COMPOUNDS OF SKEW-SYMMETRIC MATRICES

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1. Introduction. In a recent interesting paper **(3)** H. Schwerdtfeger answered a question of W. R. Utz **(4)** on the structure of the real solutions A of $A^* = B$, where A is skew-symmetric. (Utz and Schwerdtfeger call A^* the "adjugate" of A; A^* is the n-square matrix whose (i, j) entry is $(-1)^{i+j}$ times the determinant of the $(n-1)$-square matrix obtained by deleting row i and column j of A. The word "adjugate," however, is more usually applied to the matrix $(A^T)^*$, where A^T denotes the transposed matrix of A; cf. **(1, 2)**.)

The object of the present paper is to find all real n-square skew-symmetric solutions A to the equation

$$C^{n-r}(A^T) = C_r(B). \tag{1}$$

Here $C_r(A)$ is the *rth compound* matrix of A and $C^{n-r}(A)$ is the $(n-r)$*th supplementary compound* matrix of A **(5)**. Thus, for $r = 1$, (1) reduces to

$$(A^T)^* = C^{n-1}(A^T) = C_1(B) = B,$$

and the problem of determining the solutions to (1) includes the question considered by Schwerdtfeger.

The following notation will be used. Let $Q_{r,n}$ denote the set of increasing sequences of r integers chosen from $1, \ldots, n$. If ω and τ are in $Q_{r,n}$, then $A[\omega|\tau]$ is the r-square submatrix of A whose row indices are ω and column indices are τ, whereas $A(\omega|\tau)$ is the $(n-r)$-square submatrix of A whose row and column indices are ω' and τ' respectively; ω' designates the ordered set complementary to ω in $1, \ldots, n$.

Suppose $\omega \in Q_{r,n}$; then $\sigma(\omega)$ denotes the sum of the integers in ω. We shall systematically use the lexicographic ordering for the sequences in $Q_{r,n}$. Then the (ω, τ) entry of $C_r(A)$ is $d(A[\omega|\tau])$, where d indicates the determinant. Since both sets $Q_{r,n}$ and $Q_{n-r,n}$ contain $\binom{n}{r}$ sequences, we can index the entries of $C^{n-r}(A)$ with the sequences in $Q_{r,n}$ so that the (ω, τ) entry of $C^{n-r}(A)$ becomes $(-1)^{\sigma(\omega)+\sigma(\tau)} d(A(\omega|\tau))$. Various properties of these associated matrices are given in **(5**, pp. 64–67**)**, and we shall use these freely without giving specific references. The most important of these is the Laplace expansion theorem which states that

$$C_r(A)C^{n-r}(A^T) = d(A)I_N, \qquad N = \binom{n}{r}.$$

Also, both $C_r(A)$ and $C^{n-r}(A)$ are multiplicative functions of A.

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We observe that (1) always has *trivial* solutions if $\rho(A) < n - r$ and $\rho(B) < r$, where $\rho(X)$ denotes the rank of X. For in this case we would have $C^{n-r}(A^T) = 0 = C_r(B)$. We shall be interested in the *non-trivial* solutions only, i.e., those for which $\rho(A) \geqslant n - r$ or $\rho(B) \geqslant r$ so that $C^{n-r}(A^T) = C_r(B) \neq 0$.

The main results of this paper are the following two theorems.

THEOREM 1. *Let B be an n-square non-singular matrix over the real field. Every solution of* (1) *appears in the form*

$$A = (d(B))^{1/(n-r)} B^{-1}.$$

Thus there is no real solution to the problem if $n - r$ is even and $d(B) < 0$.

COROLLARY. *In the non-singular case, the solution A is skew-symmetric if and only if B is skew-symmetric and since in this case $d(B) > 0$, the solution exists always.*

THEOREM 2. *Let B be an n-square singular matrix over the reals. A necessary and sufficient condition for the existence of a non-trivial n-square real skew-symmetric matrix A such that $C^{n-r}(A^T) = C_r(B)$ is that both of the following conditions hold:*

(i) *$\rho(A) = n - r$ and $\rho(B) = r$ and $n - r$ is even;*

(ii) *there exists a real orthogonal matrix U such that the only non-vanishing r-square subdeterminant of $U^T B U$ is the first principal one; in fact,*

$$d(U^T B U[1, \ldots, r|1, \ldots, r]) = a > 0.$$

2. Proof of Theorem 1. It is easily seen that the following equations are all equivalent:

$$C^{n-r}(A^T) = C_r(B),$$

$$C_r(A) C^{n-r}(A^T) = C_r(A) C_r(B),$$

$$d(A) I_N = C_r(AB), \qquad N = \binom{n}{r}.$$

Furthermore, it is easily shown that the last equation is equivalent to $AB = kI_n$, for some suitably chosen constant k.

In order to calculate k, observe that

$$C_r(AB) = C_r(kI_n) = k^r I_N = d(A) I_N,$$
$$d(AB) = d(A)\, d(B) = d(kI_n) = k^n,$$

and the result follows.

3. Proof of Theorem 2.

LEMMA 1. *If A is a non-trivial singular skew-symmetric solution to* (1), *then $\rho(A) = n - r$ and $\rho(B) = r$.*

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Proof. A is skew-symmetric; thus there exists a real orthogonal n-square matrix U such that

$$U^TAU = \left[\begin{array}{cccc|cccc} 0 & & & & & & & \\ & \cdot & & & & 0 & & \\ & & \cdot & & & & & \\ & & & 0 & & & & \\ \hline & & & & \begin{array}{cc} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{array} & & & \\ & 0 & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \begin{array}{cc} 0 & \alpha_k \\ -\alpha_k & 0 \end{array} \end{array}\right],$$

for $\alpha_i \neq 0$, $i = 1, \ldots, k$. Now $\rho(A) = 2k$ because A is skew-symmetric. Also the zero submatrix in the upper left corner is $(n - 2k)$-square. It follows that, since A satisfies (1),

$$\begin{aligned} C^{n-r}(U^TAU) &= C^{n-r}(U^T)C^{n-r}(A)C^{n-r}(U) \\ &= C^{n-r}(U^T)C_r(B^T)C^{n-r}(U) \\ &= \{d(U)\}C_r(U^{-1})C_r(B^T)\{d(U)\}C_r(U^{-1})^T \\ &= C_r(U^{-1}B^T(U^{-1})^T) \\ &= C_r(U^TB^TU). \end{aligned}$$

Thus (1) holds if and only if $C^{n-r}(U^TAU) = C_r(U^TB^TU)$. We claim that $2k = n - r$. To prove this, first observe that A is a *non-trivial* solution to (1) and thus $\rho(A) = 2k \geqslant n - r$. Hence, the last equation reduces to

(2) $\qquad C^{n-r}(A_1) = C_r(B_1), \qquad$ where $A_1 = U^TAU$ and $B_1 = U^TB^TU$.

Now, let $\omega, \tau \in Q_{r,n}$, and let $n - 2k = m$. Then the (ω, τ) entry of $C^{n-r}(A_1)$ is $\mp\det A_1(\omega|\tau)$, and this determinant is *not* zero only if both ω and τ are of the form $(1, \ldots, m, j_1, \ldots, j_{r-m})$, where $m < j_1 < \ldots < j_{r-m} \leqslant n$. Observe that, since $2k \geqslant n - r$, $m = n - 2k \leqslant r$. Let the rows of B_1 be $u_1, \ldots, u_n$. The (ω, ω) entry of $C_r(B_1)$ is *not* zero only if $\omega = (1, \ldots, m, j_1, \ldots, j_{r-m})$, where, again, $m < j_1 < \ldots < j_{r-m} \leqslant n$. Hence, for any $(i_1, i_2, \ldots, i_r) \in Q_{r,n}$, $\{u_{i_1}, \ldots, u_{i_r}\}$ is linearly independent only if $\{1, \ldots, m\} = \{i_1, \ldots, i_m\}$. Next, observe that $\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_r\}$ is linearly independent. To prove this, we distinguish two cases. First, if $n - r$ is *even*, then the (ω, τ) entry of $C^{n-r}(A_1)$ is equal to $\mp d(A_1(\omega|\tau))$, which is different from zero for the following choice of ω, τ: $\omega = (1, \ldots, r)$, $\tau = (1, \ldots, m, m + 1, \ldots, r)$. Second, if $n - r$ is *odd*, then the (ω, τ) entry of $C^{n-r}(A_1)$ is different from zero for the following choice of ω and τ: $\omega = (1, \ldots, r)$, $\tau = (1, \ldots, m, m + 1, \ldots, r - 1, r + 1)$. (Note that, if $2k > n - r$, then $m = n - 2k \leqslant r - 1$). Similarly,

$$\{u_1, \ldots, u_{m-1}, u_{m+1}, \ldots, u_r, u_k\}$$

is linearly dependent for any k for which $r + 1 \leqslant k \leqslant n$. Hence

$$\{u_1, \ldots, u_{m-1}, u_m, u_{m+1}, \ldots, u_r, u_k\}$$

is linearly dependent. Thus u_k is linearly dependent on $\{u_1, \ldots, u_r\}$ for each k such that $r + 1 \leqslant k \leqslant n$. It follows that $\{u_1, \ldots, u_r\}$ spans the space of the u_i's, and since this set is linearly independent, it is a basis for the row space of B_1. Thus $\rho(U^T B^T U) = \rho(B_1) = r$, $\rho(B) = \rho(B_1) = r$. Hence

$$\rho(C_r(B)) = \binom{\rho(B)}{r} = 1.$$

Therefore, $\rho(C^{n-r}(A)) = \rho(C^{n-r}(A^T)) = \rho(C_r(B)) = 1$. Hence

$$\binom{\rho(A)}{n - r} = 1, \qquad \rho(A) = n - r,$$

and the lemma is proved.

LEMMA 2. *If A is a non-trivial singular skew-symmetric solution to* (1), *then there exists a real orthogonal matrix U such that the only non-vanishing r-square subdeterminant of $U^T B U$ is the first principal one; in fact,*

$$d(U^T B U[1, \ldots, r|1, \ldots, r]) > 0.$$

Proof. A is skew-symmetric, and thus the rank of A is even, say, $2k$. By Lemma 1, $\rho(A) = n - r$. Hence we have $\rho(A) = n - r = 2k$. As in the proof of Lemma 1, (1) is equivalent to

$$(3) \qquad C^{n-r}(U^T A U) = C^{n-r}\left(\left[\begin{array}{c|c} \begin{matrix} 0 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \begin{bmatrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{bmatrix} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \begin{bmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{bmatrix} \end{matrix} \end{array}\right]\right)$$

$$= C_r(U^T B^T U)$$

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for some real orthogonal matrix U, where the submatrix in the upper left corner is r-square. Thus (1) holds if and only if $C^{n-r}(U^TAU) = C_r(U^TB^TU)$. Let $U^TB^TU = H$. Then (1) is equivalent to

$$C_r(H) = C^{n-r}\left(\left[\begin{array}{ccc|cccc} 0 & & & & & & \\ & \ddots & & & 0 & & \\ & & 0 & & & & \\ \hline & & & \begin{array}{cc} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{array} & & & \\ & & & & \ddots & & \\ & 0 & & & & & \begin{array}{cc} 0 & \alpha_k \\ -\alpha_k & 0 \end{array} \end{array}\right]\right)$$

$$= \left[\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (a = \alpha_1^2 \ldots \alpha_k^2).$$

The last equation shows that the only non-vanishing r-square subdeterminant of H is the determinant of the submatrix $H[1, \ldots, r|1, \ldots, r]$; i.e. the (1,1) entry of $C_r(H)$. This in turn is just

$$d\left(\begin{bmatrix} \begin{array}{cc} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{array} & & 0 \\ & \ddots & \\ 0 & & \begin{array}{cc} 0 & \alpha_k \\ -\alpha_k & 0 \end{array} \end{bmatrix}\right) = \alpha_1^2 \ldots \alpha_k^2 = a > 0.$$

From $(U^TBU)^T = U^TB^TU = H$, it follows that the only non-vanishing subdeterminant of U^TBU is the principal one lying in rows $1, \ldots, r$. This proves the lemma.

We are now in a position to complete the proof of Theorem 2.

The necessity of the conditions follows from Lemmas 1 and 2. We prove that they are also sufficient. Suppose that both conditions (i) and (ii) of Theorem 2 hold. By (i), $\rho(A) = n - r = 2k$, say. Choose real numbers $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1^2 \ldots \alpha_k^2 = a$, where a is the *positive* number described in (ii) above. Now, using the matrix U described in (ii) and the numbers $\alpha_1, \ldots, \alpha_k$ just constructed, we define A as follows:

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$$A = U\left[\begin{array}{ccc|ccc} 0 & & & & & \\ & \ddots & & & 0 & \\ & & 0 & & & \\ \hline & & & \begin{array}{cc}0 & \alpha_1\\ -\alpha_1 & 0\end{array} & & \\ & 0 & & & \ddots & \\ & & & & & \begin{array}{cc}0&\alpha_k\\-\alpha_k&0\end{array}\end{array}\right]U^T,$$

where the zero matrix in the upper left corner is r-square, and $2k = \rho(A) = n - r$. By an argument similar to the one used in the proofs of Lemmas 1 and 2, one verifies that A is a skew-symmetric solution of $C^{n-r}(A^T) = C_r(B)$. This completes the proof.

References

1. A. C. Aitken, *Determinants and matrices* (Edinburgh, 1948).
2. T. Muir, *A treatise on the theory of determinants* (New York, 1960).
3. H. Schwerdtfeger, *On the adjugate of a matrix*, Portugal. Math., *20* (1961), 39–41.
4. W. R. Utz, *Some theorems and questions on matrices*, Portugal. Math., *18* (1959), 225–229.
5. J. H. Wedderburn, *Lectures on matrices*, Am. Math. Soc., Coll. Publ., 17 (1934).

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