbrace{xyx\ldots}_{r+2} + \underbrace{yxy\ldots}_{r+2} - (1+\zeta y) \underbrace{xy\ldots}_{r} \\
&= 1 \cdot \underbrace{yx\ldots}_{r} - 1 \cdot \underbrace{xy\ldots}_{r} + \zeta x \underbrace{yx\ldots}_{r} - \zeta y \underbrace{xy\ldots}_{r} \\
&\quad\; - \Big(\underbrace{xyx\ldots}_{r+2} - \underbrace{yxy\ldots}_{r+2} \Big) \\
&= -\Delta_r(x, y) + \zeta \Delta_{r+1}(x, y) - \Delta_{r+2}(x, y) \\
\Longrightarrow \Delta_{r+2}(x, y) &= (-1)\big((x + y - \zeta)\Delta_{r+1}(x, y) + \Delta_r(x, y)\big).
\end{aligned}$$

The claim follows by induction. $\square$

THEOREM 13. *For all $I, J \subseteq S$, $s, t \in S$ and $r \in \mathbb{N}$, define*

$$\begin{aligned}
P^r_{IJ}(s, t) &:= E_I \underbrace{x_s x_t x_s \ldots}_{r \text{ factors}} E_J \\
&= \begin{cases}
0, & r = 0, I \neq J, \\
E_I, & r = 0, I = J, \\
\displaystyle\sum_{I_1, \ldots, I_{r-1} \subseteq S} X^s_{II_1} X^t_{I_1 I_2} X^s_{I_2 I_3} \ldots X^s_{I_{r-1} J}, & r > 0, 2 \nmid r, \\
\displaystyle\sum_{I_1, \ldots, I_{r-1} \subseteq S} X^s_{II_1} X^t_{I_1 I_2} X^s_{I_2 I_3} \ldots X^t_{I_{r-1} J}, & r > 0, 2 \mid r.
\end{cases}
\end{aligned}$$

With this notation, the kernel of the quotient $\Xi \to \Omega$ is generated by the following elements.

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(α) *For all $s, t \in S$, the elements*

$$P_{IJ}^{m-1}(s,t) + a_{m-2}P_{IJ}^{m-2}(s,t) + \cdots + a_1 P_{IJ}^1(s,t) + a_0 P_{IJ}^0(s,t)$$

for all $I, J \subseteq S$, where either

- $s \in I$, $t \notin I$, $s \in J$, $t \notin J$ *and* $2 \nmid m_{st}$ *or*
- $s \in I$, $t \notin I$, $s \notin J$, $t \in J$ *and* $2 \mid m_{st}$

holds. The a_i denote the coefficients of the polynomial τ_{m-1}; that is,

$$\tau_{m-1}(T) = T^{m-1} + a_{m-2}T^{m-2} + \cdots + a_1 T + a_0.$$

(β) *For all $s, t \in S$ and all $I, J \subseteq S$ with $s, t \in I \setminus J$, the elements*

$$P_{IJ}^1(s,t) - P_{IJ}^1(t,s), P_{IJ}^2(s,t) - P_{IJ}^2(t,s), \ldots, P_{IJ}^m(s,t) - P_{IJ}^m(t,s).$$

These relations are used throughout the rest of the paper. We refer to them as the (α^{st})-relation and the (β^{st})-relation, respectively.

Proof. Consider $V := \mathbb{Z}[v^{\pm 1}]\Xi$ and fixed $s, t \in S$. Define the four subspaces

$$V_{00} := \bigoplus_{\substack{I \subseteq S \\ s \notin I, t \notin I}} VE_I, \qquad V_{01} := \bigoplus_{\substack{I \subseteq S \\ s \in I, t \notin I}} VE_I,$$

$$V_{10} := \bigoplus_{\substack{I \subseteq S \\ s \notin I, t \in I}} VE_I, \qquad V_{11} := \bigoplus_{\substack{I \subseteq S \\ s \in I, t \in I}} VE_I.$$

Note that, given an algebra A and a decomposition into pairwise orthogonal idempotents $1_A = \sum_{i=1}^n e_i$, every element $a \in A$ can be uniquely written as $a = \sum_{i,j} a_{ij}$, with $a_{ij} \in e_i A e_j$, and this additive decomposition behaves like matrices behave with respect to multiplication; that is, $(ab)_{ik} = \sum_j a_{ij} b_{jk}$.

We therefore write elements of $\mathbb{Z}[v^{\pm 1}]\Xi$ as matrices when we want to display such a decomposition in an efficient way. Note that one can view these matrices equivalently either as $d \times d$-matrices with entries in the Laurent polynomial ring $\mathbb{Z}[v^{\pm 1}]\Xi$ or as Laurent polynomials over the matrix ring $\Xi^{d \times d}$. In other words, $\mathbb{Z}[v^{\pm 1}] \otimes (\Xi^{d\times d}) = (\mathbb{Z}[v^{\pm 1}] \otimes \Xi)^{d \times d}$. It is therefore sensible to speak of the coefficient of v^k of a matrix.

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The matrices of $\iota(T_s) = -v^{-1}e_s + x_s + v(1-e_s)$ and $\iota(T_t)$ are given by

$$\iota(T_s) = -v^{-1}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0&0\\B_1&0&A_1&0\\0&0&0&0\\D_1&0&C_1&0\end{pmatrix} + v\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$$
$$= \begin{pmatrix}v&0&0&0\\B_1&-v^{-1}&A_1&0\\0&0&v&0\\D_1&0&C_1&-v^{-1}\end{pmatrix}$$

and

$$\iota(T_t) = -v^{-1}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\B_2&A_2&0&0\\D_2&C_2&0&0\end{pmatrix} + v\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$
$$= \begin{pmatrix}v&0&0&0\\0&v&0&0\\B_2&A_2&-v^{-1}&0\\D_2&C_2&0&-v^{-1}\end{pmatrix},$$

respectively, where

$$A_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\notin I, t\in J}} X^s_{IJ} \quad \text{and} \quad A_2 = \sum_{\substack{I,J\subseteq S\\ s\notin I, s\in J\\ t\in I, t\notin J}} X^t_{IJ},$$

$$B_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\notin I, t\notin J}} X^s_{IJ} \quad \text{and} \quad B_2 = \sum_{\substack{I,J\subseteq S\\ s\notin I, s\notin J\\ t\in I, t\notin J}} X^t_{IJ},$$

$$C_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\in I, t\in J}} X^s_{IJ} \quad \text{and} \quad C_2 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\in J\\ t\in I, t\notin J}} X^t_{IJ}, \qquad \text{as well as}$$

$$D_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\in I, t\notin J}} X^s_{IJ} \quad \text{and} \quad D_2 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\in I, t\notin J}} X^t_{IJ}.$$

Finally, define z to be $v + v^{-1}$.

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Step 1. We claim that for all $r \in \mathbb{N}$,

$$(*) \qquad \Delta_{r+1}(\iota(T_s), \iota(T_t)) = (-1)^r \begin{pmatrix} 0 & 0 & 0 \\ \tau_r(A)JB & \tau_r(A)J(A-z) & 0 \\ X_r & -C\tau_r(A)J & 0 \end{pmatrix}$$

holds, where

$$A := \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad C := \begin{pmatrix} C_2 & C_1 \end{pmatrix}, \qquad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$X_r := \sum_{i=0}^{r-1} (-1)^i C\tau_i(z)\tau_{r-1-i}(A)JB + (-1)^r \tau_r(z)(D_1 - D_2).$$

In order to prove this claim, define

$$E := \iota(T_s) + \iota(T_t) - (v - v^{-1}) = \begin{pmatrix} z & 0 & 0 \\ B & A & 0 \\ D_1 + D_2 & C & -z \end{pmatrix} \qquad \text{and}$$

$$F := \iota(T_s) - \iota(T_t) = \begin{pmatrix} 0 & 0 & 0 \\ JB & J(A-z) & 0 \\ D_1 - D_2 & -CJ & 0 \end{pmatrix}.$$

By Lemma 12, $\Delta_{r+1}(\iota(T_s), \iota(T_t)) = (-1)^r \tau_r(E)F$. Therefore, we inductively show that $\tau_r(E)F$ equals the matrix in $(*)$. For $r = -1$ and $r = 0$, this is clear. The induction step follows from

$$\begin{aligned}
\tau_{r+1}(E)F &= E\tau_r(E)F - \tau_{r-1}(E)F \\
&= \begin{pmatrix} z & 0 & 0 \\ B & A & 0 \\ D_1 + D_2 & C & -z \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ \tau_r(A)JB & \tau_r(A)J(A-z) & 0 \\ X_r & -C\tau_r(A)J & 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & 0 & 0 \\ \tau_{r-1}(A)JB & \tau_{r-1}(A)J(A-z) & 0 \\ X_{r-1} & -C\tau_{r-1}(A)J & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ A\tau_r(A)JB - \tau_{r-1}(A)JB & H & 0 \\ L & K & 0 \end{pmatrix},
\end{aligned}$$

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where we use the abbreviations

$$\begin{aligned} H &:= A\tau_r(A)J(A-z) - \tau_{r-1}(A)J(A-z), \\ K &:= C\tau_r(A)J(A-z) + zC\tau_r(A)J + C\tau_{r-1}(A)J \qquad \text{and} \\ L &:= C\tau_r(A)JB - zX_r - X_{r-1}. \end{aligned}$$

At the positions $(2,1)$ and $(2,2)$, the term is clearly equal to the desired result. At position $(3,2)$, we use $JA = -AJ$ and simplify the expression as follows:

$$\begin{aligned} K &= C\tau_r(A)JA - C\tau_r(A)Jz + zC\tau_r(A)J + C\tau_{r-1}(A)J \\ &= -C\tau_r(A)AJ + C\tau_{r-1}(A)J \\ &= -C\tau_{r+1}(A)J. \end{aligned}$$

Using the recursive definition of τ_{r+1}, it is also a routine calculation to show that

$$L = \sum_{i=0}^{r} (-1)^i C\tau_i(z)\tau_{r-i}(A)JB + (-1)^{r+1}\tau_{r+1}(z)(D_1 - D_2).$$

This shows $(*)$.

Step 2. Simplify the result

Now, let $\mathfrak{K} = \ker(\Xi \to \Omega)$. By definition, this ideal is generated by the coefficients of the v^γ in $\Delta_m(\iota(T_s), \iota(T_t)) \in \mathbb{Z}[v^{\pm 1}]\Xi$. Therefore, we consider the coefficients of

(1) $R_1 := \tau_{m-1}(A)JB$,
(2) $R_2 := \tau_{m-1}(A)J(A-z)$,
(3) $R_3 := \sum_{i=0}^{m-2}(-1)^i C\tau_i(z)\tau_{m-2-i}(A)JB + (-1)^{m-1}\tau_{m-1}(z)(D_1 - D_2)$ and
(4) $R_4 := C\tau_{m-1}(A)J$.

The coefficient of the highest power of v in R_2 is $-\tau_{m-1}(A)J$ because $z = v + v^{-1}$, so that the coefficient of the highest power of z is also the coefficient of the highest power of v in any Laurent polynomial. Now, R_2 is contained in $\mathfrak{K}[v^{\pm 1}]$ (remember that we view these matrices as elements of $\mathbb{Z}[v^{\pm 1}]\Xi$, so that this makes sense) if and only if $\tau_{m-1}(A) \in \mathfrak{K}$, because J is invertible. Conversely, R_1, R_2 and R_4 are in $\mathfrak{K}[v^{\pm 1}]$ if $\tau_{m-1}(A) \in \mathfrak{K}$ holds.

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Let us have a closer look at R_3: the polynomial τ_r has degree r. The coefficient of the highest power of v in R_3 equals $(-1)^{m-1}(D_1 - D_2)$. Therefore, $D_1 - D_2 \in \mathfrak{K}$, and R_3 is in $\mathfrak{K}[v^{\pm 1}]$ if and only if $D_1 - D_2 \in \mathfrak{K}$ and $R_3' = \sum_{i=0}^{m-2}(-1)^i C\tau_i(z)\tau_{r-2-i}(A)JB \in \mathfrak{K}[v^{\pm 1}]$. Looking repeatedly at the coefficient of the highest power of v and shortening the term, we get that R_3' is in $\mathfrak{K}[v^{\pm 1}]$ if and only if $C\tau_0(A)JB, C\tau_1(A)JB, \ldots, C\tau_{m-2}(A)JB \in \mathfrak{K}$. Because $\{\tau_0, \ldots, \tau_{m-2}\}$ is a $\mathbb{Z}$-basis of $\{f \in \mathbb{Z}[T] \mid \deg(f) \leqslant m-2\}$, these terms are in $\mathfrak{K}$ if and only if $CA^0JB, CA^1JB, \ldots, CA^{m-2}JB$ are.

Thus, we obtain the generating set

(α) $R_\alpha := \tau_{m-1}(A)$,
(β) $R_\beta := D_1 - D_2$ and
(γ) $R_{\gamma,k} := CA^kJB$ for $0 \leqslant k \leqslant m-2$,

for the ideal $\mathfrak{K}$.

Step 3. The relations.

Again, we decompose Ξ as $\bigoplus_I \Xi E_I$ and use $R \in \mathfrak{K}$ if and only if $E_I R E_J$ in $\mathfrak{K}$ for all $I, J \subseteq S$.

To determine $E_I R_\alpha E_J$, we consider $E_I A^k E_J$. For $k = 0$, this simplifies to $E_I A^0 E_J = \delta_{IJ} E_I = P^0_{IJ}$. For $k > 0$, we obtain

$$A^k = \begin{cases} \begin{pmatrix} (A_1A_2)^{k/2} & 0 \\ 0 & (A_2A_1)^{k/2} \end{pmatrix} & \text{if } 2 \mid k, \\ \begin{pmatrix} 0 & (A_1A_2)^{(k-1)/2}A_1 \\ (A_2A_1)^{(k-1)/2}A_2 & 0 \end{pmatrix} & \text{if } 2 \nmid k, \end{cases}$$

and substitute

$$A_1 = \sum_{\substack{I,J\subseteq S \\ s\in I, s\notin J \\ t\notin I, t\in J}} X^s_{IJ}, \qquad A_2 = \sum_{\substack{I,J\subseteq S \\ s\notin I, s\in J \\ t\in I, t\notin J}} X^t_{IJ}$$

to obtain

$$\sum_{I_0, I_1, \ldots, I_k \subseteq S} X^s_{I_0I_1} X^t_{I_1I_2} X^s_{I_2I_3} \ldots,$$

where the sum is over all I_i that satisfy $s \in I_{2i} \setminus I_{2i+1}$ and $t \in I_{2i+1} \setminus I_{2i}$ when we consider $A_1A_2A_1\ldots$. Because $X^s_{IJ} = 0$ if $s \notin I \setminus J$, only the conditions for $I = I_0$ and $I_k = J$ are not vacuous. Therefore, we could just sum over all

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paths of length k that go to I from J. We therefore obtain

$$\underbrace{A_1A_2\ldots}_{k} = \begin{cases} \displaystyle\sum_{\substack{I,J\subseteq S\\ s\in I, s\in J\\ t\notin I, t\notin J}} P^k_{IJ}(s,t) & \text{if } 2\mid k,\\ \displaystyle\sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\notin I, t\in J}} P^k_{IJ}(s,t) & \text{if } 2\nmid k.\end{cases}$$

For the other product, we similarly obtain

$$\underbrace{A_2A_1\ldots}_{k} = \begin{cases} \displaystyle\sum_{\substack{I,J\subseteq S\\ s\notin I, s\notin J\\ t\in I, t\in J}} P^k_{IJ}(t,s) & \text{if } 2\mid k,\\ \displaystyle\sum_{\substack{I,J\subseteq S\\ s\notin I, s\in J\\ t\in I, t\notin J}} P^k_{IJ}(t,s) & \text{if } 2\nmid k.\end{cases}$$

Multiplying with E_I from the left and with E_J from the right, this equals either 0 or $P^k_{IJ}(s,t)$ and $P^k_{IJ}(t,s)$, respectively. The element $E_I\tau_{m-1}(A)E_J\in\mathfrak{K}$ is, if it is not zero, equal to

$$P^{m-1}_{IJ}(s,t)+a_{m-2}P^{m-2}_{IJ}(s,t)+\cdots+a_2P^2_{IJ}(s,t)+a_1P^1_{IJ}(s,t)+a_0P^0_{IJ}(s,t),$$

where $\tau_{m-1}(T)=T^{m-1}+a_{m-2}T^{m-2}+\cdots+a_2T^2+a_1T^1+a_0$, and similarly for the symmetric situation where s,t are swapped.

The second kind of generator is easier: R_β is equal to

$$\sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\in I, t\notin J}} X^s_{IJ}-X^t_{IJ}.$$

For those I, J that do not occur in this sum, $E_IR_\beta E_J=0$. For all others, we obtain the element $X^s_{IJ}-X^t_{IJ}=P^1_{IJ}(s,t)-P^1_{IJ}(t,s)$. This is the first case in the relations of type (β).

Finally, there is only one kind of generator left, namely $R_{\gamma,k}=CA^kJB$. We already know the powers of A, and therefore obtain

$$CA^kJB=\begin{cases} C_2(A_1A_2)^{k/2}B_1-C_1(A_2A_1)^{k/2}B_2 & \text{if } 2\mid k,\\ -C_2(A_1A_2)^{(k-1)/2}A_1B_2+C_1(A_2A_1)^{(k-1)/2}A_2B_1 & \text{if } 2\nmid k.\end{cases}$$

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We substitute the definitions

$$B_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\notin I, t\notin J}} X^s_{IJ}, \qquad B_2 = \sum_{\substack{I,J\subseteq S\\ s\notin I, s\notin J\\ t\in I, t\notin J}} X^t_{IJ},$$

$$C_1 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\notin J\\ t\in I, t\in J}} X^s_{IJ}, \qquad C_2 = \sum_{\substack{I,J\subseteq S\\ s\in I, s\in J\\ t\in I, t\notin J}} X^t_{IJ}.$$

If $s, t \in I$ or $s, t \notin J$ is not satisfied, then $E_I C A^k J B E_J = 0$, because either $E_I C = 0$ or $B E_J = 0$. Otherwise,

$$\begin{aligned} E_I C A^k J B E_J &= \sum_{I_0,\ldots,I_k\subseteq S} X^t_{II_0} X^s_{I_0 I_1} \ldots X^t_{I_{k-1}I_k} X^s_{I_k J} \\ &\qquad - X^s_{I,I_0} X^t_{I_0 I_1} \ldots X^s_{I_{k-1}I_k} X^t_{I_k J} \\ &= P^{k+2}_{IJ}(t,s) - P^{k+2}_{IJ}(s,t) \end{aligned}$$

holds if $2 \mid k$ and

$$\begin{aligned} E_I C A^k J B E_J &= \sum_{I_0,\ldots,I_k\subseteq S} -X^t_{II_0} X^s_{I_0 I_1} \ldots X^s_{I_{k-1}I_k} X^t_{I_k J} \\ &\qquad + X^s_{II_0} X^t_{I_0 I_1} \ldots X^t_{I_{k-1}I_k} X^s_{I_k J} \\ &= -P^{k+2}_{IJ}(t,s) + P^{k+2}_{IJ}(s,t) \end{aligned}$$

holds if $2 \nmid k$. This provides the other elements in the relations of type (β). ∎

Because of the (β)-relation $X^s_{IJ} - X^t_{IJ} = P^1_{IJ}(s,t) - P^1_{IJ}(t,s) = 0$ in Ω, the upper index of these elements does not matter, and it is well defined to write X_{IJ} for the common value of $X^s_{IJ} \in \Omega$ for all $s \in I \setminus J$. We adopt this notation for the rest of this article.

Additionally, (α) implies that $X^s_{IJ} = X^t_{JI} = 0$ holds in Ω for all $I, J \subseteq S$, $s \in I \setminus J$, $t \in J \setminus I$ with $m_{st} = 2$ (i.e., s and t are not connected in the Dynkin diagram of (W, S)). This allows us to think of Ω as a quotient of a path algebra over a much simpler quiver, which was defined by Stembridge [8, Section 4].

DEFINITION 14. The *compatibility graph* of (W, S) is the directed graph $\mathcal{Q}_W$ with vertex set $\{I \mid I \subseteq S\}$ and a single edge $I \leftarrow J$ if and only if $I \setminus J \neq \emptyset$ and no element of $I \setminus J$ commutes with any element of $J \setminus I$.

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An edge $I \leftarrow J$ with $I \supseteq J$ is called an *inclusion edge*; all other edges are called *transversal edges*.

Note that transversal edges always occur in pairs of opposite orientation because their definition is symmetric: $I \leftarrow J$ is a transversal edge if and only if $I \setminus J \neq \emptyset$, $J \setminus I \neq \emptyset$ and all $s \in I \setminus J$ are connected to all $t \in J \setminus I$ in the Dynkin diagram of (W, S).

COROLLARY 15. *Ω is a quotient of the path algebra $\mathbb{Z}\mathcal{Q}_W$.*

Proof. Denote the vertex elements in $\mathbb{Z}\mathcal{Q}_W$ by $\tilde{E}_I$ and the edge elements by $\tilde{X}_{IJ}$. Then, $\tilde{E}_I \mapsto E_I$ and $\tilde{X}_{IJ} \mapsto X^s_{IJ}$ with any $s \in I \setminus J$ extends to a well-defined algebra homomorphism $\mathbb{Z}\mathcal{Q}_W \to \Omega$ by the (β)-relation. It is surjective because all elements $X^s_{IJ} \in \Omega$ are either zero by some (α)-type relation as seen above or contained in the image of this morphism, and Ω is generated by the elements X^s_{IJ} and E_I. $\square$

EXAMPLE 16. Figure 1 displays the compatibility graphs of the finite irreducible Coxeter groups of rank $\leqslant 4$. For the sake of clarity, inclusion edges are only displayed in rank 2 and rank 3 and only between those $I, J \subseteq S$ that satisfy $|I \setminus J| = 1$. Pairs of transversal edges $I \leftrightarrows J$ are combined into one (bold) undirected edge.

§4. The decomposition conjecture

While trying to prove Gyoja's conjecture[1] and to better understand the internal structure of Ω and W-graphs, I found a number of very similar proofs for some small types of Coxeter groups. The essence of these proofs is captured by the following four conjectural properties of Ω.

CONJECTURE 17. *Let $k \subseteq \mathbb{C}$ be a good ring for (W, S). There exists a family $(F^\lambda)_{\lambda \in \mathrm{Irr}(W)}$ of elements of $k\Omega$ with the following properties.*

(Z1) *The F^λ are pairwise orthogonal idempotents and decompose the identity*

$$\forall \lambda, \mu \in \mathrm{Irr}(W) : F^\lambda F^\mu = \delta_{\lambda\mu} F^\lambda, \qquad 1 = \sum_{\lambda \in \mathrm{Irr}(W)} F^\lambda.$$

[1] This conjecture states that the Jacobson radical of $K\Omega$ has codimension $|W|$ if K is a sufficiently large field of characteristic zero, or equivalently that two irreducible $K\Omega$-modules are isomorphic if and only if their restrictions to $K(v)H$ are isomorphic (see [4, Remark 2.18], [5, Theorem 4.3.7]).

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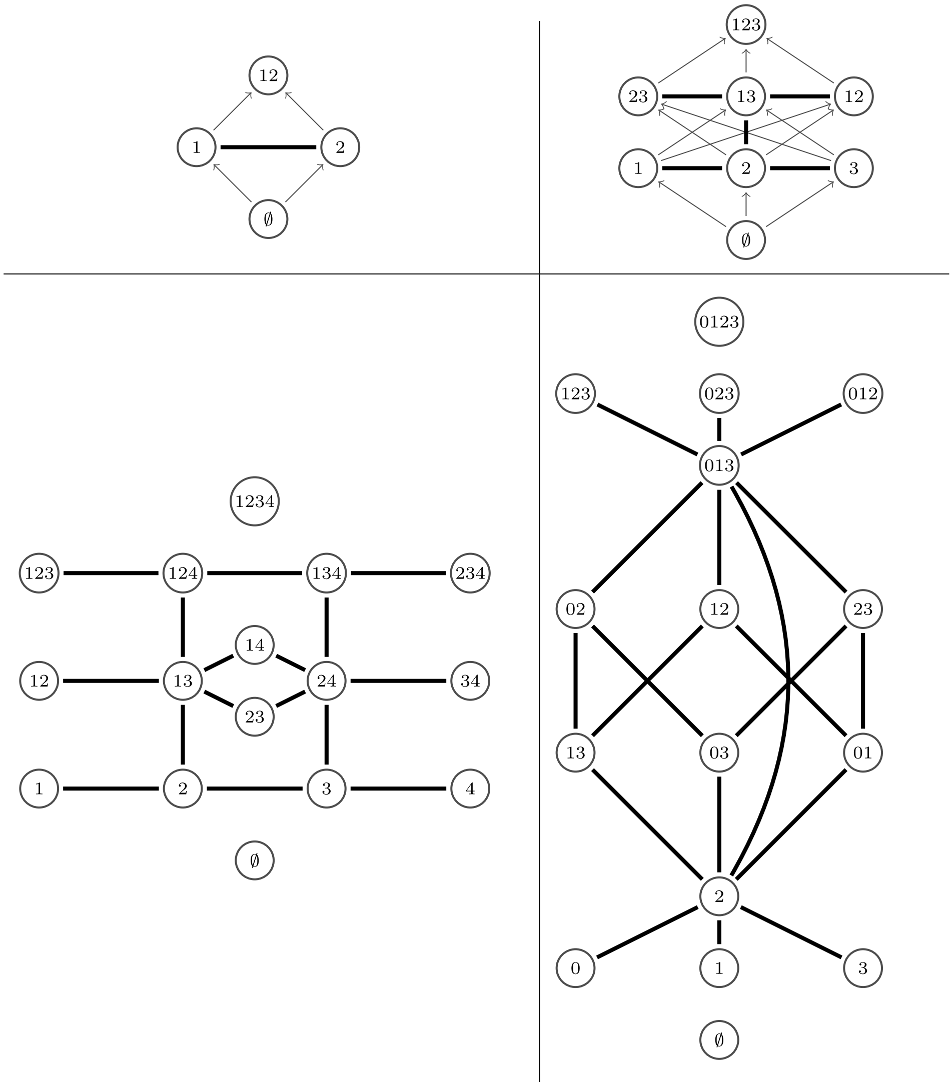

Figure 1.
Compatibility graphs for small Coxeter groups: top left for $I_2(m)$; top right for A_3, B_3 and H_3; bottom left for A_4, B_4 and F_4; bottom right for D_4.

(Z2) *This decomposition is compatible with the decomposition induced by the path-algebra structure:*

$$\forall \lambda \in \mathrm{Irr}(W) \forall I \subseteq S : E_I F^\lambda = F^\lambda E_I.$$

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(Z3) *There is a partial order* $\preceq$ *on* $\mathrm{Irr}(W)$ *such that only downward edges exist: if* $F^\lambda k\Omega F^\mu \neq 0$*, then* $\lambda \preceq \mu$*.*
(Z4) *There are surjective* k*-algebra morphisms* $\psi_\lambda : k^{d_\lambda \times d_\lambda} \twoheadrightarrow F^\lambda k\Omega F^\lambda$ *for all* $\lambda \in \mathrm{Irr}(W)$*, where* d_λ *denotes the degree of the character* λ*.*

REMARK 18. The edge terminology in (Z3) refers to the quiver Λ in the next theorem.

The decomposition conjecture is of interest because it implies several important properties of the W-graph algebra and its modules (that is, W-graphs), as the following theorem demonstrates.

THEOREM 19. *Assume that the decomposition conjecture is true for the finite Coxeter group* (W, S) *and* k *a good ring for* (W, S)*. Then, the following properties hold.*

(1) *Consider the quiver* Λ *which has* $\mathrm{Irr}(W)$ *as its set of vertices and an edge* $\lambda \leftarrow \mu$ *if and only if* $\lambda \prec \mu$*. Then,* $k\Omega$ *is a quotient of the generalized path algebra (cf. [2]) over the quiver* Λ *which has* $k^{d_\lambda \times d_\lambda}$ *as vertex algebras.*
(2) $k\Omega$ *is finitely generated as a* k*-module.*

Furthermore, if k *is a field then the following hold:*

(3) *The Jacobson radical* $\mathrm{rad}(k\Omega)$ *is generated by the elements* $F^\lambda X_{IJ} F^\mu$ *with* $\lambda \prec \mu$ *and* $k\Omega/\mathrm{rad}(k\Omega) \cong \prod_{\lambda \in \mathrm{Irr}(W)} k^{d_\lambda \times d_\lambda}$*.*
(4) *A* $k\Omega$*-module* V *is simple if and only if the restriction of* $k(v)V$ *to* $k(v)H$ *is simple. Furthermore (after reindexing the family* $(F^\lambda)_{\lambda \in \mathrm{Irr}(W)}$ *if necessary), the latter has isomorphism class* λ *if and only if* $F^\lambda V = V$ *holds.*
(5) *Every* $k\Omega$*-module* V *has a family of natural submodules* $(V^{\trianglelefteq\lambda})_{\lambda \in \mathrm{Irr}(W)}$ *such that*
   - $\lambda \preceq \mu \implies V^{\trianglelefteq\lambda} \leqslant V^{\trianglelefteq\mu}$ *and*
   - $V^{\trianglelefteq\lambda}/V^{\triangleleft\lambda}$ *is isomorphic to a direct sum of irreducibles of isomorphism class* λ*, where* $V^{\triangleleft\lambda} := \sum_{\mu \prec \lambda} V^{\trianglelefteq\mu}$*.*

REMARK 20. Given that the Kazhdan–Lusztig-W-graph is indecomposable but not irreducible, it cannot be expected that an arbitrary Ω-module decomposes as a direct sum of its irreducible constituents. The special filtration appearing in the above theorem is the next best thing one can

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hope for: one finds the irreducible constituents in the layers of a natural filtration, and even nicely grouped into isomorphism classes. This fact and the first part of the theorem, saying that Ω itself is composed of much simpler parts like matrix algebras and path algebras, motivates the name decomposition conjecture.

To the best of my knowledge, the decomposition conjecture has neither directly nor in a similar form been stated before in the literature, apart from my dissertation [5]. The above consequences of Conjecture 17 also have not been considered or proved before, even in special cases, as far as I know.

Proof. Denote with $k\widetilde{\Omega}$ this generalized path algebra, and recall that it is characterized by the following properties:

- it contains a set of pairwise orthogonal idempotents f_λ corresponding to the vertices such that $\sum_\lambda f_\lambda = 1$ and $f_\lambda k\widetilde{\Omega} f_\lambda = k^{d_\lambda \times d_\lambda}$;
- it contains a set of elements $y_{\lambda\mu}$ corresponding to the edges $\lambda \leftarrow \mu$ such that $y_{\lambda\mu} = f_\lambda y_{\lambda\mu} f_\mu$;
- it satisfies the universal mapping property with respect to these features: for any k-algebra A, any set of elements $\{f'_\lambda, y'_{\lambda\mu} | \lambda, \mu \in \operatorname{Irr}(W)\}$ satisfying these properties and any $\psi_\lambda : k^{d_\lambda \times d_\lambda} \to f'_\lambda A f'_\lambda$, there exists a unique morphism of k-algebras $\psi : k\widetilde{\Omega} \to A$ with $\psi(f_\lambda) = f'_\lambda, \psi(y_{\lambda\mu}) = y'_{\lambda\mu}$ and $\psi | f_\lambda A f_\lambda = \psi_\lambda$.

We define elements $Y_{\lambda\mu} := \sum_{I,J \subseteq S} F^\lambda X_{IJ} F^\mu \in k\Omega$. Note that $Y_{\lambda\mu} = 0$ if $\lambda \not\preceq \mu$, by (Z3). The universal property ensures that the morphisms from (Z4) $\psi_\lambda : k^{d_\lambda \times d_\lambda} \to k\Omega$ together with $f_\lambda \mapsto F^\lambda$ and $y_{\lambda\mu} \mapsto Y_{\lambda\mu}$ uniquely extend to an algebra morphism $\psi : k\widetilde{\Omega} \to k\Omega$ (this uses (Z1)).

We verify that this is an epimorphism. By construction,

$$F_I^\lambda \overset{(Z2)}{=} F^\lambda E_I F^\lambda \in F^\lambda k\Omega F^\lambda \overset{(Z4)}{=} \operatorname{im}(\psi_\lambda) \subseteq \operatorname{im}(\psi)$$

for all $I \subseteq S, \lambda \in \operatorname{Irr}(W)$. Therefore, $E_I = \sum_\lambda F_I^\lambda \in \operatorname{im}(\psi)$. Also, $X_{IJ} \overset{(Z3)}{=} \sum_\lambda F^\lambda X_{IJ} F^\lambda + \sum_{\lambda \prec \mu} Y_{\lambda\mu} \in \operatorname{im}(\psi)$ for all $I, J \subseteq S$. Because we already know that $\{E_I, X_{IJ} \mid I, J \subseteq S\}$ generates $k\Omega$, we are done.

Because $\psi : k\widetilde{\Omega} \to k\Omega$ is surjective, $k\Omega$ is finitely generated as a k-module because $k\widetilde{\Omega}$ is, and $\psi(\operatorname{rad}(k\widetilde{\Omega})) \subseteq \operatorname{rad}(k\Omega)$ holds. The morphism ψ therefore induces a surjection $k\widetilde{\Omega}/\operatorname{rad}(k\widetilde{\Omega}) \twoheadrightarrow k\Omega/\operatorname{rad}(k\Omega)$.

Now consider the case that k is a field. The radical of the generalized path algebra is then easily seen to coincide with the ideal generated by the edge

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elements $y_{\lambda\mu}$ because the quiver Λ is acyclic. (See [2, Proposition 1.3] for a more general characterization of the radical of generalized path algebras.) In fact, $k\widetilde{\Omega}/\mathrm{rad}(k\widetilde{\Omega}) = \prod_\lambda k^{d_\lambda \times d_\lambda}$.

This implies $\dim_k k\Omega/\mathrm{rad}(k\Omega) \leqslant \sum_\lambda d_\lambda^2$. We show that $\prod_\lambda k^{d_\lambda \times d_\lambda}$ is in fact a quotient of $k\Omega$ to establish equality. For each λ, choose a W-graph with edge weights in k realizing the irreducible $k(v)H$-module of isomorphism class λ (this is possible by Gyoja's work [4, Theorem 2.3]), and consider the induced $k\Omega$-module V_λ (in particular, $\dim_k V_\lambda = d_\lambda$); set $V := \bigoplus_\lambda V_\lambda$ and denote the associated representation $k\Omega \to \mathrm{End}_k(V)$ by ω. By construction, $\mathrm{im}(\omega) \subseteq \prod_\lambda k^{d_\lambda \times d_\lambda}$ holds. Now, consider $k(v)V$ as a module for $k(v)H \subseteq k(v)\Omega$ by restriction. Because $k(v)V$ contains each irreducible module of the Hecke algebra exactly once, $\omega(k(v)H) = \prod_\lambda k(v)^{d_\lambda \times d_\lambda}$ holds. By comparing dimensions, we obtain the desired equality above.

The fourth item follows from this. On one hand, a $k\Omega$-module V is certainly simple if its restriction to a subalgebra is already simple. Because every simple $k(v)H$-module can be realized by a W-graph, choosing one W-graph for each isomorphism class induces an injective map $\mathrm{Irr}(W) \cong \mathrm{Irr}(k(v)H) \to \mathrm{Irr}(k\Omega)$ with the restriction map as a left inverse. Because of $k\Omega/\mathrm{rad}(k\Omega) \cong \prod_\lambda k^{d_\lambda \times d_\lambda}$, the number of elements in both sets is the same, so that the map is actually a bijection.

Define $F^{\preceq\lambda} := \sum_{\mu \preceq \lambda} F^\mu$. By (Z3), the right ideals $F^{\preceq\lambda}k\Omega$ are actually two-sided ideals. For any $k\Omega$-module V, define $V^{\preceq\lambda} := F^{\preceq\lambda}V$. This is a submodule of V for all λ; $\lambda \preceq \mu \implies V^{\preceq\lambda} \leqslant V^{\preceq\mu}$ holds by construction, and F^λ acts as the identity on $V^{\preceq\lambda}/V^{\prec\lambda}$.

Now consider the equation $1 = \sum_\lambda F^\lambda$. It shows that there must be at least one λ with $F^\lambda V \neq 0$ if $V \neq 0$. A λ that is $\preceq$-minimal with respect to this property satisfies $0 \neq F^\lambda V = F^{\preceq\lambda}V$, so that $F^\lambda V = V$ follows if V is simple. Therefore, for each simple $k\Omega$-module V, there is exactly one λ with $F^\lambda V = V$. Conversely, $R^\lambda := F^{\preceq\lambda}k\Omega/F^{\prec\lambda}k\Omega$ is a finite-dimensional, nonzero $k\Omega$-module with $F^\lambda R^\lambda = R^\lambda$, so that for each $\lambda \in \mathrm{Irr}(W)$, there must be at least one simple $k\Omega$-module V with $F^\lambda V = V$. This establishes another bijection between $\mathrm{Irr}(k\Omega)$ and $\mathrm{Irr}(W)$. By reindexing the F^λ, one can achieve that these two bijections are in fact the same, so that $F^\lambda V = V$ holds if and only if the restriction of $k(v)V$ to $k(v)H$ is of isomorphism class λ.

Now, consider again an arbitrary V and the quotient $R^\lambda := V^{\preceq\lambda}/V^{\prec\lambda}$. Because $F^\mu R^\lambda = 0$ for all $\mu \neq \lambda$, the representation$k\Omega \to \mathrm{End}_k(W)$ must

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annihilate F^μ for all $\mu \neq \lambda$, and therefore all $F^\kappa X_{IJ} F^{\kappa'}$ with $\kappa \neq \kappa'$. Hence, the representation vanishes on the radical and R^λ is therefore semisimple. However, again $F^\mu R^\lambda = 0$ for all $\mu \neq \lambda$, so that the simple constituents of R^λ must all lie in the isomorphism class λ. ∎

§5. Proving the decomposition conjecture

The rest of the paper is devoted to proving that the W-graph decomposition conjecture holds for Coxeter groups of types $I_2(m)$, A_1–A_4 and B_3. These proofs all proceed by the same pattern: the relations from Theorem 13 are used to find orthogonal decompositions $E_I = \sum_{\lambda \in \mathrm{Irr}(W)} F_I^\lambda$ of the vertex idempotents $E_I \in k\Omega$ into smaller idempotents F_I^λ, some of which may be zero. The idempotents F^λ in the decomposition conjecture are then obtained as $F^\lambda := \sum_I F_I^\lambda$.

These decompositions are graphically represented as refinements of the compatibility graph $\mathcal{Q}_W$. That is, the single vertex corresponding to E_I is split into up to $|\mathrm{Irr}(W)|$ many vertices corresponding to the idempotents F_I^λ (some of which might be zero), and similarly the edge corresponding to the element X_{IJ} is split into up to $|\mathrm{Irr}(W)|^2$ many edges corresponding to the elements $F_I^\lambda X_{IJ} F_J^\mu$, most of which will also be zero.

Direct computations are used to show that enough edge elements are zero to satisfy the decomposition conjecture.

REMARK 21. A reviewer of this paper remarked that the computations in the rest of this paper feel like they are instances of a general algorithmic approach to the question of whether or not a particular Coxeter group satisfies the decomposition conjecture. I share this feeling, but to my frustration I have not been able to pin down such an algorithm and prove its correctness as of the time of writing this paper. Part of the complication stems from the fact that almost nothing useful about Ω is known to me in the absence of the decomposition conjecture. In particular, it is hard to algorithmically decide whether or not an element is zero without having a nice, faithful representation of Ω at hand. Even proving finite dimensionality or even that the relations in Theorem 13 are a noncommutative Gröbner basis (and therefore the problem's amenability to certain general algorithms) is beyond my capabilities as of now.

If and when these problems get resolved, the lengthy calculations in this chapter may be replaced with a computer proof.

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5.1 Auxiliary lemmas

The first lemma that is repeatedly used allows us to transport a decomposition into pairwise orthogonal idempotents from one E_I to an adjacent E_J in the compatibility graph, and immediately recognize most of the possible new edge elements as zero.

DEFINITION 22. In any algebra, define a partial order on the set of idempotents by $e \leqslant f \iff e = ef = fe$.

LEMMA 23. *Let $I, J \subseteq S$ be arbitrary but fixed subsets. Let A be a finite indexing set, and let $(e_\alpha)_{\alpha \in A}$ be pairwise orthogonal idempotents $\leqslant E_I$, with $X_{IJ}X_{JI} = \sum_{\alpha \in A} \sigma_\alpha e_\alpha$ for some $\sigma_\alpha \in k^\times$. Denote the idempotent $E_I - \sum_\alpha e_\alpha$ by e_0. With these notations, the following statements hold.*

(1) *$\widetilde{e}_\alpha := \sigma_\alpha^{-1} X_{JI} e_\alpha X_{IJ}$ and $\widetilde{e}_0 := E_J - \sum_{\alpha \in A} \widetilde{e}_\alpha$ are pairwise orthogonal idempotents $\leqslant E_J$.*
(2) *$X_{IJ}\widetilde{e}_\alpha = e_\alpha X_{IJ}$ and $X_{JI}e_\alpha = \widetilde{e}_\alpha X_{JI}$ for all $\alpha \in A \cup \{0\}$.*
(3) *$r := X_{JI}e_0X_{IJ}$ satisfies $r^2 = 0$, $r = \widetilde{e}_0 r \widetilde{e}_0$ and $X_{JI}X_{IJ} = \sum_{\alpha \in A} \sigma_\alpha \widetilde{e}_\alpha + r$. In particular, $r = 0$ holds if $X_{JI}X_{IJ}$ is an idempotent itself.*
(4) *$X_{IJ}\widetilde{e}_\alpha X_{JI} = \sigma_\alpha e_\alpha$ for all $\alpha \in A$. In other words, applying this construction twice gives back the original idempotents.*

Proof. All claims are easily verified by using the definition. For example,

$$\begin{aligned}
\widetilde{e}_\alpha \widetilde{e}_\beta &= \sigma_\alpha^{-1}\sigma_\beta^{-1} X_{JI} e_\alpha X_{IJ} X_{JI} e_\beta X_{IJ} \\
&= \sigma_\alpha^{-1}\sigma_\beta^{-1} X_{JI} e_\alpha \left(\sum_\gamma \sigma_\gamma e_\gamma \right) e_\beta X_{IJ} \\
&= \sum_\gamma \frac{\sigma_\gamma}{\sigma_\alpha \sigma_\beta} X_{JI} e_\alpha e_\gamma e_\beta X_{IJ} \\
&= \begin{cases} 0, & \alpha \neq \beta, \\ \widetilde{e}_\alpha, & \alpha = \beta. \end{cases}
\end{aligned}$$

See [5, Lemma 4.5.25] for complete proofs of the other claims. ∎

DEFINITION 24. In the above construction, the $\widetilde{e}_\alpha$ are said to be obtained by *transporting idempotents from I to J*. The e_0 and $\widetilde{e}_0$ are called *leftover idempotents* of this transport.

The following well-known result is also used repeatedly to construct the morphisms ψ_λ in conjecture (Z4).

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LEMMA 25. *The matrix algebra $k^{d\times d}$ is freely generated by the generators $\{e_{ij} \mid 1 \leqslant i, j \leqslant d, |i-j| \leqslant 1\}$ with respect to the relations*

$$e_{ii}e_{jj} = \delta_{ij}e_{ii}, \qquad 1 = \sum_{i=1}^{d} e_{ii}, \qquad e_{ii}e_{ij}e_{jj} = e_{ij} \qquad \text{and} \qquad e_{ij}e_{ji} = e_{ii}.$$

Note that this can be equivalently stated by saying that $k^{d\times d}$ is the quotient of the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r, bend right] & 2 \arrow[l, bend right] \arrow[r, bend right] & \cdots \arrow[l, bend right] \arrow[r, bend right] & d-1 \arrow[l, bend right] \arrow[r, bend right] & d \arrow[l, bend right] \end{tikzcd}$$

by the relations that declare every directed loop to be equal to (the idempotent corresponding to) its base point.

While proving (Z4), the surjectivity of the constructed morphisms is often implied by the fact that $F^\lambda k\Omega F^\lambda$ is generated as a k-algebra by the elements $F_I^\lambda = F^\lambda E_I F^\lambda$ and $F^\lambda X_{IJ} F^\lambda$. This follows from the fact that $k\Omega$ is generated by the E_I and X_{IJ} together with the observation that (Z1)–(Z3) imply that a product of the form

$$F^{\lambda_1}X_{I_1,I_2}\ldots X_{I_{k-1},I_k}F^{\lambda_k} = \sum_{\lambda_2,\ldots,\lambda_{k-1}} F^{\lambda_1}X_{I_1,I_2}F^{\lambda_2}\ldots F^{\lambda_{k-1}}X_{I_{k-1},I_k}F^{\lambda_k}$$

can only be nonzero if there are $\lambda_2, \ldots, \lambda_{k-1}$ with $F^{\lambda_j}X_{I_j,I_{j+1}}F^{\lambda_{j+1}} \neq 0$ for all $1 \leqslant j < k$. By (Z3), this implies that $\lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_k$. Thus, if $\lambda_1 = \lambda_k = \lambda$, then all intermediate λ_j must be equal to λ as well, so that $F^\lambda X_{I_1,I_2}\ldots X_{I_{k-1},I_k}F^\lambda$ is expressible as a product of elements of the form $F^\lambda X_{IJ}F^\lambda$, as claimed.

5.2 Rank 1

THEOREM 26. *The decomposition conjecture is true for all Coxeter groups (W, S) of type $A_1 \times \cdots \times A_1$.*

Proof. Groups of this particular type have the property that all $s, t \in S$ commute. In particular, there are no transversal edges in the compatibility graph but only inclusion edges, so that $\mathcal{Q}_W$ is acyclic and the trivial decomposition $E_I = E_I$ is already sufficient to satisfy (Z1)–(Z4). ∎

5.3 Rank 2

While good rings for A_n and B_n are easy to understand, the following lemma is needed to establish the existence of certain elements in a good ring for Coxeter groups of $I_2(m)$, which is used in the proof of the

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decomposition conjecture. Note that a good ring for $I_2(m)$ always contains $\mathbb{Z}[2\cos(2\pi/m), 1/m]$.

LEMMA 27. *Let $m \in \mathbb{N}_{\geqslant 1}$, and let k be a good ring for $I_2(m)$. The following assertions are true.*

(1) $2\cos(a(2\pi/m)), 4\cos(a(\pi/m))^2 \in k$ *for all* $a \in \mathbb{Z}$.
(2) $4\cos(a(\pi/m))^2 \in k^\times$ *for all* $a \in \mathbb{Z} \setminus (m/2)\mathbb{Z}$.
(3) $4\cos(a(\pi/m))^2 - 4\cos(b(\pi/m))^2 \in k^\times$ *for all* $1 \leqslant a < b \leqslant \lfloor m/2 \rfloor$.

Proof. Set $\zeta_n := \exp(2\pi i/n)$ for all $n \in \mathbb{N}_{\geqslant 1}$. With this notation, $2\cos(a(2\pi/n)) = \zeta_n^a + \zeta_n^{-a}$ holds. It follows from $T^a + T^{-a} \in \mathbb{Z}[T + T^{-1}]$ that $2\cos(a(2\pi/n)) \in k$ for all $a \in \mathbb{Z}$. The fact that $4\cos(a(\pi/m))^2 \in k$ follows from the double-angle formula $2\cos(\theta/2)^2 = \cos(\theta) + 1$.

The proofs of the second and third claims use that

$$\mathbb{Z}\left[2\cos\left(\frac{2\pi}{n}\right)\right] = \mathbb{Z}[\zeta_n + \zeta_n^{-1}] \subseteq \mathbb{Z}[\zeta_n] \subseteq \mathbb{Z}[\zeta_{nl}]$$

are integral ring extensions for all $l \in \mathbb{N}_{\geqslant 1}$, and integral extensions $R \subseteq S$ have the property $R \cap S^\times = R^\times$. Therefore, it suffices to show that the elements are units in $\mathbb{Z}[\zeta_{ml}, 1/m]$ for some $l \in \mathbb{N}_{\geqslant 1}$.

Step 1. $4\cos(a(\pi/m))^2$ is invertible for all $a \in \mathbb{Z} \setminus (m/2)\mathbb{Z}$.

This follows from

$$\prod_{1 \leqslant a < m/2} \left(2\cos\left(a\frac{\pi}{m}\right)\right)^2 = \begin{cases} 1 & \text{if } 2 \nmid m, \\ \dfrac{m}{2} & \text{if } 2 \mid m, \end{cases}$$

which is easily shown using $2\cos(a(\pi/m)) = \zeta_{2m}^a + \zeta_{2m}^{-a}$. Therefore, $4\cos(a(\pi/m))^2$ is invertible too.

Step 2. $2\sin(a(\pi/m))$ is invertible for all $a \in \mathbb{Z} \setminus m\mathbb{Z}$.

This follows from

$$\prod_{a=1}^{m-1} 2\sin\left(a\frac{\pi}{m}\right) = m,$$

which similarly can be shown using $2\sin(a(\pi/m)) = (1/i)(\zeta_{2m}^a - \zeta_{2m}^{-a})$. Hence, all $2\sin(a(\pi/m))$ are units for $a \in \mathbb{Z} \setminus m\mathbb{Z}$. This then proves the third claim because $4\cos(a(\pi/m))^2 - 4\cos(b(\pi/m))^2 = 2\sin((a+b)(\pi/m)) \cdot 2\sin((a-b)(\pi/m))$ holds. ∎

THEOREM 28. *Let m be a natural number $\geqslant 3$. The decomposition conjecture is true for all Coxeter groups of type $I_2(m)$.*

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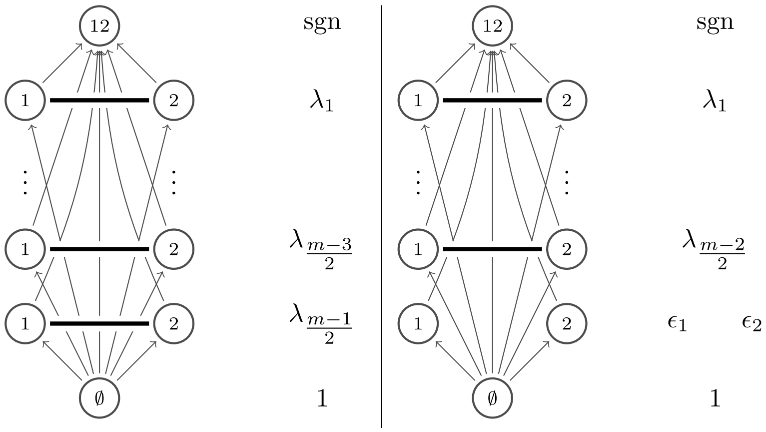

Figure 2.
Refined compatibility graphs for $I_2(m)$: left-hand side for m odd; right-hand side for m even.

Proof. The idea of the proof is to use a spectral decomposition of the loops $X_{1,2}X_{2,1}$ and $X_{2,1}X_{1,2}$, and construct a refinement of the compatibility graph as in Figure 2.

The next important observation is that there are only two transversal edges if the rank of (W, S) is two, namely $X_{1,2}$ and $X_{2,1}$. Therefore, the only relations in Ω of type (α) are

$$0 = \sum_{j=0}^{m-1} a_j \underbrace{X_{1,2}X_{2,1}\ldots}_{j} \qquad \text{and} \qquad 0 = \sum_{j=0}^{m-1} a_j \underbrace{X_{2,1}X_{1,2}\ldots}_{j},$$

where the a_j are the coefficients of τ_{m-1}.

Step 1. Preparations.

For all $n \in \mathbb{N}$, define $\widetilde{\tau}_n \in \mathbb{Z}[X]$ by

$$\widetilde{\tau}_n := \begin{cases} \tau_n(\sqrt{X}) & \text{if } 2 \mid n, \\ \tau_n(\sqrt{X})\sqrt{X} & \text{if } 2 \nmid n. \end{cases}$$

Recall that τ_n is an even polynomial if n is even and an odd polynomial if n is odd. Therefore, $\widetilde{\tau}_n$ really is a polynomial in X. It has degree $\lceil n/2 \rceil$ and is monic. Since the n zeros of τ_n are given by $2\cos((a/n+1)\pi)$ for $a = 1, \ldots, n$ (cf. [1, 22.16]), the zeros of $\widetilde{\tau}_n$ are given by $4\cos((a/n+1)\pi)^2$ for $a = 1, \ldots, \lceil n/2 \rceil$. In particular, the zeros of $\widetilde{\tau}_{m-1}$ are equal to $\sigma_a := 4\cos(a(\pi/m))^2$ for $a = 1, \ldots, \lfloor m/2 \rfloor$.

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Step 2. Construction of the idempotents.

If m is odd, then the (α)-type relations are already of the form $\widetilde{\tau}_{m-1}(X_{1,2}X_{2,1}) = 0$ and $\widetilde{\tau}_{m-1}(X_{2,1}X_{1,2}) = 0$, respectively. If m is even, then one can multiply the relation with $X_{1,2}$ and $X_{2,1}$, and obtain the same equations.

By defining

$$F_{1,a} := \prod_{\substack{b=1,\ldots,\lfloor m/2\rfloor\\ b\neq a}} \frac{X_{1,2}X_{2,1} - \sigma_b E_1}{\sigma_a - \sigma_b} \qquad \text{and}$$

$$F_{2,a} := \prod_{\substack{b=1,\ldots,\lfloor m/2\rfloor\\ b\neq a}} \frac{X_{2,1}X_{1,2} - \sigma_b E_2}{\sigma_a - \sigma_b} \tag{1}$$

for all $a = 1, \ldots, \lfloor m/2\rfloor$, we get a set of pairwise orthogonal idempotents $F_{1,a}, F_{2,a} \in k\Omega$ with

$$E_1 = \sum_{a=1}^{\lfloor m/2\rfloor} F_{1,a} \qquad \text{and} \qquad X_{1,2}X_{2,1} = \sum_{a=1}^{\lfloor m/2\rfloor} \sigma_a F_{1,a}, \qquad \text{and}$$

$$E_2 = \sum_{a=1}^{\lfloor m/2\rfloor} F_{2,a} \qquad \text{and} \qquad X_{2,1}X_{1,2} = \sum_{a=1}^{\lfloor m/2\rfloor} \sigma_a F_{2,a}.$$

Denote the irreducible characters of $W(I_2(m))$ of degree two by λ_a for $a = 1, \ldots, m - 1/2$ if m is odd and $a = 1, \ldots, m - 2/2$ if m is even. If m is even, there are two one-dimensional characters other than the trivial and the sign character, which will be denoted by ϵ_1 and ϵ_2, respectively.

Now define the idempotents $(F^\lambda)_{\lambda\in\mathrm{Irr}(W)}$ as

$$\begin{aligned} F^1 &= E_\emptyset, \\ F^{\lambda_a} &= F_{1,a} + F_{2,a}, \\ F^{\mathrm{sgn}} &= E_{\{1,2\}}, \end{aligned}$$

and, if m is even, define further

$$\begin{aligned} F^{\epsilon_1} &= F_{1,m/2} \qquad \text{and} \\ F^{\epsilon_2} &= F_{2,m/2}. \end{aligned}$$

Now, (Z1) and (Z2) hold by construction. It remains to verify (Z3) and (Z4).

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Step 3. Proving (Z3).

Now that we have the idempotents $F_{1,a}, F_{2,a}$ splitting E_1 and E_2, respectively, we can consider Ω as a quotient of the quiver which is obtained from $\mathcal{Q}_W$ by splitting the vertices labeled $\{1\}$ and $\{2\}$ into $\lfloor m/2 \rfloor$ vertices each. A priori this could lead to the edge elements $X_{1,2}$, $X_{2,1}$ being split into $\lfloor m/2 \rfloor^2$ new edge elements $F_{1,a}X_{1,2}F_{2,b}$ and $F_{2,a}X_{2,1}F_{1,b}$, respectively. We show that this does not happen and instead all edge elements not depicted in Figure 2 vanish.

This follows from Lemma 23 because $F_{1,a}$ can be obtained from $F_{2,a}$ by idempotent transporting and vice versa. Note that σ_a is invertible for $1 \leqslant a < m/2$, and $\sigma_a = 0$ for $a = m/2$. This means that $F_{1,m/2}$ and $F_{2,m/2}$ are the leftover idempotents. The lemma for idempotent transporting can be applied. Now, the following holds:

$$\sum_{a=0}^{\lfloor m/2 \rfloor} \sigma_a X_{1,2} F_{2,a} X_{2,1} = X_{1,2} \underbrace{\left(\sum_a \sigma_a F_{2,a} \right)}_{=E_2} X_{2,1} = X_{1,2} X_{2,1} = \sum_{a=0}^{\lfloor m/2 \rfloor} \sigma_a F_{1,a}.$$

Moreover, because $X_{1,2}F_{2,a}X_{2,1}$ is an idempotent, for $1 \leqslant a < m/2$, both sides of the equation $\sum_a \sigma_a X_{1,2} F_{2,a} X_{2,1} = \sum_a \sigma_a F_{1,a}$ describe the spectral decomposition of $X_{1,2}X_{2,1}$. Since the σ_a are pairwise distinct, one obtains $F_{1,a} = X_{1,2}F_{2,a}X_{2,1}$, and for symmetry reasons $X_{2,1}F_{1,a}X_{1,2} = F_{2,a}$ for all $1 \leqslant a < m/2$.

Now, Lemma 23 additionally implies $F_{1,a}X_{1,2} = X_{1,2}F_{2,a}$, so that $F_{1,a}X_{1,2}F_{2,b} = 0$ for $a \neq b$. Moreover, for symmetry reasons, also $F_{2,a}X_{2,1}F_{1,b} = 0$ for $a \neq b$.

If m is even, then it is also true that there are no edges $F_{1,m/2} \leftrightarrows F_{2,m/2}$. This can be seen as follows. By construction,

$$\prod_{1 \leqslant b < m/2} (X^2 - \sigma_b) = \frac{\widetilde{\tau}_{m-1}(X^2)}{X^2} = \frac{\tau_{m-1}(X)}{X} = \sum_{j=1}^{m-1} a_j X^{j-1}$$

holds. By inserting $X_{2,1}X_{1,2}$ for X^2 and multiplying by $X_{1,2}$, this gives

$$X_{1,2} \prod_{1 \leqslant b < m/2} (X_{2,1}X_{1,2} - \sigma_b) = \sum_{j=0}^{m-1} a_j \underbrace{X_{1,2}X_{2,1}\ldots}_{j \text{ factors}} \overset{(\alpha)}{=} 0.$$

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Now, multiplication with the denominator of (1) gives $X_{1,2}F_{2,m/2} = 0$, so that there are no edges from $F_{2,m/2}$ to any vertex labeled with $\{1\}$. Moreover, for symmetry reasons, there can be no edge from $F_{1,m/2}$ to any vertex labeled with $\{2\}$.

Therefore, the only edges that can exist are edges $F_{1,a} \leftrightarrows F_{2,a}$, edges $\emptyset \to F_{i,a}$, the edge $\emptyset \to \{12\}$ and edges $F_{i,a} \to \{12\}$. This means that (Z3) is satisfied if we define a partial order on $\mathrm{Irr}(W)$ by declaring sgn as the top element, 1 as the bottom element and all other elements as mutually incomparable.

Step 4. Proving (Z4).

For the characters of degree one, define $\psi_\lambda : k^{1\times 1} \to F^\lambda k\Omega F^\lambda$ by $\psi_\lambda(e_{11}) := F^\lambda$. This homomorphism is surjective because of the lack of closed loops based at F^λ in the quiver displayed in Figure 2. Therefore, $F^\lambda k\Omega F^\lambda = k \cdot F^\lambda$ holds and ψ_λ is surjective.

For the characters of degree two, define $\psi_{\lambda_a} : k^{2\times 2} \to F^{\lambda_a} k\Omega F^{\lambda_a}$ by

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \mapsto \begin{pmatrix} F_{1,a} & F^{\lambda_a} X_{1,2} F^{\lambda_a} \\ \sigma_a^{-1} F^{\lambda_a} X_{2,1} F^{\lambda_a} & F_{2,a} \end{pmatrix}.$$

This is a well-defined algebra homomorphism by construction of F^λ. It is surjective because $F^{\lambda_a} k\Omega F^{\lambda_a}$ is generated by the elements $F_I^{\lambda_a}$ and $F^{\lambda_a} X_{IJ} F^{\lambda_a}$, all of which are contained in the image of ψ_{λ_a}. ∎

5.4 Rank 3

THEOREM 29. *The decomposition conjecture is true for type A_3.*

We do not prove this in detail, since it is very similar to (although not formally a consequence of) the proof for A_4, which is presented in the next section. Full details can also be found in [5, Section 4.5].

THEOREM 30. *The decomposition conjecture is true for type B_3.*

Proof. We aim for a refinement of the compatibility graph as depicted in Figure 3 (where inclusion edges are again omitted for the sake of clarity). The relations of type (α) are crucial for this undertaking. We write (α^{st}) to denote that we have used the relation of type (α) belonging to the edge $s - t$ of the Dynkin diagram.

First, note that every good ring for B_3 contains $\mathbb{Z}[1/2]$, so that one is allowed to divide by two.

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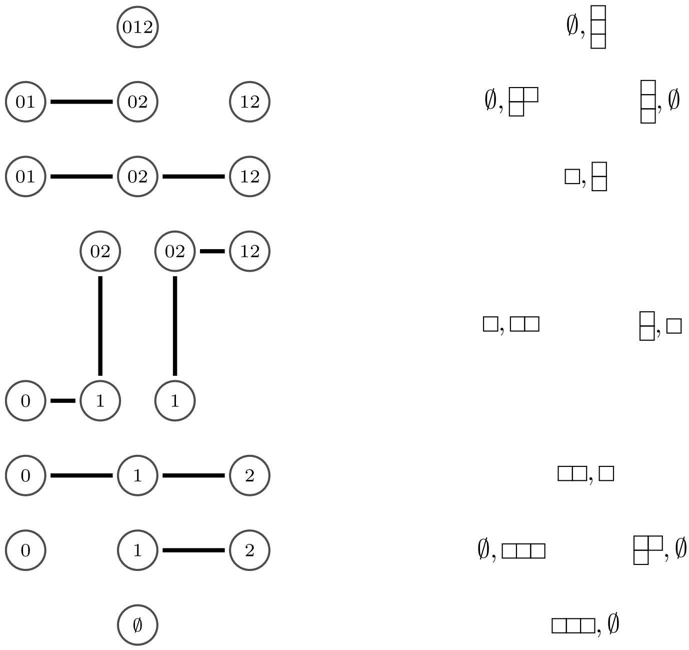

Figure 3.
Refined compatibility graph of B_3.

Step 1. (Z1) and (Z2).

We define elements $F_I^{\lambda,\mu}$ for all $I \subseteq S$ and all $(\lambda, \mu) \in \mathrm{Irr}(W)$ according to Table 1, where absent entries are understood to be defined as zero. We therefore prove that the $F_I^{\lambda,\mu}$ are pairwise orthogonal idempotents with $E_I = \sum_{\lambda,\mu} F_I^{\lambda,\mu}$.

The (α^{21})-relation $E_2 = X_{2,1}X_{1,2}$ implies that $F_1' := X_{1,2}X_{2,1}$ is an idempotent $\leqslant E_1$. The (α^{12})-relation

$$E_1 = X_{1,2}X_{2,1} + X_{1,02}X_{02,1}$$

implies that $F_1'' := X_{1,02}X_{02,1}$ also is an idempotent $\leqslant E_1$ which is orthogonal to F_1'. These two idempotents are decomposed further.

Recall that relations of type (α) use the polynomial τ_{m-1}, which for $m = 4$ has the form $\tau_{4-1}(T) = T^3 - 2T$. Therefore, (α^{01}) and (α^{10}) imply

$$0 = X_{0,1}X_{1,0}X_{0,1} + X_{0,1}X_{1,02}X_{02,1} - 2X_{0,1}, \tag{2}$$

$$0 = X_{1,0}X_{0,1}X_{1,0} + X_{1,02}X_{02,1}X_{1,0} - 2X_{1,0}. \tag{3}$$

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Table 1.
Expressions for the vertex idempotents of the refined compatibility graph of B_3 arranged in the same positions as the vertices in Figure 3.

		E_{012}		
$X_{01,02}F_{02}^{\emptyset,\ydiagram{2,1}}X_{02,01}$		$X_{02,01}X_{01,02} - F_{02}^{\ydiagram{1},\ydiagram{1,1}}$		$E_{12} - F_{12}^{\ydiagram{1},\ydiagram{1,1}} - F_{12}^{\ydiagram{1,1},\ydiagram{1}}$
$X_{01,02}F_{02}^{\ydiagram{1},\ydiagram{1,1}}X_{02,01}$		$\frac{1}{2}X_{02,12}X_{12,02} \cdot X_{02,01}X_{01,02}$		$X_{12,02}F_{02}^{\ydiagram{1},\ydiagram{1,1}}X_{02,01}$
	$X_{02,1}X_{1,02} - F_{02}^{\ydiagram{1,1},\ydiagram{1}}$		$X_{02,1}X_{1,02} \cdot X_{02,12}X_{12,02}$	$X_{12,02}F_{02}^{\ydiagram{1,1},\ydiagram{1}}X_{02,12}$
$X_{0,1}F_{1}^{\ydiagram{1},\ydiagram{2}}X_{1,0}$	$X_{1,0}X_{0,1} \cdot X_{1,02}X_{02,1}$		$X_{1,02}X_{02,1} - F_{1}^{\ydiagram{1},\ydiagram{2}}$	
$X_{0,1}F_{1}^{\ydiagram{2},\ydiagram{1}}X_{1,0}$		$\frac{1}{2}X_{1,0}X_{0,1} \cdot X_{1,02}X_{02,1}$		$X_{2,1}F_{1}^{\ydiagram{2},\ydiagram{1}}X_{1,2}$
$E_0 - F_{0}^{\ydiagram{1},\ydiagram{2}} - F_{0}^{\ydiagram{2},\ydiagram{1}}$		$X_{1,0}X_{0,1} - F_{1}^{\ydiagram{1},\ydiagram{2}}$		$X_{2,1}F_{1}^{\ydiagram{2},\ydiagram{1}}X_{1,2}$
		$E_{\emptyset}$		

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By setting $f := X_{1,0}X_{0,1}$, multiplying the first equation by $X_{1,0}$ from the left and the second by $X_{0,1}$ from the right, we obtain

$$0 = f^2 + fF_1'' - 2f, \tag{4}$$

$$0 = f^2 + F_1''f - 2f. \tag{5}$$

Thus,

$$f'' := fF_1'' = F_1''f \qquad \text{and} \qquad f' := fF_1' = F_1'f$$

are idempotents. We multiply (4) with F_1'' and (5) with F_1', and obtain

$$0 = f''^2 - f'', \tag{6}$$

$$0 = f'^2 - 2f'. \tag{7}$$

This gives us the following decomposition into orthogonal idempotents:

$$E_1 = F_1' + F_1'' = \underbrace{\left(\tfrac{1}{2}f'\right)}_{=F_1^{\ydiagram{2},\ydiagram{1}}} + \underbrace{\left(F_1' - \tfrac{1}{2}f'\right)}_{=F_1^{\ydiagram{2,1},\emptyset}} + \underbrace{\left(f''\right)}_{=F_1^{\ydiagram{1},\ydiagram{2}}} + \underbrace{\left(F_1'' - f''\right)}_{=F_1^{\ydiagram{1,1},\ydiagram{1}}}. \tag{8}$$

With these notations, $X_{1,0}X_{0,1} = 2F_1^{\ydiagram{2},\ydiagram{1}} + F_1^{\ydiagram{1},\ydiagram{2}}$ holds. We see that the other idempotents are now related either by transporting of idempotents along $\{1\} \to \{0\}$ or $\{1\} \to \{2\}$, or by applying the antiautomorphism δ to previously constructed elements. In particular, the $F_I^{\lambda,\mu}$ defined in Table 1 are pairwise orthogonal idempotents.

Step 3. Verifying (Z3).

We check that in Figure 3 only upward edges appear, so that the partial ordering on $\mathrm{Irr}(W)$ can be read off from the picture. We in fact show that the only edges not depicted in Figure 3 are inclusion edges.

The following holds:

$$\begin{aligned}
X_{0,1}F_1^{\ydiagram{2,1},\emptyset} &= X_{0,1}\left(F_1' - \tfrac{1}{2}f'\right) \\
&= X_{0,1}\left(E_1 - \tfrac{1}{2}X_{1,0}X_{0,1}\right)F_1' \\
&= \tfrac{1}{2}(2X_{0,1} - X_{0,1}X_{1,0}X_{0,1})F_1' \\
&\overset{(1)}{=} \tfrac{1}{2}(X_{0,1}F_1'')F_1' \\
&= 0, \\
X_{0,1}F_1^{\ydiagram{1,1},\ydiagram{1}} &= X_{0,1}(F_1'' - f'')
\end{aligned}$$

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$$
\begin{aligned}
&= X_{0,1}(E_1 - X_{1,0}X_{0,1})F_1''\\
&= (X_{0,1} - X_{0,1}X_{1,0}X_{0,1})F_1''\\
&\overset{(1)}{=} (X_{0,1}F_1'' - X_{0,1})F_1''\\
&= 0.
\end{aligned}
$$

This means that there cannot be edges from $F_1^{\yng(2,1),\emptyset}$ or $F_1^{\yng(1,1),\yng(1)}$ to vertices labeled with $\{0\}$. Since the idempotents labeled by $\{0\}$ were defined by transport of idempotents, it follows from Lemma 23 that $F_0^{\emptyset,\yng(3)}X_{0,1} = 0$ holds; that is, there are no edges from vertices labeled with $\{1\}$ to $F_0^{\emptyset,\yng(3)}$.

Analogously, both $F_1^{\yng(2,1),\emptyset}X_{1,0} = 0$ and $F_1^{\yng(1,1),\yng(1)}X_{1,0} = 0$ also hold. Therefore, there cannot be edges from vertices labeled with $\{0\}$ to $F_1^{\yng(2,1),\emptyset}$ or $F_1^{\yng(1,1),\yng(1)}$. Again, it follows from Lemma 23 that $X_{1,0}F_0^{\emptyset,\yng(3)} = 0$ holds; that is, there are no edges from $F_0^{\emptyset,\yng(3)}$ to vertices labeled with $\{1\}$.

Because the idempotents $F_0^{\yng(1),\yng(2)}$ and $F_0^{\yng(2),\yng(1)}$ were defined by transport of idempotents, there are no edges $F_0^{\yng(1),\yng(2)} \leftrightarrows F_1^{\yng(2),\yng(1)}$ or $F_0^{\yng(2),\yng(1)} \leftrightarrows F_1^{\yng(1),\yng(2)}$. Similarly, there are no edges $F_2^{\yng(2,1),\emptyset} \leftrightarrows F_1^{\yng(2),\yng(1)}$ or $F_2^{\yng(2),\yng(1)} \leftrightarrows F_1^{\yng(2,1),\emptyset}$.

Now, we use the symmetry given by δ and obtain the same result for vertices labeled with $\{01\}$, $\{02\}$ and $\{12\}$.

It remains to verify that there are no edges $F_1^{\yng(1),\yng(2)} \leftrightarrows F_{02}^{\yng(1,1),\yng(1)}$ or $F_1^{\yng(1,1),\yng(1)} \leftrightarrows F_{02}^{\yng(1),\yng(2)}$. To this end, we prove that the idempotents $F_{02}^{\yng(1,1),\yng(1)}$ and $F_{02}^{\yng(1),\yng(2)}$ are also given by a transport of idempotents. This follows from an application of the (α^{10})-relation:

$$
0 = X_{02,1}X_{1,0}X_{0,1} + X_{02,1}X_{1,02}X_{02,1} + X_{02,12}X_{12,02}X_{02,1} - 2X_{02,1}. \tag{9}
$$

Multiplying with $X_{1,02}$ from the left, and using $X_{1,0}X_{0,1} = F_1^{\yng(1),\yng(2)} + 2F_1^{\yng(2),\yng(1)}$ as well as $X_{02,12}X_{12,02} = F_{02}^{\yng(1,1),\yng(1)} + 2F_{02}^{\yng(1),\yng(1,1)}$ and $F_{02}'' := X_{02,1}X_{1,02} = \delta(F_1'')$, we obtain

$$
\begin{aligned}
0 &= X_{02,1}(F_1^{\yng(1),\yng(2)} + 2F_1^{\yng(2),\yng(1)})X_{1,02} + F_{02}''F_{02}'' + (F_{02}^{\yng(1,1),\yng(1)} + 2F_{02}^{\yng(1),\yng(1,1)})F_{02}'' - 2F_{02}''\\
&= X_{02,1}(F_1^{\yng(1),\yng(2)} + 2F_1^{\yng(2),\yng(1)})X_{1,02} + F_{02}^{\yng(1,1),\yng(1)}F_{02}'' + 2\underbrace{F_{02}^{\yng(1),\yng(1,1)}F_{02}''}_{=0} - F_{02}''\\
&= X_{02,1}(F_1^{\yng(1),\yng(2)} + 2F_1^{\yng(2),\yng(1)})X_{1,02} + F_{02}^{\yng(1,1),\yng(1)} - F_{02}''\\
&= X_{02,1}(F_1^{\yng(1),\yng(2)} + 2F_1^{\yng(2),\yng(1)})X_{1,02} + (-F_{02}^{\yng(1),\yng(2)}).
\end{aligned}
$$

Hence, we obtain

$$
F_{02}^{\yng(1),\yng(2)} = X_{02,1}F_1^{\yng(1),\yng(2)}X_{02,1} + 2 \cdot X_{02,1}F_1^{\yng(2),\yng(1)}X_{1,02}.
$$

The (α^{21})-relation

$$0 = X_{02,1}X_{1,2}$$

implies $X_{02,1}F_1' = 0$, so that $X_{02,1}F_1^{\ydiagram{2},\ydiagram{1}} = 0$, because $F_1^{\ydiagram{2},\ydiagram{1}} \leqslant F_1'$. Therefore, we obtain

$$F_{02}^{\ydiagram{1},\ydiagram{2}} = X_{02,1}F_1^{\ydiagram{1},\ydiagram{2}}X_{1,02}.$$

That is, $F_{02}^{\ydiagram{1},\ydiagram{2}}$ is a transported idempotent along the edge $\{02\} \leftrightarrows \{1\}$. Because of $F_1'' = F_1^{\ydiagram{1},\ydiagram{2}} + F_1^{\ydiagram{1,1},\ydiagram{1}}$,we also obtain

$$F_{02}^{\ydiagram{1,1},\ydiagram{1}} = X_{02,1}F_1^{\ydiagram{1,1},\ydiagram{1}}X_{1,02},$$

which, together with Lemma 23, implies that there are no edges other than the ones displayed in Figure 3 between vertices labeled with $\{1\}$ and $\{02\}$. This shows that (Z3) holds.

Step 4. Verifying (Z4).

There is not much to do for the characters of degree one. We define $\psi_{\lambda,\mu} : \mathbb{Z}[1/2]^{1\times 1} \to F^{\lambda,\mu}\mathbb{Z}[1/2]\Omega F^{\lambda,\mu}$ to be the only possible morphism, namely $\psi_{\lambda,\mu}(e_{11}) := F^{\lambda,\mu}$. The surjectivity of these maps is automatic because the four components for the one-dimensional characters in the refined compatibility graph have no edges, and therefore $F^{\lambda,\mu}\mathbb{Z}[1/2]\Omega F^{\lambda,\mu} = \mathbb{Z}[1/2]F^{\lambda,\mu}$.

Table 2 lists all of the morphisms $\psi_{\lambda,\mu} : \mathbb{Z}[\frac{1}{2}]^{d_{\lambda,\mu}\times d_{\lambda,\mu}} \to F^{\lambda,\mu}\mathbb{Z}[\frac{1}{2}]\Omega F^{\lambda,\mu}$ for the characters of degree two and three, where we use the notation $X_{IJ}^{\lambda,\mu} := F^{\lambda,\mu}X_{IJ}F^{\lambda,\mu}$.

We have used again that $\mathbb{Z}[1/2]^{d\times d}$ is the $\mathbb{Z}[1/2]$-algebra given by the presentation in Lemma 25. These relations are satisfied by construction of the $F^{\lambda,\mu}$, and therefore all of the maps in the table are well-defined algebra morphisms.

The construction of $\psi_{\lambda,\mu}$ ensures that all idempotents $F_I^{\lambda,\mu}$ for all $I \subseteq S$ and all $F^{\lambda,\mu}X_{IJ}F^{\lambda,\mu}$ for transversal edges $I \leftrightarrows J$ are contained in the image of $\psi_{\lambda,\mu}$. For $(\lambda,\mu) \in \{(\emptyset,\ydiagram{2,1}),(\ydiagram{2,1},\emptyset),(\ydiagram{2},\ydiagram{1}),(\ydiagram{1},\ydiagram{1,1})\}$, this is already enough to guarantee surjectivity, because all edges in the component of $F^{\lambda,\mu}$ are transversal edges.

For $\chi_{\ydiagram{1},\ydiagram{2}}$, on the other hand, there could be an inclusion edge $\{0\} \to \{0,2\}$, and for $\chi_{\ydiagram{1,1},\ydiagram{1}}$,there could be an inclusion edge $\{1\} \to \{1,2\}$. To complete the proof, we show that this is not the case by using the (β^{20})-relation:

$$\begin{aligned} X_{02,0}^{\ydiagram{1},\ydiagram{2}} &= F^{\ydiagram{1},\ydiagram{2}}X_{02,0}F_0^{\ydiagram{1},\ydiagram{2}} \\ &= F^{\ydiagram{1},\ydiagram{2}}X_{02,0}(X_{0,1}F_1^{\ydiagram{1},\ydiagram{2}}X_{1,0}) \end{aligned}$$

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Table 2.
Morphisms $\psi_{\lambda,\mu} : \mathbb{Z}[1/2]^{d_{\lambda,\mu}\times d_{\lambda,\mu}} \twoheadrightarrow F^{\lambda,\mu}\mathbb{Z}[1/2]\Omega F^{\lambda,\mu}$ for $2 \leqslant d_{\lambda,\mu} \leqslant 3$.

Character $\chi_{\lambda,\mu}$	Map $\psi_{\lambda,\mu}$		
$\chi_{\ydiagram{2,1},\emptyset}$	$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_1^{\ydiagram{2,1},\emptyset} & X_{1,2}^{\ydiagram{2,1},\emptyset} \\ X_{2,1}^{\ydiagram{2,1},\emptyset} & F_2^{\ydiagram{2,1},\emptyset} \end{pmatrix}$
$\chi_{\emptyset,\ydiagram{2,1}}$	$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_{02}^{\emptyset,\ydiagram{2,1}} & -X_{02,01}^{\emptyset,\ydiagram{2,1}} \\ -X_{01,02}^{\emptyset,\ydiagram{2,1}} & F_{01}^{\emptyset,\ydiagram{2,1}} \end{pmatrix}$
$\chi_{\ydiagram{2},\ydiagram{1}}$	$\begin{pmatrix} e_{11} & e_{12} & \\ e_{21} & e_{22} & e_{23} \\ & e_{32} & e_{33} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_0^{\ydiagram{2},\ydiagram{1}} & X_{0,1}^{\ydiagram{2},\ydiagram{1}} & \\ \frac{1}{2}X_{1,0}^{\ydiagram{2},\ydiagram{1}} & F_1^{\ydiagram{2},\ydiagram{1}} & X_{1,2}^{\ydiagram{2},\ydiagram{1}} \\ & X_{2,1}^{\ydiagram{2},\ydiagram{1}} & F_2^{\ydiagram{2},\ydiagram{1}} \end{pmatrix}$
$\chi_{\ydiagram{1},\ydiagram{1,1}}$	$\begin{pmatrix} e_{11} & e_{12} & \\ e_{21} & e_{22} & e_{23} \\ & e_{32} & e_{33} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_{12}^{\ydiagram{1},\ydiagram{1,1}} & -X_{12,02}^{\ydiagram{1},\ydiagram{1,1}} & \\ -\frac{1}{2}X_{02,12}^{\ydiagram{1},\ydiagram{1,1}} & F_{02}^{\ydiagram{1},\ydiagram{1,1}} & -X_{02,01}^{\ydiagram{1},\ydiagram{1,1}} \\ & -X_{01,02}^{\ydiagram{1},\ydiagram{1,1}} & F_{01}^{\ydiagram{1},\ydiagram{1,1}} \end{pmatrix}$
$\chi_{\ydiagram{1},\ydiagram{2}}$	$\begin{pmatrix} e_{11} & e_{12} & \\ e_{21} & e_{22} & e_{23} \\ & e_{32} & e_{33} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_0^{\ydiagram{1},\ydiagram{2}} & X_{0,1}^{\ydiagram{1},\ydiagram{2}} & \\ -X_{1,0}^{\ydiagram{1},\ydiagram{2}} & F_1^{\ydiagram{1},\ydiagram{2}} & X_{1,02}^{\ydiagram{1},\ydiagram{2}} \\ & X_{02,1}^{\ydiagram{1},\ydiagram{2}} & F_{02}^{\ydiagram{1},\ydiagram{2}} \end{pmatrix}$
$\chi_{\ydiagram{1,1},\ydiagram{1}}$	$\begin{pmatrix} e_{11} & e_{12} & \\ e_{21} & e_{22} & e_{23} \\ & e_{32} & e_{33} \end{pmatrix}$	$\mapsto$	$\begin{pmatrix} F_{12}^{\ydiagram{1,1},\ydiagram{1}} & X_{12,02}^{\ydiagram{1,1},\ydiagram{1}} & \\ X_{02,12}^{\ydiagram{1,1},\ydiagram{1}} & F_{02}^{\ydiagram{1,1},\ydiagram{1}} & -X_{02,1}^{\ydiagram{1,1},\ydiagram{1}} \\ & -X_{1,02}^{\ydiagram{1,1},\ydiagram{1}} & F_1^{\ydiagram{1,1},\ydiagram{1}} \end{pmatrix}$

$$
\begin{aligned}
&= F^{\ydiagram{1},\ydiagram{2}}(X_{02,0}X_{0,1})F_1^{\ydiagram{1},\ydiagram{2}}X_{1,0} \\
&\overset{(\beta^{20})}{=} F^{\ydiagram{1},\ydiagram{2}}(X_{02,2}X_{2,1} + X_{02,12}X_{12,1} - X_{02,01}X_{01,1})F_1^{\ydiagram{1},\ydiagram{2}}X_{1,0} \\
&= (X_{02,2}^{\ydiagram{1},\ydiagram{2}}X_{2,1}^{\ydiagram{1},\ydiagram{2}} + X_{02,12}^{\ydiagram{1},\ydiagram{2}}X_{12,1}^{\ydiagram{1},\ydiagram{2}} - X_{02,01}^{\ydiagram{1},\ydiagram{2}}X_{01,1}^{\ydiagram{1},\ydiagram{2}})F_1^{\ydiagram{1},\ydiagram{2}}X_{1,0}.
\end{aligned}
$$

All summands within the brackets disappear because the $\ydiagram{1},\ydiagram{2}$ component in the graph of Figure 3 has no vertices labeled $\{2\}$, $\{12\}$ or $\{01\}$. Using the symmetry given by δ, the equation $F_{12}^{\ydiagram{1,1},\ydiagram{1}}X_{12,1} = 0$ also holds. ∎

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5.5 Rank 4

THEOREM 31. *The decomposition conjecture is true for type A_4.*

Proof. We use an analogous strategy to that used before and use the relations of type (α). Again, we write (α^{st}) to denote that we have used the relation of type (α) belonging to the edge $s - t$ of the Dynkin diagram, and similarly for (β)-type relations.

Our goal is to decompose the compatibility graph as in Figure 4(a) (inclusion edges have been omitted for the sake of clarity).

Step 1. Verifying (Z1) and (Z2).

We define idempotents $F_I^\lambda \leqslant E_I$ for all $I \subseteq S, \lambda \in \operatorname{Irr}(W)$, and set $F^\lambda := \sum_I F_I^\lambda$. First, note that, by Lemma 25, the idempotents of a matrix algebra are given by evaluating loops in the quiver. Looking at the quiver, we want to arrive at Figure 4(a). We therefore define the idempotents F_I^λ either as one of the E_I at the boundary of the compatibility graph or as loops connecting inner vertices to those outer vertices. More precisely, we use the definitions in Figure 4(b), where all F_I^λ not appearing there are understood to be defined as zero.

Once these elements have been defined, we have to prove that they are in fact idempotents and $E_I = \sum_\lambda F_I^\lambda$ is an orthogonal decomposition. (Z2) will then be satisfied because $F^\lambda E_I = F_I^\lambda = E_I F^\lambda$ holds by definition.

The (α^{12})-relation implies

$$E_1 = X_{1,2}X_{2,1},$$

from which it follows that $F_2^{\ydiagram{4,1}}$ is an idempotent $\leqslant E_2$; namely, the idempotent obtained by transport of $E_1 \leqslant E_1$ along the edge $\{1\} \to \{2\}$. From the (α^{12})-relation

$$E_2 = X_{2,1}X_{1,2} + X_{2,13}X_{13,2},$$

we deduce that $F_2^{\ydiagram{3,2}}$ is the leftover idempotent of this transport. By applying the nontrivial graph automorphism, we obtain that $F_3^{\ydiagram{4,1}}$ and $F_{24}^{\ydiagram{3,2}}$ are idempotents as well, and by applying the antiautomorphism δ, we find that $F_{134}^{\ydiagram{2,1,1,1}}, F_{13}^{\ydiagram{2,2,1}}, F_{124}^{\ydiagram{2,1,1,1}}$ and $F_{2,4}^{\ydiagram{2,2,1}}$ are idempotents too. Moreover, because Lemma 23 also gives us orthogonality with the leftover idempotent, we are done with all except the two-element subsets of S.

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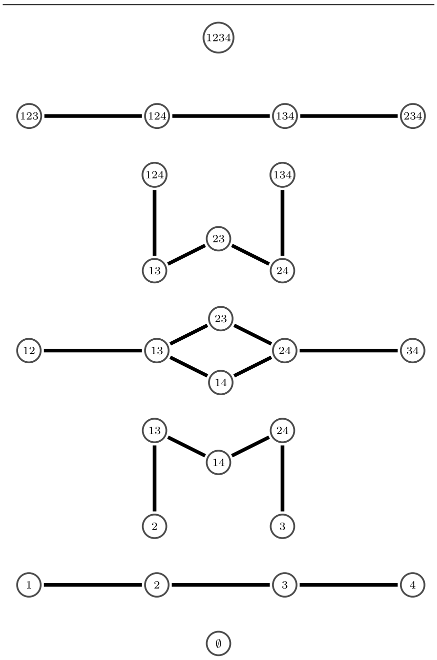
(a)



(b)

λ	I	F_I^λ
$\ydiagram{1,1,1,1}$	$\{1,2,3,4\}$	E_{1234}
$\ydiagram{2,1,1}$	$\{1,2,3\}$	E_{123}
	$\{1,2,4\}$	$X_{124,123}X_{123,124}$
	$\{1,3,4\}$	$X_{134,234}X_{234,134}$
	$\{2,3,4\}$	E_{234}
$\ydiagram{2,2,1}$	$\{1,2,4\}$	$X_{124,13}X_{13,124}$
	$\{1,3\}$	$X_{13,124}X_{124,13}$
	$\{2,3\}$	$X_{23,13}X_{13,124}X_{124,13}X_{13,23}$
	$\{2,4\}$	$X_{24,134}X_{134,24}$
	$\{1,3,4\}$	$X_{134,24}X_{24,134}$
$\ydiagram{3,1,1}$	$\{1,2\}$	E_{12}
	$\{1,3\}$	$X_{13,12}X_{12,13}$
	$\{2,3\}$	$X_{23,13}X_{13,12}X_{12,13}X_{13,23}$
	$\{1,4\}$	$X_{14,24}X_{24,34}X_{34,24}X_{24,14}$
	$\{2,4\}$	$X_{24,34}X_{34,24}$
	$\{3,4\}$	E_{34}
$\ydiagram{3,2}$	$\{2\}$	$X_{2,13}X_{13,2}$
	$\{1,3\}$	$X_{13,2}X_{2,13}$
	$\{1,4\}$	$X_{14,13}X_{13,2}X_{2,13}X_{13,14}$
	$\{2,4\}$	$X_{24,3}X_{3,24}$
	$\{3\}$	$X_{3,24}X_{24,3}$
$\ydiagram{4,1}$	$\{1\}$	E_1
	$\{2\}$	$X_{2,1}X_{1,2}$
	$\{3\}$	$X_{3,4}X_{4,3}$
	$\{4\}$	E_4
$\ydiagram{5}$	$\emptyset$	$E_\emptyset$

Figure 4.
(a) The refined compatibility graph for A_4 and (b) vertex idempotents of the refined compatibility graph for A_4.

By transporting $F_2^{\ydiagram{3,2}} = X_{2,13}X_{13,2}$ along $\{2\} \to \{13\}$, we obtain the idempotent

$$\begin{aligned} X_{13,2}F_2^{\ydiagram{3,2}}X_{2,13} &= X_{13,2}E_2X_{2,13} - X_{13,2}F_2^{\ydiagram{4,1}}X_{2,13} \\ &= X_{13,2}X_{2,13} - X_{13,2}X_{2,1}\underbrace{X_{1,2}X_{2,13}}_{=0 \text{ by } (\alpha^{12})} \\ &= F_{13}^{\ydiagram{3,2}}. \end{aligned}$$

By applying δ and the graph automorphism, we find that $F_{13}^{\ydiagram{2,2,1}}, F_{24}^{\ydiagram{3,2}}$ and $F_{134}^{\ydiagram{2,2,1}}$ are also idempotents.

The (α^{23})-relation

$$E_{12} = X_{12,13}X_{13,12}$$

implies that $F_{13}^{\ydiagram{3,1,1}}$ is the idempotent obtained by transporting E_{12} along $\{12\} \to \{13\}$.

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Considering the (α^{32})-relations

$$\begin{aligned} E_{13} &= X_{13,2}X_{2,13} + X_{13,12}X_{12,13} + X_{13,124}X_{124,13} = F_{13}^{\ydiagram{3,2}} + F_{13}^{\ydiagram{3,1,1}} + F_{13}^{\ydiagram{2,2,1}}, \\ 0 &= X_{2,13}X_{13,12} = X_{2,13}X_{13,124}, \\ 0 &= X_{12,13}X_{13,2} = X_{12,13}X_{13,124}, \\ 0 &= X_{124,13}X_{13,2} = X_{124,13}X_{13,12}, \end{aligned}$$

we find that $F_{13}^{\ydiagram{3,2}}, F_{13}^{\ydiagram{3,1,1}}, F_{13}^{\ydiagram{2,2,1}}$ constitute an orthogonal decomposition of E_{13}.

By applying the graph automorphism, we find that $F_{24}^{\ydiagram{3,1,1}}, F_{24}^{\ydiagram{3,2}}$ and $F_{24}^{\ydiagram{2,2,1}}$ are pairwise idempotents as well.

We are now almost done. We still need to look at the innermost vertices $\{2, 3\}$ and $\{1, 4\}$ of the compatibility graph. The following (α^{34})-relation holds:

$$E_{13} = X_{13,14}X_{14,13} + X_{13,124}X_{124,13}.$$

This means that $X_{13,14}X_{14,13} = F_{13}^{\ydiagram{3,2}} + F_{13}^{\ydiagram{3,1,1}}$. Transporting these two idempotents along $\{13\} \to \{14\}$, we obtain $F_{14}^{\ydiagram{3,2}}$ and $F_{14}^{\ydiagram{3,1,1}}$. By symmetry, $F_{23}^{\ydiagram{3,1,1}}$ and $F_{23}^{\ydiagram{2,2,1}}$ are idempotents as well.

From the (α^{43})- and (α^{21})-relations

$$E_{14} = X_{14,13}X_{13,14} \qquad \text{and} \qquad E_{23} = X_{23,13}X_{13,23},$$

we can infer that the two leftover idempotents for these transports vanish. Therefore, we get orthogonal decompositions $E_{14} = F_{14}^{\ydiagram{3,2}} + F_{14}^{\ydiagram{3,1,1}}$ and $E_{23} = F_{23}^{\ydiagram{3,1,1}} + F_{23}^{\ydiagram{2,2,1}}$.

Step 2. Verifying (Z3).

We prove that the only edges between the components not displayed in Figure 4(a) are inclusion edges, from which it follows that the dominance ordering on $\{\lambda \vdash 5\}$ is the sought-after partial ordering. Because we have constructed all idempotents by transport of idempotents, most transversal edges split into parallel edges. This eliminates almost all possible transversal edges between different components.

The (α^{32})-relation

$$X_{13,2}X_{2,3} = 0$$

implies that $F_2^{\ydiagram{3,2}} X_{2,3} = 0$, so that there is no transversal edge emanating from $E_3 = F_3^{\ydiagram{4,1}} + F_3^{\ydiagram{3,2}}$ and going to $F_2^{\ydiagram{3,2}}$. By symmetry, there are no

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transversal edges going from E_2 to $F_3^{\ydiagram{3,2}}$, which shows that there are only inclusion edges between the $\ydiagram{4,1}$ and the $\ydiagram{3,2}$ component. Applying δ, we find the same between the $\ydiagram{2,2,1}$ and the $\ydiagram{2,1,1,1}$ component.

The only other possibility is transversal edges of the form $\{14\} \leftrightarrows \{24\}$ and $\{23\} \leftrightarrows \{24\}$, because we have not used idempotent transport along these edges. Instead, we worked with $\{13\} \leftrightarrows \{14\}$ and $\{13\} \leftrightarrows \{23\}$.

Consider the (β^{24})-relation

$$X_{24,23}X_{23,13} + X_{24,2}X_{2,13} = X_{24,14}X_{14,13} + X_{24,134}X_{134,13},$$

which implies

$$\begin{aligned}
&F_{24}^{\ydiagram{3,2}} \cdot X_{24,14} \cdot F_{14}^{\ydiagram{3,1,1}} \\
&\quad = F_{24}^{\ydiagram{3,2}} \cdot \underbrace{X_{24,14} \cdot X_{14,13}} F_{13}^{\ydiagram{3,1,1}} X_{13,14} \\
&\quad \overset{(\beta^{24})}{=} F_{24}^{\ydiagram{3,2}} \cdot (-X_{24,134}X_{134,13} + X_{24,23}X_{23,13} + X_{24,2}X_{2,13}) F_{13}^{\ydiagram{3,1,1}} X_{24,14} \\
&\quad = -X_{24,3} \underbrace{X_{3,24}X_{24,134}}_{=0 \text{ by } (\alpha^{32})} X_{134,13} F_{13}^{\ydiagram{3,1,1}} X_{24,14} \\
&\qquad + X_{24,3} \underbrace{X_{3,24}X_{24,23}}_{=0 \text{ by } (\alpha^{43})} X_{23,13} F_{13}^{\ydiagram{3,1,1}} X_{24,14} \\
&\qquad + X_{24,3}X_{3,24}X_{24,2} \underbrace{X_{2,13} F_{13}^{\ydiagram{3,1,1}}}_{=0} X_{24,14} \\
&\quad = 0.
\end{aligned}$$

Similarly, combining the (β)- and (α)-relations, one shows that the transversal edge $\{14\} \to \{24\}$ splits into a pair of parallel edges, as displayed in Figure 4(a), and all four of the possible cross-component edges are indeed zero. Applying δ, we find that the same holds between the $\ydiagram{3,1,1}$ and the $\ydiagram{2,2,1}$ component.

This shows that, even for the edges $\{14\} \leftrightarrows \{24\}$ and $\{23\} \leftrightarrows \{24\}$, the idempotents on both sides are given by transporting idempotents, and hence there can only be parallel edges, as depicted in Figure 4(a). The only other possible edges are those not depicted in this picture; in other words, the inclusion edges.

Step 3. Verifying (Z4).

We construct surjective homomorphisms $\psi_\lambda : \mathbb{Z}^{d_\lambda\times d_\lambda} \to F^\lambda \Omega F^\lambda$. We use the presentation of $\mathbb{Z}^{d_\lambda\times d_\lambda}$ from Lemma 25.

We use the abbreviation $X_{IJ}^\lambda := F^\lambda X_{IJ} F^\lambda$. By construction, the equation $X_{IJ}^\lambda X_{JI}^\lambda = F_I^\lambda$ holds for all transversal edges $I \leftrightarrows J$ if F_I^λ and F_J^λ are both nonzero, as well as $X_{IJ}^\lambda = X_{JI}^\lambda = 0$ otherwise.

If we denote the vertices of a component in Figure 4(a) with its index set (which is possible without conflicts since no index set occurs more than once), then

$$\psi_\lambda : \mathbb{Z}^{d_\lambda\times d_\lambda} \to F^\lambda \Omega F^\lambda,\, e_{II} \mapsto F_I^\lambda,\, e_{IJ} \mapsto X_{IJ}^\lambda$$

defines a morphism $\psi_\lambda : \mathbb{Z}^{d_\lambda\times d_\lambda} \to F^\lambda \Omega F^\lambda$ for those components that are straight lines without their inclusion edges; that is, all components except the one labeled with $\lambda = \ydiagram{3,1,1}$.We prove surjectivity of ψ_λ, which is equivalent to showing that all X_{IJ}^λ are contained in the image ψ_λ. For the transversal edges, this is clear from the construction. Therefore, we are done for $\lambda = \ydiagram{5}$, $\ydiagram{4,1}$, $\ydiagram{2,1,1,1}$ and $\ydiagram{1,1,1,1,1}$.For $\lambda = \ydiagram{3,2}$, we must consider the inclusion edges $X_{13,3}$ and $X_{24,2}$. We use the relations of type (β):

$$\begin{aligned}
X_{24,2}^{\ydiagram{3,2}} &= F_{24}^{\ydiagram{3,2}} \cdot X_{24,2} \cdot F_2^{\ydiagram{3,2}} \\
&= X_{24,3}X_{3,24} \cdot (X_{24,2} \cdot X_{2,13})X_{13,2} \\
&\overset{(\beta^{42})}{=} X_{24,3}X_{3,24}(X_{24,14}X_{14,13} + X_{24,134}X_{134,13} - X_{24,23}\underbrace{X_{23,13})X_{13,2}}_{=0} \\
&= X_{24,3}X_{3,24}X_{24,14}X_{14,13}X_{13,2} + X_{24,3}\underbrace{X_{3,24}X_{24,134}}_{=0}X_{134,13}X_{13,2} \\
&= X_{24,3}^{\ydiagram{3,2}}X_{3,24}^{\ydiagram{3,2}}X_{24,14}^{\ydiagram{3,2}}X_{14,13}^{\ydiagram{3,2}}X_{13,2}^{\ydiagram{3,2}} \quad \text{because } X_{24,2}^{\ydiagram{3,2}} \in F^{\ydiagram{3,2}}\Omega F^{\ydiagram{3,2}} \\
&\in \operatorname{im}(\psi_{\ydiagram{3,2}}).
\end{aligned}$$

By applying the graph automorphism, we obtain $X_{13,3}^{\ydiagram{3,2}} \in \operatorname{im}(\psi_{\ydiagram{3,2}})$,and by applying the antiautomorphism δ, we obtain $X_{124,24}^{\ydiagram{2,2,1}}$, $X_{134,13}^{\ydiagram{2,2,1}} \in \operatorname{im}(\psi_{\ydiagram{2,2,1}})$. Therefore, all X_{IJ}^λ are contained in the image of ψ_λ for $\lambda = \ydiagram{3,2}$, $\ydiagram{2,2,1}$, and surjectivity holds in both cases.

It remains to handle the case $\lambda = \ydiagram{3,1,1}$. We sort the two-element sets in the order $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, and claim that the following

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homomorphism $\psi_{\ydiagram{3,1,1}} : \mathbb{Z}^{6\times 6} \to F^{\ydiagram{3,1,1}} \Omega F^{\ydiagram{3,1,1}}$ is well defined:

$$\begin{pmatrix} e_{11} & e_{12} & & & & \\ e_{21} & e_{22} & e_{23} & & & \\ & e_{32} & e_{33} & e_{34} & & \\ & & e_{43} & e_{44} & e_{45} & \\ & & & e_{54} & e_{55} & e_{56} \\ & & & & e_{65} & e_{66} \end{pmatrix}$$

$$\mapsto \begin{pmatrix} F^\lambda_{12} & X^\lambda_{12,13} & & & & \\ X^\lambda_{13,12} & F^\lambda_{13} & X^\lambda_{13,14} & & & \\ & X^\lambda_{14,13} & F^\lambda_{14} & X^\lambda_{14,13}X^\lambda_{13,23} & & \\ & & X^\lambda_{23,13}X^\lambda_{13,14} & F^\lambda_{23} & X^\lambda_{23,24} & \\ & & & X^\lambda_{24,23} & F^\lambda_{24} & X^\lambda_{24,34} \\ & & & & X^\lambda_{34,24} & F^\lambda_{34} \end{pmatrix}.$$

Most relations from Lemma 25 are satisfied by construction of the idempotents. We still need to verify

$$X^\lambda_{14,13}X^\lambda_{13,23} \cdot X^\lambda_{23,13}X^\lambda_{13,14} = F^\lambda_{14} \qquad \text{and}$$
$$X^\lambda_{23,13}X^\lambda_{13,14} \cdot X^\lambda_{14,13}X^\lambda_{13,23} = F^\lambda_{23}.$$

These equations follow from the (α)-relations $E_{13} = X_{13,23}X_{23,13}$ and $E_{24} = X_{24,14}X_{14,24}$.

We verify the surjectivity of ψ_λ. By construction, most X^λ_{IJ} are already contained in the image. We only have to consider the edges between $F^\lambda_{14} \leftrightarrows F^\lambda_{24}$ and $F^\lambda_{13} \leftrightarrows F^\lambda_{23}$:

$$\begin{aligned} X^\lambda_{23,13} &= X^\lambda_{23,13}F^\lambda_{13} \\ &= X^\lambda_{23,13}(X^\lambda_{13,14}X^\lambda_{14,13}) \\ &= (X^\lambda_{23,13}X^\lambda_{13,14})X^\lambda_{14,13} \in \mathrm{im}(\psi_\lambda) \end{aligned}$$

and

$$\begin{aligned} X^\lambda_{24,14} &= X^\lambda_{24,14}F^\lambda_{14} \\ &= X^\lambda_{24,14}(X^\lambda_{14,13}X^\lambda_{13,14}) \\ &= (X^\lambda_{24,14}X^\lambda_{14,13})X^\lambda_{13,14} \end{aligned}$$

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$$\begin{aligned} &\overset{(\beta)}{=} (X^{\lambda}_{24,23} X^{\lambda}_{23,13}) X^{\lambda}_{13,14} \\ &= X^{\lambda}_{24,23} (X^{\lambda}_{23,13} X^{\lambda}_{13,14}) \in \operatorname{im}(\psi_{\lambda}). \end{aligned}$$

Applying δ, we find $X^{\lambda}_{13,23}, X^{\lambda}_{14,24} \in \operatorname{im}(\psi_{\lambda})$ as well. Therefore, ψ_{λ} is surjective. ∎

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