

DEHN SURGERIES ON SOME CLASSICAL LINKS

ALBERTO CAVICCHIOLI, FULVIA SPAGGIARI AND AGNESE ILARIA TELLONI

Dipartimento di Matematica, Università di Modena e Reggio Emilia,

Via Campi 213/B, 41100 Modena, Italy

(cavicchioli.alberto@unimore.it; spaggiari.fulvia@unimore.it;
agneseilaria.telloni@unimore.it)

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Abstract We consider orientable closed connected 3-manifolds obtained by performing Dehn surgery on the components of some classical links such as Borromean rings and twisted Whitehead links. We find geometric presentations of their fundamental groups and describe many of them as 2-fold branched coverings of the 3-sphere. Finally, we obtain some topological applications on the manifolds given by exceptional surgeries on hyperbolic 2-bridge knots.

Keywords: 3-manifolds; links; Dehn surgery; exceptional surgeries; group presentations; cyclic branched coverings

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1. Introduction

Dehn surgery is a basic method for constructing closed 3-manifolds. It was introduced by Dehn to construct homology spheres. In the early 1960s, Lickorish [7] and Wallace [17] showed that every closed orientable 3-manifold can be obtained by performing Dehn surgery on the components of some link in the 3-sphere. Kirby [5] found an equivalence relation, on the class of all framed links (i.e. links with rational coefficients), with the property that two such links are equivalent if and only if they represent the same surgery manifold. We study the closed manifolds obtained by performing Dehn surgery on the components of some classical links, such as Borromean rings and twisted Whitehead links. We obtain geometric presentations for the fundamental groups of the constructed manifolds and describe some covering properties of them. Some applications to exceptional surgeries on hyperbolic 2-bridge knots complete the paper.

2. Dehn surgeries on Borromean rings

In this section we consider the manifolds

$$M = \mathcal{B}(m/n; p/q; h/k) \quad \text{and} \quad \bar{M} = \bar{\mathcal{B}}(m/n; p/q; h/k)$$

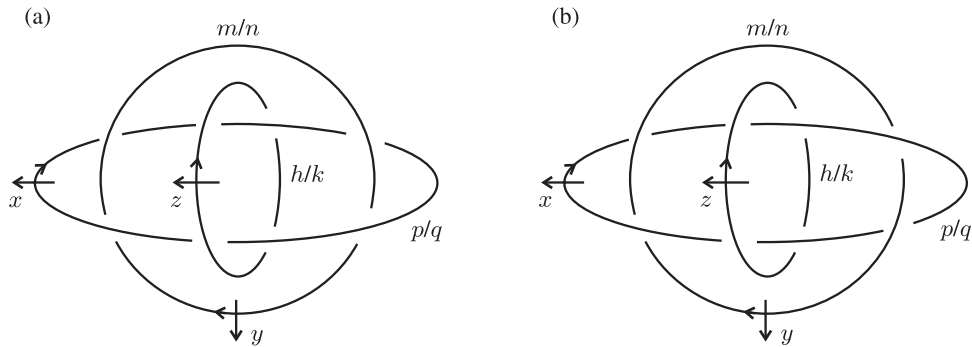


Figure 1. Dehn surgery description of the manifolds $M = \mathcal{B}(m/n; p/q; h/k)$ and $\bar{M} = \bar{\mathcal{B}}(m/n; p/q; h/k)$.

obtained by performing Dehn surgery on the oriented components of the Borromean rings \mathcal{B} and of the link $\bar{\mathcal{B}}$, which are illustrated in parts (a) and (b), respectively, of Figure 1.

Of course, we always assume that $\gcd(m, n) = \gcd(p, q) = \gcd(h, k) = 1$. It is well known that such manifolds are 2-fold cyclic branched coverings of the lens space $L(h, k)$ [11]. We now determine geometric presentations for the fundamental groups of M and \bar{M} . A group presentation is said to be *geometric* if it arises from a Heegaard diagram of a closed connected (orientable) 3-manifold. If it is geometric, then the presentation also corresponds to a spine of the considered manifold. A Wirtinger presentation of the link group $\pi(\mathcal{B})$ (respectively, $\pi(\bar{\mathcal{B}})$) has generators x, y, z and relations $w_1x = xw_1$, $w_2y = yw_2$ and $w_3z = zw_3$ (respectively, $\bar{w}_1x = x\bar{w}_1$, $\bar{w}_2y = y\bar{w}_2$ and $\bar{w}_3z = z\bar{w}_3$), where

$$\begin{aligned} w_1 &= y^{-1}x^{-1}z^{-1}xyx^{-1}z, & \bar{w}_1 &= z^{-1}yzxy, \\ w_2 &= x^{-1}zxz^{-1}, & \bar{w}_2 &= zxyxy^{-1}x^{-1}z^{-1}x, \\ w_3 &= yxy^{-1}x^{-1}, & \bar{w}_3 &= yxy^{-1}x^{-1}. \end{aligned}$$

The meridians \mathbf{m}_i and the longitudes ℓ_i of the components of \mathcal{B} are

$$\begin{aligned} \mathbf{m}_1 &= x, & \ell_1 &= xy^{-1}x^{-1}z^{-1}xyx^{-1}z, \\ \mathbf{m}_2 &= y, & \ell_2 &= x^{-1}zxz^{-1}, \\ \mathbf{m}_3 &= z, & \ell_3 &= zxyx^{-1}y^{-1}z^{-1}, \end{aligned}$$

where $[\mathbf{m}_i, \ell_i] = 1$ for $i = 1, 2, 3$.

The meridians \mathbf{m}_i and the longitudes ℓ_i of the components of $\bar{\mathcal{B}}$ are

$$\begin{aligned} \mathbf{m}_1 &= x, & \ell_1 &= xy^{-1}x^{-1}z^{-1}y^{-1}z, \\ \mathbf{m}_2 &= y, & \ell_2 &= x^{-1}zxyx^{-1}y^{-1}x^{-1}z^{-1}, \\ \mathbf{m}_3 &= z, & \ell_3 &= zxyx^{-1}y^{-1}z^{-1}, \end{aligned}$$

where $[\mathbf{m}_i, \ell_i] = 1$ for $i = 1, 2, 3$.

A finite presentation for the fundamental group of the closed connected orientable 3-manifold $M = \mathcal{B}(m/n; p/q; h/k)$ (respectively, $\bar{M} = \bar{\mathcal{B}}(m/n; p/q; h/k)$) is obtained

from that of $\pi(\mathcal{B}) = \pi_1(\mathbb{S}^3 \setminus \mathcal{B})$ (respectively, $\pi(\bar{\mathcal{B}}) = \pi_1(\mathbb{S}^3 \setminus \bar{\mathcal{B}})$) by adding the relations

$$\begin{aligned} \mathbf{m}_1^p \ell_1^q &= 1, \\ \mathbf{m}_2^m \ell_2^n &= 1, \\ \mathbf{m}_3^h \ell_3^k &= 1. \end{aligned}$$

We now improve the presentations of $\pi_1(M)$ and $\pi_1(\bar{M})$. Since the integers of the pairs (p, q) , (m, n) and (h, k) are coprime, there are integers (α, β) , (γ, δ) and (ξ, η) such that

$$\begin{aligned} q\alpha - p\beta &= 1, \\ n\gamma - m\delta &= 1, \\ k\xi - h\eta &= 1. \end{aligned}$$

Let us define

$$\begin{aligned} a &:= \mathbf{m}_1^\alpha \ell_1^\beta, \\ b &:= \mathbf{m}_2^\gamma \ell_2^\delta, \\ c &:= \mathbf{m}_3^\xi \ell_3^\eta. \end{aligned}$$

We then have

$$\begin{aligned} a^q &= (\mathbf{m}_1^\alpha \ell_1^\beta)^q = \mathbf{m}_1^{\alpha q} \ell_1^{\beta q} = \mathbf{m}_1^{\alpha q} (\mathbf{m}_1^p \ell_1^q)^{\beta/q} = \mathbf{m}_1 = x, \\ b^n &= (\mathbf{m}_2^\gamma \ell_2^\delta)^n = \mathbf{m}_2^{\gamma n} \ell_2^{\delta n} = \mathbf{m}_2^{\gamma n} (\mathbf{m}_2^m \ell_2^n)^{\delta/n} = \mathbf{m}_2 = y, \\ c^k &= (\mathbf{m}_3^\xi \ell_3^\eta)^k = \mathbf{m}_3^{\xi k} \ell_3^{\eta k} = \mathbf{m}_3^{\xi k} (\mathbf{m}_3^h \ell_3^k)^{\eta/k} = \mathbf{m}_3 = z, \\ a^{-p} &= (\mathbf{m}_1^\alpha \ell_1^\beta)^{-p} = \mathbf{m}_1^{-\alpha p} \ell_1^{-\beta p} = (\mathbf{m}_1^p \ell_1^q)^{-\alpha/p} \ell_1^{-\beta} = \ell_1^{-\alpha}, \\ b^{-m} &= (\mathbf{m}_2^\gamma \ell_2^\delta)^{-m} = \mathbf{m}_2^{-\gamma m} \ell_2^{-\delta m} = (\mathbf{m}_2^m \ell_2^n)^{-\gamma/m} \ell_2^{-\delta} = \ell_2^{-\gamma}, \\ c^{-h} &= (\mathbf{m}_3^\xi \ell_3^\eta)^{-h} = \mathbf{m}_3^{-\xi h} \ell_3^{-\eta h} = (\mathbf{m}_3^h \ell_3^k)^{-\xi/h} \ell_3^{-\eta} = \ell_3^{-\xi}. \end{aligned}$$

Substituting these relations into the relators of $\pi(\mathcal{B}) = \pi_1(\mathbb{S}^3 \setminus \mathcal{B})$ and $\pi(\bar{\mathcal{B}}) = \pi_1(\mathbb{S}^3 \setminus \bar{\mathcal{B}})$ and using the corresponding formulae for the longitudes ℓ_i , $i = 1, 2, 3$, we get the following result.

Theorem 2.1. *The fundamental groups of the surgery manifolds*

$$M = \mathcal{B}(m/n; p/q; h/k) \quad \text{and} \quad \bar{M} = \bar{\mathcal{B}}(m/n; p/q; h/k)$$

admit the finite balanced presentations

$$\begin{aligned} \pi_1(M) &= \langle a, b, c : a^{p+q} b^{-n} a^{-q} c^{-k} a^q b^n a^{-q} c^k = 1, b^m a^{-q} c^k a^q c^{-k} = 1, \\ &\qquad\qquad\qquad c^h a^q b^n a^{-q} b^{-n} = 1 \rangle, \\ \pi_1(\bar{M}) &= \langle a, b, c : a^{p+q} b^{-n} a^{-q} c^{-k} b^{-n} c^k = 1, b^m a^{-q} c^k a^q b^n a^{-q} b^{-n} a^{-q} c^{-k} = 1, \\ &\qquad\qquad\qquad c^h a^q b^n a^{-q} b^{-n} = 1 \rangle. \end{aligned}$$

These presentations are geometric since they are induced by genus-3 Heegaard diagrams of the considered manifolds.

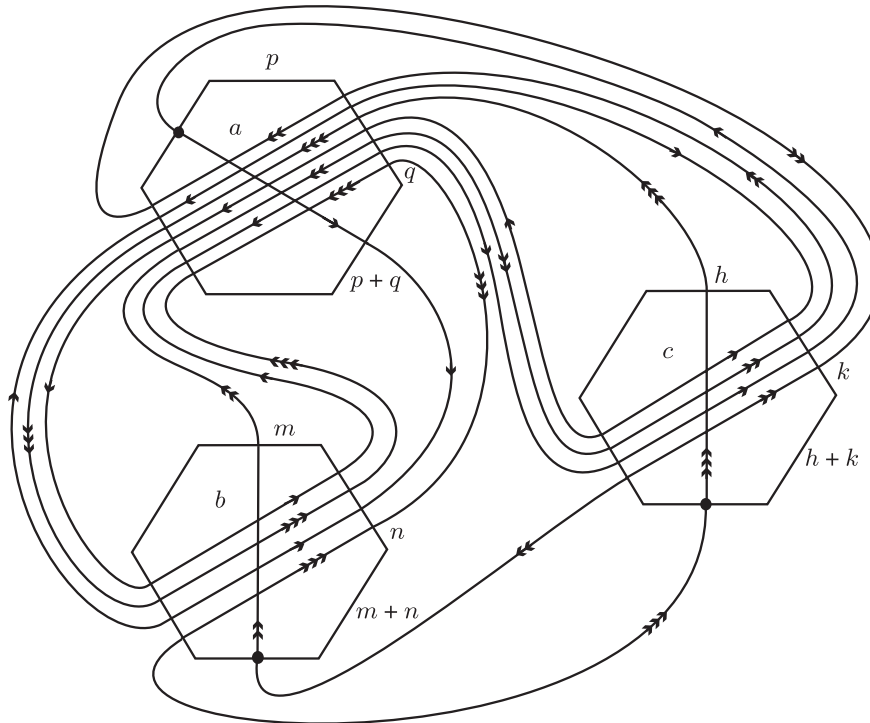


Figure 2. An RR system of genus 3 inducing the presentation of $\pi_1(M)$.

To prove the last sentence of Theorem 2.1, it suffices to draw suitable genus-3 ‘rail-road systems’ (RR systems), which induce precisely the above presentations (see Figures 2 and 3). For the theory of RR systems we refer the reader to [13] and [14, 15].

3. Dehn surgeries on twisted Whitehead links

Following [2], we use the notation $[b_1, \dots, b_n]$ to denote the partial fraction decomposition $1/(b_1 - 1/(b_2 - \dots - 1/b_n))$.

A *twisted Whitehead link* \mathcal{W} is a 2-bridge link $\mathbf{b}(\alpha, \beta)$ associated to a rational number $\beta/\alpha = [2, r, -2]$ for some $r \neq 0$. Hence we have $\mathcal{W}_r = \mathbf{b}(4r, 2r + 1)$. Recall that $\mathbf{b}(\alpha, -\beta)$ is equivalent to the mirror image of $\mathbf{b}(\alpha, \beta)$, and $\mathbf{b}(\alpha, \beta - \alpha)$ is equivalent to the link $\mathbf{b}(\alpha, \beta)$ with opposite orientation on one of the two components. Then $\mathcal{W}_{-r} = \mathbf{b}(4r, 2r - 1)$ is equivalent to the mirror image $\mathcal{W}_r^* = \mathbf{b}(4r, -2r - 1)$ of \mathcal{W}_r , and it is enough to consider \mathcal{W}_r with $r \geq 1$. When $r = 1$, \mathcal{W}_r is the torus link $\mathbf{b}(4, 1) = T(4, 2)$ of type $(4, 2)$. For surgery along a torus knot, see [12]. Thus we consider the case $r \geq 2$. When $r = 2$, \mathcal{W}_r is the Whitehead link $\mathcal{W} = \mathbf{b}(8, 3)$. See Figure 4 for \mathcal{W}_r .

Notice that after $-1/\ell$ surgery, $\ell \geq 1$, on the component of \mathcal{B} (respectively, $\bar{\mathcal{B}}$) with meridian z (see Figure 1), the other components become the twisted Whitehead links \mathcal{W}_r ,

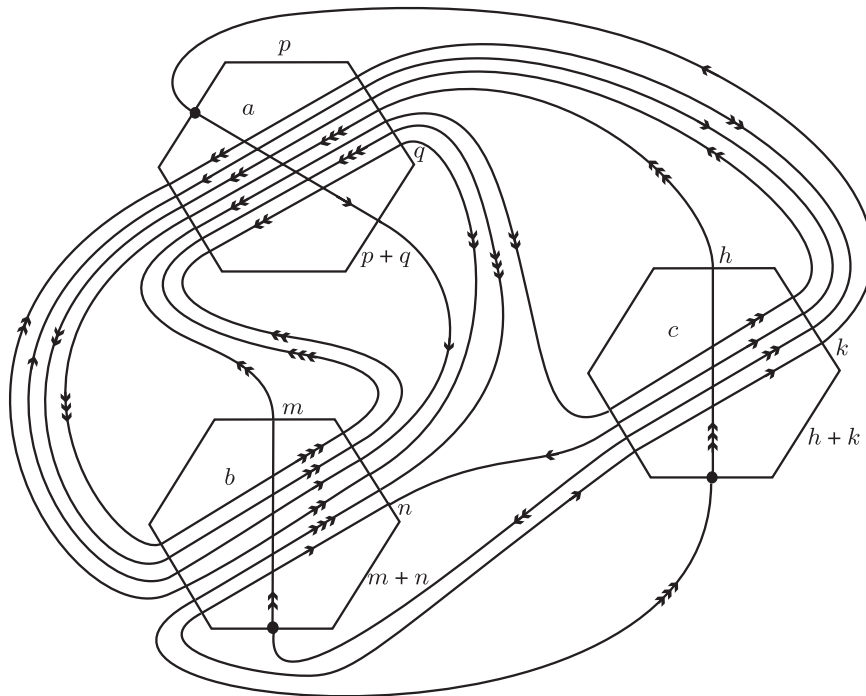


Figure 3. An RR system of genus 3 inducing the presentation of $\pi_1(\bar{M})$.

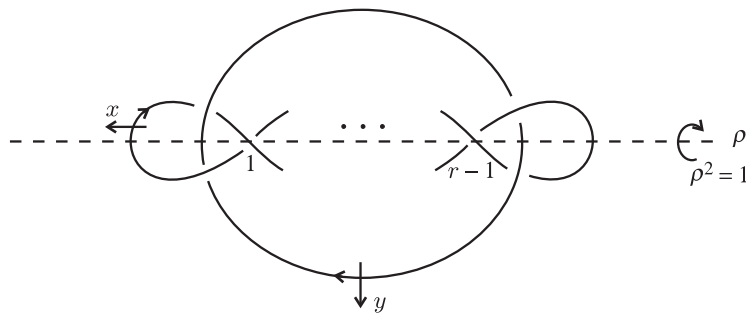


Figure 4. The twisted Whitehead link $\mathcal{W}_r = \mathbf{b}(4r, 2r + 1)$, $r \geq 2$.

where $r = 2\ell$ (respectively, $r = 2\ell + 1$). Let us denote by $\mathcal{W}_r(m/n; p/q)$ the closed connected orientable 3-manifold that is obtained by m/n and p/q Dehn surgeries on the components of \mathcal{W}_r . From the previous remark, we have $\mathcal{W}_{2\ell}(m/n; p/q) = \mathcal{B}(m/n; p/q; -1/\ell)$ and $\mathcal{W}_{2\ell+1}(m/n; p/q) = \mathcal{B}(m/n; p/q; -1/\ell)$ for every $\ell \geq 1$. As an immediate consequence of Theorem 2.1 we obtain the following.

Theorem 3.1. *Let $\mathcal{W}_r(m/n; p/q)$ be the closed 3-manifold obtained by Dehn surgery on the twisted Whitehead link \mathcal{W}_r , $r \geq 2$. The fundamental group of $\mathcal{W}_r(m/n; p/q)$ then*

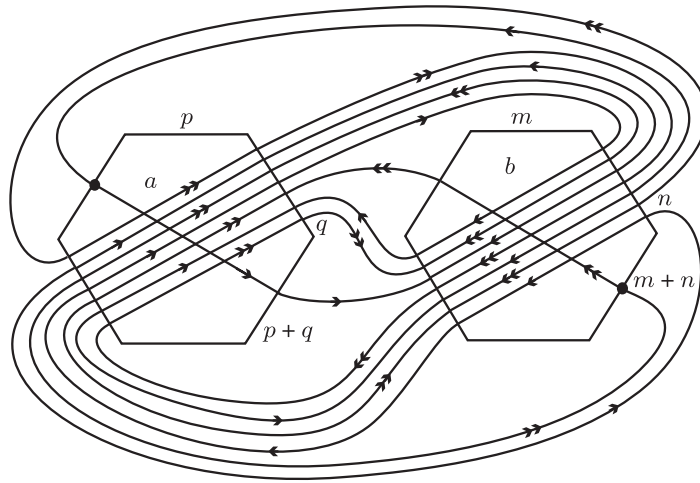


Figure 5. An RR system inducing the presentation of $\pi_1(\mathcal{W}(m/n; p/q))$.

admits the finite geometric presentation

$$\langle a, b: a^{p+q}(b^{-n}a^{-q}b^n a^q)^\ell a^{-q}(a^q b^n a^{-q} b^{-n})^\ell = 1, \\ b^{m+n}(a^{-q} b^{-n} a^q b^n)^\ell b^{-n}(b^n a^q b^{-n} a^{-q})^\ell = 1, \quad r = 2\ell,$$

$$\langle a, b: a^{p+q}(b^{-n}a^{-q}b^n a^q)^\ell b^{-n}a^{-q}b^{-n}(a^q b^n a^{-q} b^{-n})^\ell = 1, \\ b^{m+n}(a^{-q} b^{-n} a^q b^n)^\ell a^{-q} b^{-n} a^{-q}(b^n a^q b^{-n} a^{-q})^\ell = 1, \quad r = 2\ell + 1.$$

In particular, the Heegaard genus of $\mathcal{W}_r(m/n; p/q)$ is less than or equal to 2, and if the manifold admits a hyperbolic structure, then the Heegaard genus is exactly 2.

Dehn surgeries on the Whitehead link $\mathcal{W} = \mathcal{W}_2$ were studied by Mednykh and Vesnin in [9] and [8]. The Heegaard genus of the surgery manifolds $\mathcal{W}(m/n; p/q)$ was discussed in [8]. The following theorem completes the results of the quoted papers and gives a different proof of Theorem 3 of [8] (see also Proposition 1 of [9]).

Theorem 3.2. *Let $\mathcal{W}(m/n; p/q)$ be the closed 3-manifold obtained by m/n and p/q Dehn surgeries on the Whitehead link. The fundamental group of $\mathcal{W}(m/n; p/q)$ then admits the finite balanced presentation*

$$\langle a, b: a^{p+q} b^{-n} a^{-q} b^n a^q b^n a^{-q} b^{-n} = 1, b^{m+n} a^{-q} b^{-n} a^q b^n a^q b^{-n} a^{-q} = 1 \rangle.$$

Such a presentation is geometric: that is, it corresponds to a spine of the manifold (or, equivalently, arises from a Heegaard diagram of genus 2). In particular, if $\mathcal{W}(m/n; p/q)$ admits a hyperbolic structure, then it has Heegaard genus 2.

An RR system inducing the above presentations is depicted in Figure 5.

Following [2], let $K = K_{[b_1, b_2]}$ denote a 2-bridge knot in \mathbb{S}^3 that corresponds to a continued fraction $\beta/\alpha = [b_1, b_2] = 1/(b_1 - 1/b_2)$. Then α is odd, and we may assume that β is even and $1 < \beta < \alpha$. As remarked in the quoted paper, at least one of the b_i is

even, and we may set $b_1 = 2n$ for some integer n since $K_{[b_1, b_2]}$ is equivalent to $K_{[b_2, b_1]}$. Now, doing a p/q surgery on $K_{[2n, r]}$ is the same as doing p/q and $-1/n$ surgeries on the components of \mathcal{W}_r : that is, $K_{[2n, r]}(p/q) = \mathcal{W}_r(-1/n; p/q)$. Recall also that $K_{[2n, \pm 2]}$ is a twist knot and that $K_{[2, -2]}$ is the figure-eight knot. A finite geometric presentation for the fundamental group of the manifold $K_{[2n, r]}(p/q)$ can be obtained from the presentations given in Theorem 3.1 ($r \geq 2$) by taking $m = -1$.

4. Covering properties of the manifolds $\mathcal{W}_r(m/n; p/q)$

The twisted Whitehead link \mathcal{W}_r is *strongly invertible* (see Figure 4), i.e. there is an orientation-preserving involution ρ of \mathbb{S}^3 that induces in each component of \mathcal{W}_r an involution with two fixed points. A well-known theorem of Montesinos [11] applies to our case, and we can state that the manifolds $\mathcal{W}_r(m/n; p/q)$ are 2-fold coverings of \mathbb{S}^3 branched over a link of at most three components. We now apply the Montesinos algorithm, given in [11], to describe explicitly the branch sets of the above 2-fold branched coverings. Let $\mathcal{L}_r(m/n; p/q)$ denote the branch set of the 2-fold branched covering $\mathcal{W}_r(m/n; p/q)$ of \mathbb{S}^3 that corresponds to the involution ρ shown in Figure 4. Let $\mathbf{m}_1 = x$ and $\mathbf{m}_2 = y$ be the meridians of the components L_1 and L_2 of \mathcal{W}_r , respectively. The pair (\mathbf{m}_2, ℓ_2) , where ℓ_2 is the longitude of L_2 , is a preferred frame, i.e. $\ell_2 \sim 0$ in the exterior space $\mathbb{S}^3 \setminus L_2$ with linking number $\text{lk}(\mathbf{m}_2, \ell_2) = +1$. The pair (\mathbf{m}_1, ℓ_1^*) , where ℓ_1^* is the longitude of L_1 , is not a preferred frame since $\ell_1^* \sim -(r - 1)\mathbf{m}_1$ in $\mathbb{S}^3 \setminus L_1$. To have a preferred frame, we take the pair (\mathbf{m}_1, ℓ_1) , where $\ell_1 = \ell_1^* + (r - 1)\mathbf{m}_1$. This gives (in multiplicative notation) $\mathbf{m}_1 = x$, $\mathbf{m}_2 = y$, and $\ell_1 = (xyx^{-1}y^{-1})^\ell(xy^{-1}x^{-1}y)^\ell$ and $\ell_2 = (x^{-1}y^{-1}xy)^\ell(xy^{-1}x^{-1}y)^\ell$ if $r = 2\ell$, and $\ell_1 = (xy^{-1}x^{-1}y)^\ell xy^{-1}x^{-1}y^{-1}(xyx^{-1}y^{-1})^\ell$ and $\ell_2 = x^{-1}y^{-1}(xyx^{-1}y^{-1})^\ell x^{-1}y(xy^{-1}x^{-1}y)^\ell$ if $r = 2\ell + 1$, for every $\ell \geq 1$. For simplicity we only treat the case $r = 2\ell$, $\ell \geq 1$; one can easily obtain the analogous results for $r = 2\ell + 1$ by replacing \mathcal{W}_r with the twisted link $\tilde{\mathcal{B}}_r$ constructed in the same manner as \mathcal{W}_r . Let V be a regular tubular neighbourhood of the link \mathcal{W}_r in \mathbb{S}^3 . Without loss of generality, we can choose V , the meridians \mathbf{m}_1 and \mathbf{m}_2 and the longitudes ℓ_1 and ℓ_2 on ∂V to be invariant under the involution ρ . The quotient space of \mathbb{S}^3 under ρ is illustrated in Figure 6.

The image of V under the projection $\pi: \mathbb{S}^3 \rightarrow \mathbb{S}^3/\rho$ consists of two disjoint 3-balls: B_1 and B_2 , say. Each ball B_i intersects the image (under π) of the axis of ρ in two disjoint arcs. To obtain the branch set $\mathcal{L}_r(m/n; p/q)$ via the Montesinos algorithm, we can move by isotopy the B_i along the images $\pi(\ell_i)$ of the longitudes ℓ_i , and replace them by m/n and p/q rational tangles as in Figure 7. Using Reidemeister’s moves we can redraw the link $\mathcal{L}_r(m/n; p/q)$ in a more convenient form (see Figure 8).

Theorem 4.1. *Let $\mathcal{M} = \mathcal{W}_r(m/n; p/q)$ be the closed connected orientable 3-manifold obtained by m/n and p/q surgeries on the twisted Whitehead link \mathcal{W}_r , $r = 2\ell$, $\ell \geq 1$. Then \mathcal{M} is the 2-fold covering of the 3-sphere branched over the link $\mathcal{L}_r(m/n; p/q)$ pictured in Figure 8.*

For $r = 2$ we have the following result proved by Mednykh and Vesnin in [9, Theorem 2] (see also [8]).

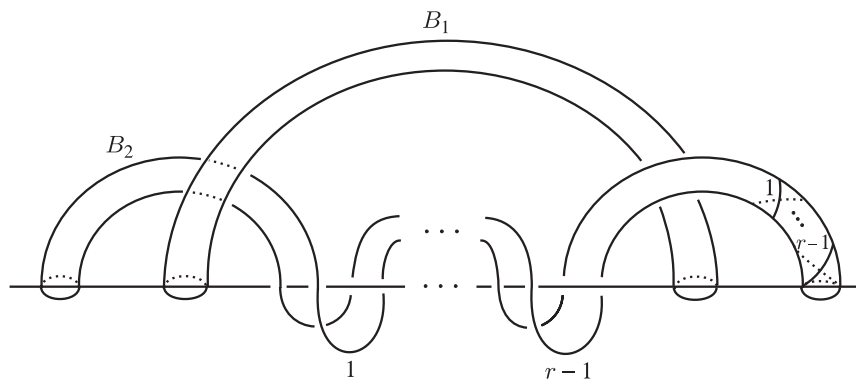


Figure 6. The quotient $(\mathbb{S}^3 \setminus \text{int } V)/\rho$.

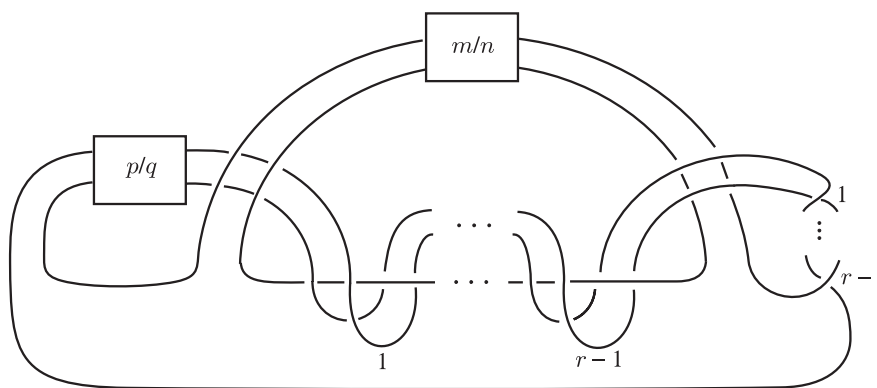


Figure 7. The link $\mathcal{L}_r(m/n; p/q)$.

Theorem 4.2. *Let $\mathcal{M} = \mathcal{W}(m/n; p/q)$ be the closed manifold obtained by m/n and p/q surgeries on the Whitehead link \mathcal{W} . Then \mathcal{M} is the 2-fold covering of \mathbb{S}^3 branched over the link $\mathcal{L}(m/n; p/q)$ pictured in Figure 9.*

From Theorem 4.1 with $m = -1$ we immediately obtain a representation of the surgery manifolds $K_{[2n, 2\ell]}(p/q) = \mathcal{W}_{2\ell}(-1/n; p/q)$ as 2-fold branched coverings of the 3-sphere.

5. Exceptional surgeries on hyperbolic 2-bridge knots

Let K be a hyperbolic knot and let $K(\gamma)$, $\gamma \neq \infty$, be the closed manifold obtained by γ Dehn surgery on K . When $K(\gamma)$ is not hyperbolic, the surgery is said to be *exceptional*. Each hyperbolic knot has only finitely many exceptional surgeries by Thurston’s Hyperbolic Dehn Surgery Theorem [16]. It was conjectured that, except for the figure-eight knot $K_{[2, -2]}$ and the $(-2, 3, 7)$ -pretzel knot, any hyperbolic knot admits at most six exceptional surgeries [6, Problem 1.77 (A) (1)]. On the other hand, the resulting manifold obtained by an exceptional surgery is expected to be a lens space, a Seifert fibred space over the 2-sphere with three exceptional fibres (referred to as a *small Seifert fibred*

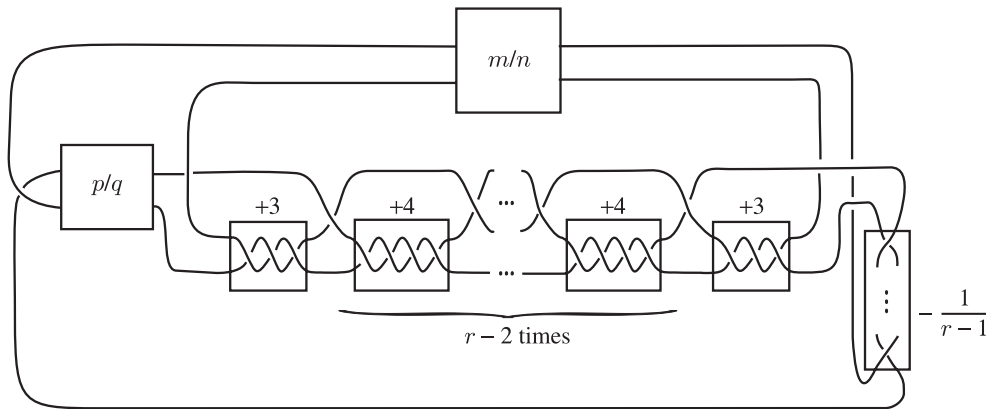


Figure 8. The link $\mathcal{L}_r(m/n; p/q)$, $r = 2\ell$, $\ell \geq 1$.

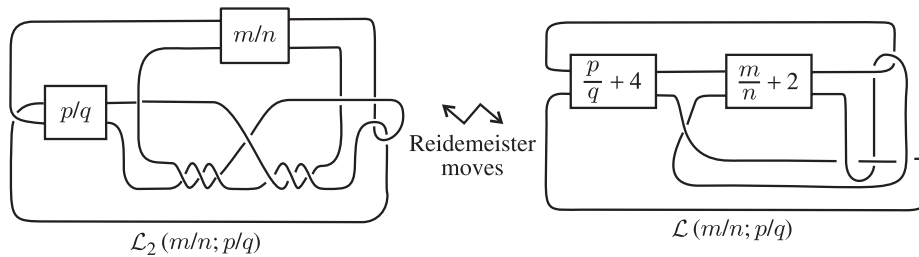


Figure 9. The link $\mathcal{L}(m/n; p/q)$.

manifold), or a toroidal manifold (i.e. a closed manifold that contains an incompressible torus). Dehn surgeries on 2-bridge knots were studied by Brittenham and Wu in [2]. The following is the main theorem of that paper. (As it says in [2], part (4) is due to Thurston and it is included for the sake of completeness.)

Theorem 5.1 (Brittenham and Wu [2]). *Let K be a hyperbolic 2-bridge knot.*

- (1) *If $K \neq K_{[b_1, b_2]}$ for any integers b_1 and b_2 , then K admits no exceptional surgery.*
- (2) *If $K = K_{[b_1, b_2]}$ with $|b_1|, |b_2| > 2$, then $K(\gamma)$ is exceptional for exactly one γ , which yields a toroidal manifold. When both b_1 and b_2 are even, $\gamma = 0$. If b_1 is odd and b_2 is even, then $\gamma = 2b_2$.*
- (3) *If $K = K_{[2n, \pm 2]}$ and $|n| > 1$, then $K(\gamma)$ is exceptional for exactly five γ . $K(\gamma)$ is toroidal for $\gamma = 0, \mp 4$ and is small Seifert fibred for $\gamma = \mp 1, \mp 2, \mp 3$.*
- (4) *If $K = K_{[2, -2]}$ is the figure-eight knot, then $K(\gamma)$ is exceptional for only nine γ . $K(\gamma)$ is toroidal for $\gamma = 0, 4, -4$ and is small Seifert fibred for $\gamma = -1, -2, -3, 1, 2, 3$.*

The results in § 4 allow us to complete Theorem 5.1 with some additional information.

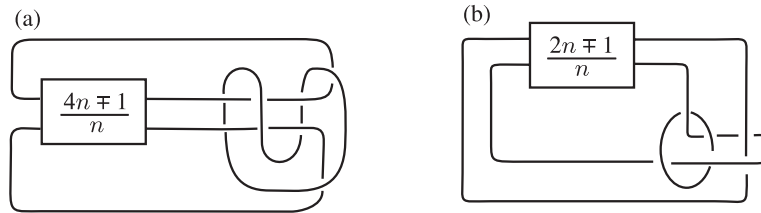


Figure 10. The branch sets of the toroidal manifolds $K(0)$ and $K(\mp 4)$.

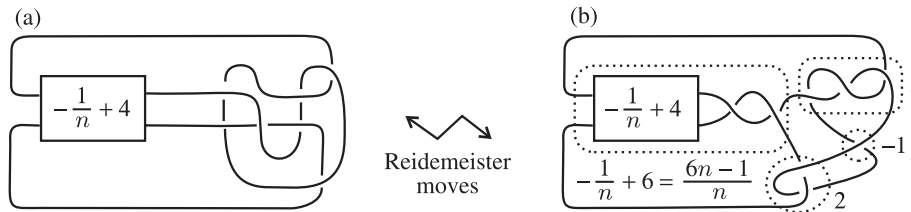


Figure 11. The link $\mathcal{L}(-1/n; -1) \simeq \mathbf{m}(-1; \frac{1}{2}; \frac{1}{3}; n/(6n - 1))$.

Theorem 5.2. *If $K = K_{[2n, \pm 2]}$ and $|n| > 1$, then we have*

$$\begin{aligned} K(\mp 1) &\cong (O0o: -1(2, 1)(3, 1)(6n \mp 1, n)), \\ K(\mp 2) &\cong (O0o: -1(2, 1)(4, 1)(4n \mp 1, n)), \\ K(\mp 3) &\cong (O0o: -1(3, 1)(3, 1)(3n \mp 1, n)). \end{aligned}$$

The toroidal manifolds $K(0)$ and $K(\mp 4)$ are the 2-fold coverings of \mathbb{S}^3 branched over the links pictured in parts (a) and (b), respectively, of Figure 10.

Theorem 5.2 also covers the case of the figure-eight knot $K = K_{[2, -2]}$ noting that $K(\gamma) = K(-\gamma)$ since it is amphicheiral.

Proof of Theorem 5.2. If $K = K_{[2n, \pm 2]}$, then $K(\gamma) \cong \mathcal{W}(\mp 1/n, \gamma)$. Since

$$K_{[2n, -2]}(\gamma) = K_{[2(-n), 2]}^*(\gamma) = K_{[2(-n), 2]}(-\gamma),$$

where K^* denotes the mirror image of K , it is enough to consider the case $K = K_{[2n, 2]}$ for some integer n . If $\gamma = -1$, then $K(-1) \cong \mathcal{W}(-1/n; -1)$ is the 2-fold covering of \mathbb{S}^3 branched over the link $\mathcal{L}(-1/n; -1)$ shown in Figure 11 (a). By Reidemeister moves, this link is equivalent to the Montesinos link $\mathbf{m}(-1; \frac{1}{2}; \frac{1}{3}; n/(6n - 1))$ as shown in Figure 11 (b) (for such links we refer, for example, to [3, Chapter 12]). By [10] it follows that the surgery manifold $K(-1)$ is homeomorphic to the Seifert fibred space defined by the Seifert invariants $(O0o: -1(2, 1)(3, 1)(6n - 1, n))$.

If $\gamma = -2$, then $K(-2) = \mathcal{W}(-1/n; -2)$ is the 2-fold covering of \mathbb{S}^3 branched over the link $\mathcal{L}(-1/n; -2)$ shown in Figure 12 (a). By Reidemeister moves, this link is equivalent to the Montesinos link $\mathbf{m}(-1; \frac{1}{2}; \frac{1}{4}; n/(4n - 1))$ as shown in Figure 12 (b). Then [10] implies that $K(-2)$ is the fibred space defined by the Seifert invariants $(O0o: -1(2, 1)(4, 1)(4n - 1, n))$.

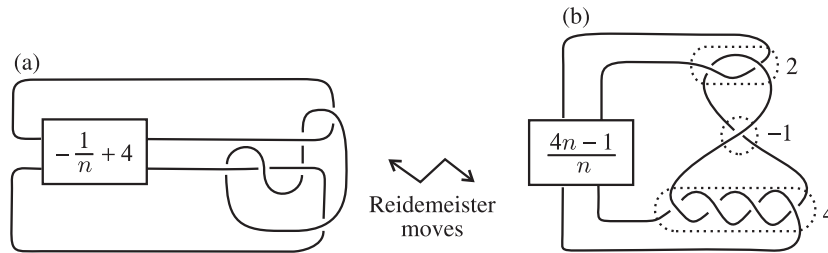


Figure 12. The link $\mathcal{L}(-1/n; -2) \simeq \mathbf{m}(-1; \frac{1}{2}; \frac{1}{4}; n/(4n - 1))$.

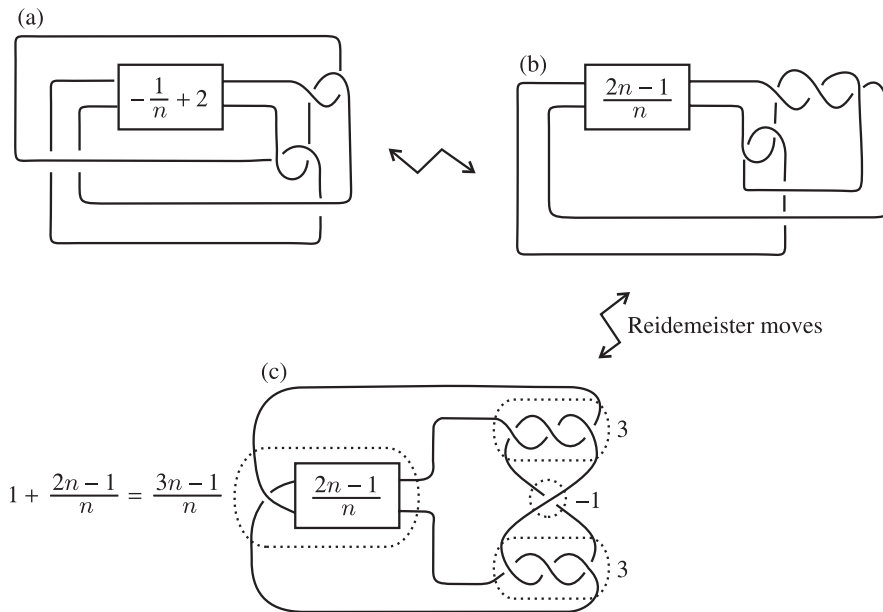


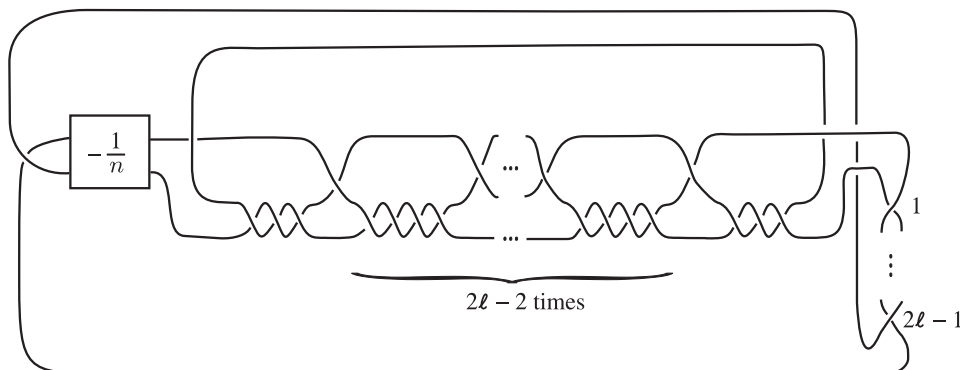
Figure 13. The link $\mathcal{L}(-1/n; -3) \simeq \mathbf{m}(-1; \frac{1}{3}; \frac{1}{3}; n/(3n - 1))$.

If $\gamma = -3$, then $K(-3) = \mathcal{W}(-1/n; -3)$ is the 2-fold covering of \mathbb{S}^3 branched over the link $\mathcal{L}(-1/n; -3)$ shown in Figure 13 (a). By a sequence of Reidemeister moves, this link is equivalent to the Montesinos link $\mathbf{m}(-1; \frac{1}{3}; \frac{1}{3}; n/(3n - 1))$, as shown in parts (b) and (c) of Figure 13. Then $K(-3)$ is the small Seifert manifold $(O0o: -1(3, 1)(3, 1)(3n - 1, n))$. The last sentence of the theorem follows from Theorem 4.2. \square

From Theorem 4.1 we immediately obtain the following result.

Theorem 5.3. *The toroidal manifold $K_{[2n, 2\ell]}(0) = \mathcal{W}_{2\ell}(-1/n; 0)$, $\ell > 1$, is the 2-fold covering of the 3-sphere branched over the link $\mathcal{L}_{2\ell}(-1/n; 0)$ depicted in Figure 14.*

We observe that the manifold investigated by Birman and Montesinos in [1] is just the surgery manifold $K_{[4, 4]}(-4) = \mathcal{W}_4(-\frac{1}{2}; -4)$. This manifold is hyperbolic of volume 4.229135386 and its symmetry group is the octahedral group D_4 (the SNAPPEA program [18] can be used to calculate this). From the above theorems and using SNAPPEA,

Figure 14. The link $\mathcal{L}_{2\ell}(-1/n; 0)$.

one can obtain many interesting series of hyperbolic homology spheres of Heegaard genus 2. For example, the surgery manifolds $K_{[2n,2]}(-1/n) = \mathcal{W}(-1/n; -1/n)$, $n \geq 2$, give a family of distinct hyperbolic homology 3-spheres of Heegaard genus 2, symmetry group D_4 , and strictly increasing volumes as $n \rightarrow +\infty$.

In a forthcoming paper [4] we shall study some covering properties of the hyperbolic surgery manifolds arising from 2-bridge knots $K \neq K_{[b_1, b_2]}$, and further generalizations of them.

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References

1. J. S. BIRMAN AND J. M. MONTESINOS, On minimal Heegaard splittings, *Michigan Math. J.* **27** (1980), 47–57.
2. M. BRITTENHAM AND Y. Q. WU, The classification of exceptional Dehn surgeries on 2-bridge knots, *Commun. Analysis Geom.* **9**(1) (2001), 97–113.
3. G. BURDE AND H. ZIESCHANG, *Knots* (Walter de Gruyter, Berlin, 1985).
4. A. CAVICCHIOLI, F. SPAGGIARI AND A. I. TELLONI, On the fundamental group of some surgery manifolds, in preparation.
5. R. KIRBY, A calculus for framed links in \mathbb{S}^3 , *Invent. Math.* **45** (1978), 35–56.
6. R. KIRBY, Problems in low-dimensional topology, in *Geometric topology* (ed. R. Kirby), pp. 35–473 (American Mathematical Society, Providence, RI, 1997).
7. W. B. R. LICKORISH, A representation of orientable combinatorial 3-manifolds, *Annals Math.* **76**(2) (1962), 531–540.
8. A. D. MEDNYKH AND A. YU. VESNIN, On Heegaard genus of three-dimensional hyperbolic manifolds of small volume, *Sb. Math. J.* **37**(5) (1996), 893–897.
9. A. D. MEDNYKH AND A. YU. VESNIN, Covering properties of small volume hyperbolic 3-manifolds, *J. Knot Theory Ram.* **7**(3) (1998), 381–392.
10. J. M. MONTESINOS, Variedades de Seifert que son recubridores ciclicos ramificados de dos hojas, *Bol. Soc. Mat. Mex.* **18** (1973), 1–32.

11. J. M. MONTESINOS, Surgery on links and double branched covers of \mathbb{S}^3 , in *Knots, groups and 3-manifolds* (ed. L. P. Neuwirth), Annals of Mathematics Studies, Volume 84 (Princeton University Press, 1975).
12. L. MOSER, Elementary surgery along a torus knot, *Pac. J. Math.* **38** (1971), 737–745.
13. R. P. OSBORNE AND R. S. STEVENS, Group presentations corresponding to spines of 3-manifolds, I, *Am. J. Math.* **96** (1974), 454–471.
14. R. P. OSBORNE AND R. S. STEVENS, Group presentations corresponding to spines of 3-manifolds, II, *Trans. Am. Math. Soc.* **234** (1977), 213–243.
15. R. P. OSBORNE AND R. S. STEVENS, Group presentations corresponding to spines of 3-manifolds, III, *Trans. Am. Math. Soc.* **234** (1977), 245–251.
16. W. THURSTON, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Am. Math. Soc.* **6** (1982), 357–381.
17. A. H. WALLACE, Modifications and cobounding manifolds, *Can. J. Math.* **12** (1960), 503–528.
18. J. R. WEEKS, Computer program SNAPPEA and tables of volumes and isometries of knots, links, and manifolds (available by anonymous ftp from www.geometrygames.org in directory `pub/software/snappea/`).