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A BOUNDARY RIGIDITY PROBLEM FOR HOLOMORPHIC MAPPINGS ON SOME WEAKLY PSEUDOCONVEX DOMAINS

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ABSTRACT. In this paper, we study the boundary version of the classical Cartan theorem. We show that for some weakly pseudoconvex domains, when a holomorphic self-mapping has a sufficiently high order of contact (which depends only on the geometric properties of the domains) with the identical map at some boundary point, then it must coincide with the identity.

0. **Introduction.** Let D be a bounded domain in C^n, and let $\mathrm{Hol}(D,D)$ denote the set of holomorphic mappings from D to itself. The well-known rigidity theorem of Cartan states that if $f \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z - z_0|)$, as $z \longrightarrow z_0$ for some point $z_0 \in D$, then $f(z) \equiv z$.

A new rigidity problem of holomorphic mappings originating from the work of Burns-Krantz attempts to create a boundary version of the above fundamental result. It can be formulated as follows:

PROBLEM 0.1. Let D be a bounded domain in C^n, and let p be a point on ∂D. Does there exist a positive number m_p, depending only on the geometric properties of ∂D at p, so that for any $f \in \mathrm{Hol}(D,D)$, if

$$f(z) = z + o(|z-p|^{m_p}), \quad \text{as } z \longrightarrow p,$$

then $f(z) \equiv z$ over D?

The following is the first result in this direction obtained in [BK].

THEOREM 0.2 (BURNS-KRANTZ, [BK]). *Let D be a bounded strongly pseudoconvex domain in C^n, and let $p \in \partial D$. If $f \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z-p|^3)$ as $z \longrightarrow p$, then $f(z) \equiv z$.*

The proof of Cartan's theorem has no possibility of being extended to the boundary situation, for its main ingredients are normal family arguments and classical Cauchy estimates that can not apply when $z_0 \notin D$. To prove their theorem, Burns-Krantz used some sort of 'continuum extremal disks' method, which works pretty well for those domains that can be embedded into the domains satisfying the following properties:

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PROPERTY A. There exist a large number of holomorphic retracts smoothly passing through the point p under study.

PROPERTY B. The holomorphic retracts in Property A are uniquely determined by their tangent directions at p.

In [H1], we verified that the 'continuum extremal disks' method may also work for some weakly pseudoconvex domains (for example, egg domains) which satisfy the above two properties (also see [BK]).

This work is a continuation of [BK] and [H1]. Our main purpose is to improve Burns-Krantz's argument so that it is valid for a class of weakly pseudoconvex domains that satisfy either a portion of Property A or the following:

PROPERTY C. There exists one uniquely determined holomorphic retract passing through the point p.

Here are two typical results in this note:

THEOREM 0.3. *Let D be a bounded domain in C^n defined as*

$$D = \{z = (z_1, z') \in C^1 \times C^{n-1} : |z_1|^2 + h(z') < 1\},$$

where h is a nonnegative smooth function with value zero if and only if $z' = 0$. Denote $(1, 0')$ by p. If $f \in \mathrm{Hol}(D, D)$ is such that $f(z) = z + o(|z-p|^3)$ as $z \longrightarrow p$, then $f(z)$ is the identity.

THEOREM 0.4. *Let D be a bounded smoothly convex domain of finite type in C^n, and let $p \in \partial D$. Then there exists a number m, depending only on the geometric properties of ∂D near p, such that for every $f \in \mathrm{Hol}(D, D)$, if $f(z) = z + o(|z-p|^m)$ as $z \longrightarrow p$, then $f(z) \equiv z$ on D (see §3 for the determination of this m).*

Our ideas can be briefly described as follows: For the domains with Property (C), with some assumptions about the behavior of $f \in \mathrm{Hol}(D, D)$ at $p \in \partial D$, we first show that f is the identity on some holomorphic curve. Then we study the eigenvalues of the Jacobian of f on this curve and apply the Hopf lemma to obtain the conclusion. For the domains satisfying a portion of Property A, when $f \in \mathrm{Hol}(D, D)$ has a sufficiently high order of contact with the identity at p, we prove that f must be a biholomorphism. We then show that it is the identity by the results in [H1] and [BP2].

The paper is organized as follows: In Section 1, we study the behaviors of complex geodesics on finite type domains. For example, we will prove that all complex geodesics on finite type domains can be extended Hölder continuously up to the boundary. As a result, we will obtain some existence theorems for complex geodesics with prescribed data. This section is crucial for our later discussion. In Section 2, we start by recalling an elementary result on the unit disc. Then we present a basic theorem by the first aforementioned idea and discuss some of its applications, including Theorem 0.3. Section 3 is devoted to the proof of Theorem 0.4 along the lines of our second aforementioned idea.

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1. **Complex geodesics on finite type domains.** For the sake of brevity, we first fix some notation.

In all that follows, the symbol Δ will stand for the unit disc in C^1, and the symbol $|\cdot|$ will denote the euclidean norm in C^n $(n \geq 1)$.

If D_1, D_2 are two bounded domains in C^n, we use $\mathrm{Hol}(D_1, D_2)$ to denote the space of all holomorphic mappings from D_1 to D_2. For the sequence $\{f_k\} \in \mathrm{Hol}(D_1, D_2)$ and the point $w \in \bar{D}_2$, the expression '$f_k \longrightarrow w$' will mean that $\{f_k\}$ converges to the constant map w uniformly on compacta. Moreover, when $f \in \mathrm{Hol}(D_1, D_2)$ with $D_1 = D_2$, we denote by f^k the k-th iterate of f defined inductively by $f^1 = f, \ldots, f^k = f \circ f^{k-1}$ for $k = 2, 3, \ldots$.

We recall the concepts of Kobayashi metric and Kobayashi distance on a domain $D \subset\subset C^n$.

Let $z \in D$, and let X be a holomorphic vector in $T_z D$. The Kobayashi infinitesimal metric $\kappa_D(z, X)$ is defined as

$$\kappa_D(z, X) = \inf\{|\xi| : \exists \phi \in \mathrm{Hol}(\Delta, D) \text{ so that } \phi(0) = z \text{ and } d_0\phi(\xi) = X\}.$$

For every two points $z_1, z_2 \in D$, the Kobayashi distance between them is given by the following integral form:

$$K_D(z_1, z_2) = \inf_{\substack{\gamma:[0,1] \to D \\ \gamma(0) = z_1 \\ \gamma(1) = z_2}} \int_\gamma \kappa_D\big(\gamma(t), \gamma'(t)\big)\, dt.$$

For the disk Δ, it is an elementary fact that

$$\kappa_\Delta(\tau, a) = \frac{|a|}{(1 - |\tau|^2)}, \quad K_\Delta(0, \tau) = \frac{1}{2} \log \frac{1 + |\tau|}{1 - |\tau|}, \qquad \text{for every } a \in C^1 \text{ and } \tau \in \Delta.$$

Let $D \subset\subset C^n$, and let $\delta(z)$ denote the distance of z to ∂D. Then $\delta^*(z)$, defined by $-\delta(z)$ inside D and $\delta(z)$ outside D, is a standard defining function D. A point $p \in \partial D$ is said to be *a smooth point* if δ^* is smooth at p and $d_p\delta^* \neq 0$. A smooth boundary point p is said to be *a finite type point* in the sense of D'Angelo [D] if

$$t_p = \sup_{\substack{\alpha \in \mathrm{Hol}(\Delta, C^n) \\ \alpha(0) = p \\ \alpha \text{ nontrivial}}} \frac{\mathrm{ord}_0\, \delta^*\big(\alpha(\tau)\big)}{\mathrm{ord}_0\, \alpha(\tau)} < \infty.$$

D is called *a domain* of finite type if all boundary points are of finite type. In this case, by D'Angelo's semi-continuity theorem, $t_D = \sup_{p \in \partial D} t_p < \infty$.

For the Kobayashi metric on a finite type domain, we have the following important result:

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THEOREM 1.1 ([G], [CA], [DF3], [CO]). *Let D be a bounded pseudoconvex domain of finite type in C^n. Then there exist a positive number $\epsilon \leq \frac{1}{2}$ and a constant C depending only on D such that for every point $z \in D$ and holomorphic vector $X \in T_z^{(1,0)}D$, it holds that*

$$\kappa_D(z,X) \geq \frac{C|X|}{\delta^{\epsilon}(z)}.$$

Here, when $\dim(D) = 2$, *then ϵ can be chosen as t_D^{-1}; while D is strongly pseudoconvex, ϵ can be chosen as $\frac{1}{2}$.*

For simplicity, we shall call such an ϵ *a K-admissible number* of D.

We also need the concepts of 'big' and 'small' horospheres introduced in Abate [Ab2]. Let $D \subset\subset C^n$ be a bounded domain, $p \in \partial D$, $z \in D$, and let $R > 0$. The small horosphere $E_z^D(p,R)$ and the big horosphere $F_z^D(p,R)$ of center p, pole z, and radius R are defined as

$$E_z^D(p,R) = \left\{a \in D : \overline{\operatorname{Lim}}_{w \to p}\Big(K_D(a,w) - K_D(z,w)\Big) < \frac{1}{2}\log R\right\}$$

$$F_z^D(p,R) = \left\{a \in D : \underline{\operatorname{Lim}}_{w \to p}\Big(K_D(a,w) - K_D(z,w)\Big) < \frac{1}{2}\log R\right\}$$

On Δ, for every $z \in \Delta$ and $R > 0$, it is known [A3] (see p. 46) that $E_0^{\Delta}(1,R) = F_0^{\Delta}(1,R) = E(1,R) = \{\tau \in \Delta : (|1-\tau|^2)/(1-|\tau|^2) < R\}$, which is a euclidean disc of radius $R/(1+R)$.

PROPOSITION 1.2. *Let D be a smooth bounded domain in C^n ($n \geq 1$), and let $p \in \partial D$. If $f \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z-p|)$, as $z \longrightarrow p$ along the normal direction, then for every $z_0 \in D$ and $R > 0$, $f\big(E_{z_0}(p,R)\big) \subset F_{z_0}(p,R)$.*

PROOF. By Proposition 2.4.15 of [Ab3] (see p. 270) we need only show that there exists a sequence $\{z_k\} \subset D$ converging to p such that $\{f(z_k)\}$ converges to p and

$$\overline{\operatorname{Lim}}_{k \to \infty}\Big(K_D(z_0,z_k) - K_D\big(z_0,f(z_k)\big)\Big) \leq 0.$$

Let n_p be the inward normal vector of ∂D at p, $z_k = p + \frac{n_p}{k}$, and let $\gamma_k : I = [0,1] \longrightarrow D$ be the segment from z_k to $f(z_k)$. Obviously, when $k >> 1$, then z_k and γ_k lie in D. Denote by $B(a,r)$ the euclidean ball of center a and radius r. Then, when $k >> 1$, for every $X \in T_{\gamma_k(t)}^{(1,0)}D$, we have

$$\kappa_D\big(\gamma_k(t),X\big) \leq \kappa_{B(\gamma_k(t),\frac{1}{2k})}\big(\gamma_k(t),X\big) \leq C|X|k.$$

Here C is a positive constant independent of k and t. Hence,

$$\begin{aligned} K_D(z_0,z_k) - K_D\big(z_0,f(z_k)\big) &\leq K_D\big(z_k,f(z_k)\big) \leq \int_{\gamma_k} \kappa_D\big(\gamma_k(t),\gamma_k'(t)\big)\,dt \\ &\leq Ck|f(z_k) - z_k| = o(1), \quad \text{as } k \longrightarrow \infty. \end{aligned}$$

Since $f(z_k)$ obviously converges to p, this completes the proof. ∎

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COROLLARY 1.3. *Let D be a bounded smooth domain in C^n, and let $p \in \partial D$. Suppose that there exist a point $z_0 \in D$ and $R > 0$ so that $\emptyset \neq E^D_{z_0}(p,R)$ and $\overline{F^D_{z_0}}(p,R) \cap \partial D = \{p\}$. Assume that $f(z) \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z-p|)$, as $z \longrightarrow p$, and such that $\{f^k\}$ compactly diverges. Then $f^k \longrightarrow p$.*

PROOF. Let g be a limit point of the sequence $\{f^k\}$. By the hypothesis, we see that g is a holomorphic mapping from D to ∂D and there is a subsequence $\{f^{k_j}\}$ of $\{f^k\}$ with $f^{k_j} \longrightarrow g$ on compacta. Notice that, for every natural number k, f^k satisfies the hypotheses in Proposition 1.2. It thus follows that $f^k\big(E^D_z(p,R)\big) \subset F^D_z(p,R)$ for every $z \in D$ and $R > 0$. Hence $g\big(E^D_{z_0}(p,R)\big) \subset \overline{F^D_{z_0}}(p,R) \cap \partial D$ ($= \{p\}$, by the hypothesis). This implies $g \equiv p$. So $f^k \longrightarrow p$. ∎

REMARK 1.4. We observe that, according to a result of Abate [A2] (Theorem 1.7), if D is a bounded strongly pseudoconvex domain, then $\overline{F^D_{z_0}}(p,R) \cap \partial D = \{p\}$ for every $p \in \partial D$ and $R > 0$. Moreover, it was proved in [H2] (see the proof of Lemma 7 there) that for every open neighborhood U of $p \in \partial D$, there exist $z_0 \in D$ and $R > 0$ so that $\emptyset \neq E^D_{z_0}(p,R) \subset F^D_{z_0}(p,R) \subset U$.

COROLLARY 1.5. *Let D be a bounded smooth domain in C^1, $z_0 \in D$, and let $p \in \partial D$. If $f \in \mathrm{Hol}(D,D)$ is such that $f(z_0) = z_0$ and $f(z) = z + o(|z-p|)$ as $z \longrightarrow p$, then $f(z) \equiv z$ over D.*

PROOF. Consider the sequence $\{f^k\}$. By the hypothesis, the above remark, and Corollary 1.3, it is easy to see that $\{f^k\}$ cannot converge to just a single point on $\bar{D}$. Thus, by the Heins iteration theorem (see Theorem 0.3 of [A2], for instance) f must be a biholomorphism and hence, in view of Theorem 1 of [K1] (or Theorem 1 of [H1]) we can conclude that $f \equiv \mathrm{id}$. ∎

Before turning to the rest of the propositions of this section, we recall that a holomorphic map $\phi\colon \Delta \longrightarrow D$ is called a *complex geodesic* (respectively, a *complex infinitesimal geodesic*) of D if, for every pair $\tau_1, \tau_2 \in \Delta$ (respectively, $\tau \in \Delta$, $a \in C^1$), we have $K_\Delta(\tau_1,\tau_2) = K_D\big(\phi(\tau_1),\phi(\tau_2)\big)$ (respectively, $K_\Delta(\tau,a) = K_D\big(\phi(\tau), a\phi'(t)\big)$). We also recall that a complex geodesic $\phi\colon \Delta \longrightarrow D$ is said to be *normalized* if $\delta\big(\phi(0)\big) = \max_{\tau\in\Delta} \delta\big(\phi(\tau)\big)$.

PROPOSITION 1.6. *Let D be a bounded pseudoconvex domain of finite type in C^n, and let ϵ be a K-admissible number for D. Then, for every compact subset $K \subset\subset D$, there exists a constant $C(K)$ such that for every complex geodesic $\phi\colon \Delta \longrightarrow D$ with $\phi(0) \in K$, we have*

$$|\phi(\tau_1) - \phi(\tau_2)| \leq C(K)|\tau_1 - \tau_2|^\epsilon, \quad \textit{for all } \tau_1, \tau_2 \in \Delta.$$

PROOF. We note that ϕ is also a complex infinitesimal geodesic (for example, see the discussion of [Ab3], p. 347). Hence, it follows from Theorem 1.1 that for every $\tau \in \Delta$,

$$\kappa_\Delta(\tau,1) = \kappa_D\big(\phi(t),\phi'(\tau)\big) \geq C_1(D)\frac{|\phi'(\tau)|}{\delta^\epsilon\big(\phi(\tau)\big)},$$

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and therefore

$$\begin{aligned}|\phi'(\tau)| &\leq C_2(D)\,\kappa_\Delta(\tau,1)\,\delta^\epsilon\big(\phi(\tau)\big),\\ &\leq C_2(D)\,\delta^\epsilon\big(\phi(\tau)\big)/(1-|\tau|^2).\end{aligned} \tag{1.3}$$

On the other hand, from Theorem 2.3.51 of [Ab3], we know that for every $z_1, z_2 \in D$ it holds that

$$K_D(z_1, z_2) \leq C'(z_1) - \frac{1}{2}\log\delta(z_2),$$

where $C'(z_1)$ is independent of z_2 and is bounded on every compact set. Thus, by the fact that ϕ preserves the Kobayashi metrics, we see that

$$\frac{1}{2}\log\frac{1+|\tau|}{1-|\tau|} = K_\Delta(\tau,0) = K_D\big(\phi(\tau),\phi(0)\big) \leq C'\big(\phi(0)\big) - \frac{1}{2}\log\delta\big(\phi(\tau)\big),$$

and hence

$$\delta\big(\phi(\tau)\big) \leq C_3(1-|\tau|^2). \tag{1.4}$$

Combining (1.3) and (1.4), we now obtain

$$|\phi'(\tau)| \leq C_4(1-|\tau|^2)^{\epsilon-1},$$

where C_4 shares the same property as does C'. So from the classical Hardy-Littlewood theorem (p. 324, Theorem 2.6.26 of [Ab3]) the proposition follows. ∎

Let us denote by F the set of normalized complex geodesics of D.

PROPOSITION 1.7. *Under the same hypotheses as Proposition 1.6, there exists a constant $C(D)$ depending only on D such that for every $\phi \in F$, and $\tau_1, \tau_2 \in \Delta$, we have*

$$|\phi(\tau_1) - \phi(\tau_2)| \leq C(D)|\tau_1 - \tau_2|^{\epsilon/2}.$$

PROOF. First, by a general fact proved in [CHL] (which holds for all bounded smooth domains), there exists a constant C_1 depending only on D so that

$$\delta\big(\phi(\tau)\big) \leq C_1(1-|\tau|^2)^{1/2}$$

for every $\phi \in F$. Hence, as argued above, we have

$$|\phi'(\tau)| \leq C_2(1-|\tau|^2)^{-1-\epsilon/2}$$

for some positive constant C_2 depending only on D. Once again, by the Hardy-Littlewood theorem, our proof is complete. ∎

REMARK 1.8. When $\mathrm{Dim}(D) = 2$, as we have seen in Theorem 1.1, ϵ can be taken to be $1/t_D$.

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The following example shows that, in general, we can not expect that all complex geodesics on a domain of finite type can be smoothly extended up to the boundary:

For the natural number k and $a \in \Delta$, let E_k be the egg domain:

$$E_k = \{(z,w) \in C^2 : |z|^2 + |w|^{2k} < 1\},$$

and let $\phi_a \colon E(1, |a|^{-1}) \longrightarrow E_k$ defined by

$$\phi_a(\tau) = \big(\tau, |a|^{1/2k}(1-\tau)^{1/k}\big).$$

Then ϕ_a is a complex geodesic. But it only can be extended $1/k$-Hölder continuously up to the boundary (except in the trivial case that $a = 0$).

PROPOSITION 1.9. *Let D be a smooth convex domain of finite type in C^n. Then*

(a) For each pair $z \in D$ and $p \in \partial D$, there is a complex geodesic ϕ of D so that $\phi(0) = z$ and $\phi(1) = p$.

(b) For every $p \in \partial D$ and $q \in \partial D$ with $p \neq q$, there is a complex geodesic ϕ satisfying that $\phi(e^{i\theta_1}) = p$ and $\phi(e^{i\theta_2}) = q$ for some θ_1 and $\theta_2 \in R^1$.

PROOF. The argument is the same as that for strongly linear convex domains [CHL], except that we use Proposition 1.6, Proposition 1.7, and Royden-Wong's results [RW] instead of the machinery established by Lempert [L1]. For example, to prove (a), we pick out a sequence $\{z_k\} \in D$ converging to p, and find a complex geodesic ϕ_k (see Theorem 2.6.19 of [A3], p. 321) so that $\phi_k(0) = z$, $z_k \in \phi_k\big((0,1)\big)$ for $k = 1, 2, \ldots$. By the fact that ∂D contains only trivial holomorphic curves, it follows from a normal family argument that some subsequence of $\{\phi_n\}$ converges to a complex geodesic ϕ which, in view of Proposition 1.6, must satisfy the given conditions: $\phi(0) = z$ and $\phi(1) = p$. ∎

COROLLARY 1.10. *Let D be a bounded convex domain of finite type, and let $p \in \partial D$. Then for every $z_0 \in D$ and $R > 0$, it holds that $\overline{F^D_{z_0}(p,R)} \supset \{p\}$.*

PROOF. By Proposition 1.9, we can find a Hölder continuous curve $\gamma(t) \colon I = [0,1] \longrightarrow \bar{D}$ with $\gamma(0) = z_0$, $\gamma(1) = p$ such that, for every $0 \le x_1 \le x_2 \le x_3 < 1$,

$$K_D\big(\gamma(x_1), \gamma(x_2)\big) + K_D\big(\gamma(x_2), \gamma(x_3)\big) = K_D\big(\gamma(x_1), \gamma(x_3)\big).$$

Now, from the definition, it obviously follows that $\gamma(t) \in F^D_{z_0}(p,R)$ when t is close enough to 1. ∎

2. **A basic result and it applications.** We begin this section by proving the following lemma, which was partially obtained in [L2], [CHL], and was proved in the form of Theorem 0.1 in [BK]. The argument here essentially follows the ideas of Burns-Krantz's.

LEMMA 2.1. *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f(z) = z + o(|z-1|)$ as $z \longrightarrow 1$. Then either*

(a) $f(\tau) \equiv \tau$, or

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(b) $f^k \longrightarrow 1$ *and* $\overline{\mathrm{Lim}}_{\tau\to 1}\frac{|f(\tau)-\tau|}{|1-\tau|^3} > 0$.

PROOF. Assume that f is not the identity. Then we shall show that (b) is the only possibility.

We first claim that f has no interior fixed point. Otherwise, without loss of generality, we may assume that $f(0) = 0$. Hence, it follows from the Schwartz lemma that $|f(\tau)| < |\tau|$ for $\tau \neq 0$. Therefore

$$\mathrm{Re}\left(\frac{f(\tau)-\tau}{f(\tau)+\tau}\right) = \frac{|f(\tau)|^2 - |\tau|^2}{|\tau + f(\tau)|^2} < 0, \quad \text{for } \tau \neq 0.$$

We note that the displayed function is harmonic on Δ and continuously approaches its maximal value 0 at the rate of $o(|\tau - 1|)$ when $\tau \longrightarrow 1$. So, by the Hopf lemma, we can conclude that $\mathrm{Re}\left(\frac{f(\tau)-\tau}{f(\tau)+\tau}\right) \equiv 0$, *i.e.*, $f(\tau) \equiv \tau$. This is a contradiction.

Secondly, by the Wolff-Denjoy theorem (see Theorem 1.3.9 of [Ab3], p. 78) and Corollary 1.3, we see that $f^k \longrightarrow 1$. Moreover, by Proposition 2.1, we see that $f\big(E(1,R)\big) \subset E(1,R)$ for every $R > 0$ where, as stated in Section 1, $E(1,R) = \left\{\tau \in \Delta : \frac{1-|\tau|^2}{|\tau-1|^2} > R\right\}$. Hence,

$$\frac{1-|f(\tau)|^2}{|1-f(\tau)|^2} \geq \frac{1-|\tau|^2}{|1-\tau|^2}, \quad \text{for } \tau \in \Delta. \tag{2.1}$$

Let the harmonic function $\xi(\tau)\colon \Delta \longrightarrow R^1$ be defined by

$$\xi(\tau) = \mathrm{Re}\left(\frac{1+\tau}{1-\tau} - \frac{1+f(\tau)}{1-f(\tau)}\right).$$

Then (2.1) yields $\xi(\tau) \leq 0$ on Δ.

If $\overline{\lim}\frac{|f(\tau)-\tau|}{|1-\tau|^3} = 0$, *i.e.*, if $f(\tau) = \tau + o(|1-\tau|^3)$ as $\tau \longrightarrow 1$, then, from a direct computation, it follows that $\xi(\tau) = 0 + o(|\tau - 1|)$ as $\tau \longrightarrow 1$. Thus, again by Hopf's lemma, we have $\xi(\tau) \equiv 0$, *i.e.*, $f(\tau) \equiv \tau$ over Δ. That contradicts the assumption that f is not the identity.

This completes the proof. ∎

We now state and prove our basic theorem.

THEOREM 2.2. *Let $D \subset\subset C^n$ be a bounded domain, and let $\phi\colon \Delta \longrightarrow D$ be a proper holomorphic mapping which is ϵ-Hölder continuous up to the boundary with $\phi(1) = p \in \partial D$. Suppose that there exists a defining function ρ of D which is smooth over $\bar{\phi}(\Delta)$ with $d\rho \neq 0$ on $\phi(\partial\Delta)$ and that there exists a positive number $\mu \leq 1$ so that $-(-\rho \circ \phi)^\mu$ is subharmonic. Then, if $f \in \mathrm{Hol}(D,D)$ fixed $\phi(\Delta)$, and if*

$$f(z) = z + o(|z-p|^{2/\mu\epsilon}), \quad \text{as } z \longrightarrow p,$$

we can conclude that $f(z) \equiv z$ over D.

PROOF. Obviously we can assume $n > 1$.

Consider the holomorphic function λ on Δ defined by

$$\lambda(\tau) = \left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + \cdots + \frac{\partial f_n}{\partial z_n}\right) \circ \phi(\tau),$$

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which is the sum of the eigenvalues of the Jacobian of f at $\phi(\tau)$. Since f fixes $\phi(\Delta)$, then from the classical Carathéodory-Cartan-Kaup-Wu theorem (see [K2], for example) it follows that $\mathrm{Re}\big(\lambda(\tau)\big) \leq n$ and that equality holds at some point if and only if f is the identity.

Let $h(\tau) = -\big(-\rho \circ \phi(\tau)\big)^{\mu}$, defined on Δ. Since h is subharmonic and attains the maximal value 0 on $\partial\Delta$, it follows from Hopf's lemma that $-h(\tau) \geq C_1(1-|\tau|^2)$, and hence that $-\rho \circ \phi(\tau) \geq C_2(1-|\tau|^2)^{1/\mu}$ for $\tau \in \Delta$. Recalling that $-\rho \approx \delta$ near $\phi(\Delta)$ and

$$E(1,1) = \left\{\tau : \frac{1-|\tau|^2}{|1-\tau|^2} > 1\right\},$$

we now find that

$$\delta \circ \phi(\tau) \geq C_3(|1-\tau|)^{2/\mu} \quad \text{on } E(1,1). \tag{2.2}$$

Let $U_j = \{z_j \in C^1 : |z_j - \phi_j(\tau)| < C_3/\sqrt{2n} \cdot |1-\tau|^{2/\mu}\}$ for $j = 1, \ldots, n$, and let $P = U_1 \times \cdots \times U_n$. Then, since the distance from each point in P to $\phi(\tau) = \big(\phi_1(\tau), \ldots, \phi_n(\tau)\big)$ is less than $\frac{1}{2}C_3(|1-\tau|)^{2/\mu}$, it follows easily that $P \subset\subset D$. Moreover, for every point $z = (z_1, \ldots, z_n) \in P$, it holds that $|z-p| \leq |z-\phi(\tau)| + |\phi(\tau)-p| \leq C_4|1-\tau|^{2/\mu} + C_5|1-\tau|^{\epsilon} \approx C_5|1-\tau|^{\epsilon}$.

So, by making use of the Cauchy formula, we have

$$\begin{aligned}
\left|\frac{\partial f_j}{\partial z_j} \circ \phi(\tau) - 1\right| &= \left|\frac{1}{2\pi}\int_{\partial U_j} \frac{f_j\big(\phi_1(\tau), \ldots, \xi, \ldots, \phi_n(\tau)\big) - \xi}{\big(\xi - \phi_j(\tau)\big)^2}\, d\xi\right| \\
&\leq C_6 \sup_{\xi \in \partial U_j} \big(|f_j(\ldots, \xi, \ldots) - \xi|\big)/(|1-\tau|^{2/\mu}) \\
&\leq o(|1-\tau|^{2/\mu})/(|1-\tau|^{2/\mu}) \\
&= o(1), \quad \text{as } \tau \in E(1,1) \longrightarrow 1.
\end{aligned}$$

Thus we have proved that $\mathrm{Re}\,\lambda(\tau)$ continuously approaches its maximal value n as $\tau \in E(1,1)$ goes to 1.

On the other hand, when we restrict τ to $(0,1)$, it is easy to see, from the above argument, that $\delta\big(\phi(\tau)\big) \geq C_7|1-\tau|^{1/\mu}$, and thus that $\mathrm{Re}\,\lambda(\tau) = n + o(|1-\tau|^{1/\mu})$. So applying Hopf's lemma to $\mathrm{Re}\,\lambda(\tau)$ on $E(1,1)$ at 1, we conclude that $\mathrm{Re}\,\lambda(\tau) \equiv n$. This completes the proof. ∎

The first application of this theorem is our aforementioned Theorem 0.3.

PROOF OF THEOREM 0.3. Let $f(z) = \big(f_1(z), \tilde{f}(z)\big) \in C^1 \times C^{n-1}$ satisfy the hypotheses in Theorem 0.3, and let $\phi : \Delta \longrightarrow D$ be defined by $\phi(\tau) = (\tau, 0)$. Obviously, $f_1 \circ \phi \in \mathrm{Hol}(\Delta, \Delta)$ and $f_1 \circ \phi(\tau) = \tau + o(|\tau - 1|^3)$ as $\tau \longrightarrow 1$. From Lemma 2.1, it follows that $f_1 \circ \phi(\tau) \equiv \tau$. Moreover, since

$$\big|f_1\big(\phi(\tau)\big)\big|^2 + h\Big(\tilde{f}\big(\phi(\tau)\big)\Big) < 1,$$

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we can conclude that $\mathrm{Lim}_{\tau\to\partial\Delta} h\Big(\tilde{f}\big(\phi(\tau)\big)\Big) = 0$. From the given hypotheses on h, it now follows that $\sup_{\tau\in\partial\Delta}\big|\tilde{f}\big(\phi(\tau)\big)\big| = 0$. Therefore, by the maximum principle, $\tilde{f}\big(\phi(\tau)\big) \equiv 0$. So we have proved that f fixes $\phi(\Delta)$.

To finish the argument, we now need only apply Theorem 2.2 with $\epsilon = 1$, $\mu = 1$, and $\rho = |z_1|^2 + h(z') - 1$. ■

COROLLARY 2.3. *Let*

$$D = \Big\{(z_1,\ldots,z_n) \in C^n : |z_1|^2 + \sum_{j=2}^{n} e\cdot\exp\Big(-\frac{1}{|z_j|}\Big) < 1\Big\},$$

and let $p = (1,0,\ldots,0)$. If $f \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z-p|^3)$ as $z \longrightarrow p$, then f must be the identity.

COROLLARY 2.4. *Let $P = (p_0,\ldots,p_n)$ be a set of positive integers with $p_j > 1$ for $j > 1$. Let*

$$D_P = \Big\{Z = (z,w) \in C^{p_0} \times C^n : |z|^2 + \sum_{\alpha} C_\alpha |w^\alpha|^2 < 1\Big\},$$

where C_α's are non-negative constants which make D_P a bounded domain, and the sum is taken over those multi-indices such that

$$\sum_{j>0} \frac{\alpha_j}{p_j} = 1,$$

Denote by q the boundary point $(1,0,\ldots,0)$ of D_P. If $f \in \mathrm{Hol}(D_P,D_P)$ is such that $f(Z) = Z + o(|Z-q|^3)$ as $z \longrightarrow q$, then $f(Z) \equiv Z$ over D_P.

Another application of Theorem 2.2 is the following generalized Burns-Krantz theorem:

THEOREM 2.5 (BURNS-KRANTZ). *Let $D \subset\subset C^n$ ($n \geq 1$) be a bounded domain, and let $p \in \partial D$ be a strongly pseudoconvex point. If $f \in \mathrm{Hol}(D,D)$ is such that $f(z) = z + o(|z-p|^3)$ as $z \longrightarrow p$, then $f(z) \equiv z$ over D.*

PROOF. Since p is a strongly pseudoconvex point, then by a standard argument (for example, see [K2]), we can find local holomorphic coordinates $\{U, h(z)\}$ centered at p so that $h(U\cap\partial D)$ is a strongly convex hypersurface in C^n. Noting that f is continuous at p, we may choose two strongly convex domains Ω_1, Ω_2 satisfying the following properties:

(a) $\Omega_1 \subset \Omega_2 \subset h(U\cap D)$, and $h\circ f\circ h^{-1}(\Omega_1) \subset \Omega_2$;

(b) $\partial\Omega_1 \cap h(\partial D\cap U) = \partial\Omega_1 \cap \partial\Omega_2$;

(c) $\partial\Omega_1 \cap h(\partial D\cap U)$ is a piece of strongly convex hypersurface containing p.

From a result of Lempert [L1], there is a complex geodesic ϕ of Ω_2 so that $\phi(\bar{\Delta}) \subset \Omega_1$ and $\phi(1) = p$. Let $\pi\colon \Omega_2 \longrightarrow \Delta$ be a holomorphic left inverse of this ϕ (for the existence of such a mapping, see [L3]). By the above properties of Ω_1 and Ω_2, we therefore get a well-defined holomorphic map $\xi(\tau) \in \mathrm{Hol}(\Delta,\Delta)$ defined as $\xi(\tau) = \pi\circ h\circ f\circ h^{-1}\circ\phi(\tau)$. Because ϕ, π are smooth up to the boundary ([L1], [L3]) it is easy to check that $\xi(\tau) = \tau + o(|1-\tau|^3)$

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as $\tau \longrightarrow 1$. Thus, from Lemma 2.1, it follows that $\xi(\tau) \equiv \tau$. Thus, by the monotonicity property of Kobayashi distance, we see that $\tilde{\phi} \stackrel{\Delta}{=} h \circ f \circ h^{-1} \circ \phi$ is also a complex geodesic of Ω_2 satisfying $\tilde{\phi}(1) = \phi(1)$, $\tilde{\phi}'(1) = \phi(1)$, and $\tilde{\phi}''(1) = \phi''(1)$. So by the uniqueness property of complex geodesics on strongly convex domains [L1], $h \circ f \circ h^{-1} \circ \phi \equiv \phi$, *i.e.*, f fixes $h^{-1} \circ \phi(\Delta)$. Now the proof easily follows from Theorem 2.2 with $\epsilon, \mu = 1$, and an obvious choice of ρ. ∎

The following theorem is an analogue of Lemma 2.1 for strongly convex domains.

THEOREM 2.6. *Let D be a strongly convex domain in C^n $(n > 1)$ and $p \in \partial D$. Suppose that $f \in \mathrm{Hol}(D, D)$ is such that $f(z) = z + o(|z-p|^2)$ as $z \longrightarrow p$. Then, either*

(a) $f(z) \equiv z$ over D, or

(b) $f^k \longrightarrow p$ and $\overline{\mathrm{Lim}}_{z \to p} \frac{|f(z)-z|}{|z-p|^3} > 0$.

PROOF. Suppose that $f(z)$ is not the identity. We prove then that (b) is the only possibility.

In a manner similar to the argument in Lemma 2.1, we first show that f has no fixed point on D. Actually, if $z_0 \in D$ is such that $f(z_0) = z_0$, then we may construct a complex geodesic ϕ of D satisfying $\phi(0) = z_0$ and $\phi(1) = p$. Let π be a holomorphic left inverse of ϕ, and let $\xi(\tau) = \pi \circ f \circ \phi(\tau)$. Since $\phi, \pi \in C^\infty(\bar{\Delta})$, we have

$$\begin{aligned} \xi(\tau) &= \pi\Big(\phi(\tau) + o\big(|\phi(\tau) - p|^2\big)\Big) = \pi\big(\phi(\tau)\big) + O\Big(o\big(|\phi(\tau) - \phi(1)|\big)\Big) \\ &= \tau + o(|\tau - 1|^2), \quad \text{as } \tau \longrightarrow 1. \end{aligned}$$

Noting that $\xi(0) = 0$, we see, by Lemma 2.1, that $\xi(\tau) \equiv \tau$. From the monotonicity of Kobayashi distance, it therefore follows that $f \circ \phi$ is also a complex geodesic. Again by the uniqueness property, we see that $f \circ \phi \equiv \phi$, *i.e.*, f fixes $\phi(\Delta)$. From Theorem 2.2 with $\epsilon, \mu = 1$, it now follows that $f(z) \equiv z$. This is a contradiction.

As soon as we know that f is fixed point free on D, then, from Abate's iteration theorem [A2], Corollary 1.3, and Burns-Krantz's theorem, it follows that (b) is the only possibility. This completes the proof. ∎

COROLLARY 2.7. *Let D be a strongly convex domain in C^n $(n > 1)$, and let $p \in \partial D$. If $f \in \mathrm{Hol}(D, D)$ is such that $f(z_0) = z_0$ for some $z_0 \in D$ and $f(z) = z + o(|z-p|^2)$ as $z \longrightarrow p$, then $f(z) \equiv z$.*

REMARK 2.8. The following example shows that Corollary 2.7 cannot be improved in general (compare with Lemma 2.1):

Let B be the unit ball in C^n $(n > 1)$, $p = (1, 0, \ldots, 0) \in \partial B$, and let $f \in \mathrm{Hol}(B, B)$ be defined by

$$f(z_1, \ldots, z_n) = (z_1, z_1 z_2, \ldots, z_1 z_n).$$

Obviously, $f(z_1, 0, \ldots, 0) = (z_1, 0, \ldots, 0)$, and $f(z) = z + 0(|z-p|)$ as $z \longrightarrow p$. But f is not the identity.

REMARK 2.9. We believe that Corollary 2.7 is valid even for general strongly pseudoconvex domains, although we have not been able to prove this at the present time (see Theorem 2 of [H2] for a special case).

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3. **Some results on domains of finite type.** The main purpose of this section is to prove Theorem 0.4. Let us start with the following:

LEMMA 3.1. *Let D be a bounded smooth pseudoconvex domain in C^n, and let $\phi\colon \Delta \longrightarrow D$ be a complex geodesic which is Hölder continuous up to the boundary. For every τ_1 and $\tau_2 \in \partial\Delta$ with $\tau_1 \neq \tau_2$, it holds that $\phi(\tau_1) \neq \phi(\tau_2)$.*

PROOF. Seeking a contradiction, we suppose not. Then, without loss of generality, we can assume that $\phi(1) = \phi(-1) = p \in \partial D$.

Let ϵ be the Hölder index of ϕ, and let $\epsilon' = \frac{2}{2+\epsilon}$. By Remark (b) of §1 in [DF1], we can find a small neighborhood U of p, and a defining function ρ of $U \cap D$ so that $-(-\rho)^{\epsilon'}$ is plurisubharmonic on $U \cap D$. Since ϕ is continuous up to the boundary, we can choose two small neighborhoods V_1 containing 1 and V_2 containing -1 such that $\phi(V_i \cap \Delta) \subset D \cap U$ for $i = 1, 2$.

As in the proof of Theorem 2.2, by applying the Hopf lemma to $-(-\rho \circ \phi)^{\epsilon'}$ at 1 and -1, we find that $-\rho \circ \phi(\tau) \geq C_1(1 - |\tau|^2)^{1/\epsilon'}$ for $\tau \in (-1, 1)$, and hence that

$$\delta\big(\phi(\tau)\big) \geq C_2(1 - |\tau|^2)^{\epsilon/2+1}. \tag{3.1}$$

On the other hand, by Theorem 2.3.56 of [Ab3] (p. 233), we know that

$$K_D\big(\phi(\tau), \phi(-\tau)\big) \leq \frac{1}{2}\log\left(1 + \frac{|\phi(\tau) - \phi(-\tau)|}{\delta\big(\phi(\tau)\big)}\right) + \frac{1}{2}\log\left(1 + \frac{|\phi(\tau) - \phi(-\tau)|}{\delta\big(\phi(-\tau)\big)}\right) + C, \tag{3.2}$$

where C is a constant and $\tau \in (-1, 1)$.

Recall that

$$K_D\big(\phi(\tau), \phi(-\tau)\big) = K_\Delta(\tau, -\tau) = \log\frac{1+\tau}{1-\tau} \quad \text{for } \tau \in (0, 1), \tag{3.2$'$}$$

and that

$$|\phi(\tau) - \phi(-\tau)| \leq |\phi(\tau) - \phi(1)| + |\phi(-\tau) - \phi(-1)| \leq C_3\big((1-\tau)^\epsilon\big) \quad \big(\tau \in (0, 1)\big). \tag{3.3}$$

From (3.2) and (3.2)$'$, it then follows that

$$\log\left((1-\tau) + \frac{|\phi(\tau) - \phi(-\tau)|}{\delta\big(\phi(\tau)\big)}(1-\tau)\right) + \log\left((1-\tau) + \frac{|\phi(\tau) - \phi(-\tau)|}{\delta\big(\phi(-\tau)\big)}(1-\tau)\right) > C_4,$$

where $\tau \in (0, 1)$ and C_4 is a constant. Combining this inequality with (3.1) and (3.3), we then see that

$$\log\big((1-\tau) + C_5(1-\tau)^{\epsilon/2}\big) \geq C_6,$$

for some constants $C_5 > 0$ and C_6, and for $\tau \in (0, 1)$. This is obviously a contradiction. Thus our proof is complete. ∎

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THEOREM 0.4. *Suppose that D is a bounded convex domain of finite type in C^n and $p \in \partial D$. Let $\epsilon > 0$ be a K-admissible number of D, and let $m > 5/\epsilon$. If $f \in \mathrm{Hol}(D, D)$ is such that $f(z) = z + 0(|z-p|^m)$ as $z \longrightarrow p$, then $f(z) \equiv z$ on D.*

PROOF. We divide our discussion into three steps.

(a) We first show that, with the given assumptions, for every complex geodesic ϕ with $\phi(1) = p$, then $f \circ \phi$ is also a complex geodesic with $f \circ \phi(1) = p$.

In fact, by Royden-Wong's results (see Theorem 2.6.22 of [Ab3], p. 322), we can, for such a ϕ, find its holomorphic left inverse $\pi: D \longrightarrow \Delta$ satisfying that $\pi \circ \phi(\tau) \equiv \tau$. As argued before, we consider $\xi(\tau) \in \mathrm{Hol}(\Delta, \Delta)$ defined by $\xi(\tau) = \pi \circ f \circ \phi(\tau)$. For every $1 > \mu \approx 1$, as in the proofs of Theorem 2.2 and Lemma 3.1, by an appropriate application of Hopf's lemma, we can find a $R > 0$ small enough so that

$$\delta\big(\phi(\tau)\big) \geq C_1(|1-\tau|)^{2/\mu} \quad \text{for some constant } C_1 \text{ and } \tau \in E(1,R).$$

Denote by $\gamma_\tau(t): I \longrightarrow D$ the segment from $\phi(\tau)$ to $f \circ \phi(\tau)$. We can then see that

$$\delta\big(\gamma_\tau(t)\big) \geq C_2(|1-\tau|)^{2/\mu} \quad \text{for some constant } C_2 \text{ and } \tau \in E(1,R).$$

So it follows from the Cauchy estimates that

$$\Big|\pi'\big(\gamma_\tau(t)\big)\Big| \leq C_3(|1-\tau|^{-2/\mu}) \quad \text{on } E(1,R),$$

and thus that

(3.4)
$$\begin{aligned}|\xi(\tau)-\tau| &= |\pi \circ f \circ \phi(\tau) - \pi \circ \phi(\tau)| \leq |f \circ \phi(\tau) - \phi(\tau)|\, C_4|1-\tau|^{-2/\mu} \\ &\leq o\big(|\phi(\tau)-p|^m\big) \cdot |1-\tau|^{-2/\mu} \leq o(|1-\tau|^{\epsilon m - 2/\mu}), \quad \text{as } \tau \in E(1,R) \longrightarrow 1.\end{aligned}$$

Since $m > 5/\epsilon$, we can choose μ so close to 1 that $\epsilon m - 2/\mu > 3$. Hence, by noting the fact that $\xi\big(E(1,R)\big) \subset E(1,R)$ and $E(1,R)$ is also a disk (see Proposition 1.2), it follows from Lemma 2.1 that ξ is the identity on $E(1,R)$, thus on Δ. Hence we can conclude, by the monotonicity property of Kobayashi distance, that $f \circ \phi$ is also a complex geodesic with the same left inverse π as ϕ.

(b) Secondly, we show that f is a proper holomorphic mapping.

Seeking a contradiction, we suppose not. Then there exists a sequence $\{z_n\} \subset D$ converging to $q \in \partial D$ such that $f(z_n)$ converges to some $z_0 \in D$ (clearly, $q \neq p$). Let ϕ_n be a normalized complex geodesic of D satisfying that $p \in \phi_n(\partial\Delta)$ and $\phi_n(\tau_n) = z_n$ for some $\tau_n \in \Delta$ ($n = 1, 2, \ldots$) (see (a) of Proposition 1.9 for the existence). Now, it obviously holds that $\inf_n\big\{\text{diameter}\big(\phi_n(\Delta)\big)\big\} > 0$. Hence, we can find some $K \subset\subset D$ so that $\phi_n(0) \in K$ for every n (see [CHL]). From the completeness of the Kobayashi distance on D, it follows that $\tau_n \longrightarrow \partial\Delta$.

On the other hand, by Step (a), $K_\Delta(0,\tau_n) = K_D\big(\phi_n(0), \phi_n(\tau_n)\big) = K_D\big(f \circ \phi_n(0), f \circ \phi_n(\tau_n)\big) = K_D\big(f \circ \phi_n(0), f(z_n)\big)$. This is a contradiction: for $K_\Delta(0,\tau_n) \longrightarrow \infty$ but $K_D\big(f \circ \phi_n(0), f(z_n)\big), < M < \infty$.

(c) Lastly, we show that f is the identity.

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Since f is now a proper holomorphic map, then by the result of [BC] or [DF2] and by the fact that D satisfies Condition R (see [Be1], [K2] and their references), f can be extended smoothly up to the boundary. If f is not a biholomorphism, then the degree of f is bigger than 1. Let $\{a_n\} \subset D$ be a sequence of regular values of f, which converges to p, and let sequences $\{w_n\}$ and $\{z_n\} \subset D$ be such that $f(z_n) = f(w_n) = a_n$ and $z_n \neq w_n$ for each n. Notice that f is a local diffeomorphism near p and $f(p) = p$. We can assume, without loss of generality, that $z_n \longrightarrow p$ and $w_n \longrightarrow q$ with $q \neq p$. Obviously, it holds that $f(p) = f(q)$. By Proposition 0.9, we may construct a complex geodesic ϕ with $\phi(\tau_1) = p$ and $\phi(\tau_2) = q$ for some $\tau_1 \neq \tau_2$ on $\partial\Delta$. From Step (a), it therefore follows that $f \circ \phi$ is a complex geodesic with $f \circ \phi(\tau_1) = f \circ \phi(\tau_2)$. This contradicts Lemma 3.1. Hence we have proved that f must be a biholomorphism.

From Step (a) and the fact that for every natural number k, f^k still satisfies the hypotheses of our theorem, it follows that

$$\pi \circ f^k \circ \phi(\tau) \equiv \tau, \tag{3.5}$$

where, as we defined before, π is some fixed holomorphic left inverse of ϕ. Now, if $\{f^k\}$ is a precompact family, then by Theorem 1 of [H1], we can conclude that $f = \mathrm{id}$. If $\{f^k\}$ is non-compact, then (3.5), Bell's theorem [Be2] yields that $f^k \longrightarrow p$ on $C^\infty(\bar{D} - \{p\})$. Thus, by Theorem 4 of [H1], the rank of the Levi form of ∂D attains its minimal value at p. On the other hand, according to the recent result of Bedford-Pinchuk [BP1] and [BP2], such a D must be biholomorphic to some standard domain D_P as in Corollary 2.4. Thus from Corollary 2.4 and the fact that there is a biholomorphism from D to some D_P which sends p to $q = (1, 0, \ldots, 0)$ (also see Proposition 3 of [H1]), it still follows that $f(z) \equiv z$ over D. That contradicts the assumption that $\{f^k\}$ is noncompact.

The proof is now complete. ∎

REMARK 3.3. In the proof of Theorem 0.4, we see that the only reason for choosing m to be bigger than $5/\epsilon$ is to guarantee all ξ functions to be the identity. Let us denote by ϵ_p the *local K-admissible number* of D at p, *i.e.*, the number which makes Theorem 1.1 valid near p. Since it can be similarly shown that every complex geodesic of D passing through p can be extended ϵ_p-Hölder continuously across p then, from (3.4), it can be seen that to obtain $\xi(\tau) \equiv \mathrm{id}$, it is enough to take $m > 5/\epsilon_p$. In particular, by Catlin's theorem [Ca], we now have the following:

COROLLARY 3.4. *Let $D \subset\subset C^2$ be a bounded convex domain of finite type, and let $p \in \partial D$ with type t_p. If $f \in \mathrm{Hol}(D, D)$ satisfies $f(z) = z + o(|z-p|^{5t_p+v})$ as $z \longrightarrow p$ for some $v > 0$, then $f(z) \equiv z$.*

REMARK 3.5. It is likely that by an argument similar to that in the proof of Theorem 2.5, we can weaken the assumption of the global convexity in Theorem 0.4 to the local convexity near the point p under study. However, by Kohn-Nirenberg's example [K2], even in dimension 2, these domains with such a nice property are still very

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restricted. At this time, we have no idea how to construct the Burns-Krantz rigidity theorem for the general finite type domains, although there is considerable evidence that something must be true.

REMARK 3.6. The exponents appearing in Theorem 0.4 and Corollary 3.4 are not sharp. We believe that the best one should be 3.

ADDED IN PROOF. For m-convex domains in C^n, Mercer ([M]) has independently obtained results analogous to our Proposition 1.6, Proposition 1.7, and Proposition 1.9.

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