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SOLUBLE RIGHT ORDERABLE GROUPS ARE LOCALLY INDICABLE

I. M. CHISWELL AND P. H. KROPHOLLER

ABSTRACT. The object of this paper is to show that every soluble right orderable group is locally indicable. The proof identifies an interesting connection between the theory of right orderable groups and the theory of amenable groups and bounded cohomology.

1. **Introduction.** A right ordered group is a group G with a total ordering $\leq$ such that for any $x, y, z \in G$,

$$x \leq y \Rightarrow xz \leq yz.$$

An important and useful result, which we restate as Lemma 2 below, asserts that a group G is right orderable if and only if it is isomorphic to a group of permutations of some totally ordered set. We refer the reader to Chapter 7 of [6] for further information.

Right orderable groups arise naturally in various contexts, and include all lattice ordered groups and all locally indicable groups, (see Theorem 7.1.4 of [6] and Exercise 9, Chapter 13 of [7]). A group is said to be *locally indicable* if every non-trivial finitely generated subgroup admits a homomorphism onto the infinite cyclic group and for some time it was an open question whether the classes of right orderable groups and locally indicable groups coincide. However, Bergman [1] recently gave examples of right orderable groups which are perfect: one such group is the Seifert fibered 3-manifold group Γ with presentation

$$\Gamma = \langle a, b, c\,;\, a^2 = b^3 = c^7 = abc \rangle. \tag{1}$$

Here, we prove that such counterexamples cannot arise among soluble-by-finite groups:

THEOREM A. *Let G be a soluble-by-finite group. Then G is right orderable if and only if G is locally indicable.*

This generalises a result of Rhemtulla [8] that right orderable polycyclic groups are poly (infinite cyclic) and solves Problem $21'$ of [5].

We shall say that a group is RO-simple if it is non-trivial, right orderable but has no non-trivial proper right orderable quotients. Theorem A is proved by first reducing to the case when G is finitely generated and RO-simple and then applying the following result.

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THEOREM B. *Let G be a finitely generated* RO*-simple group. Then*
(i) *every abelian normal subgroup of G is central;*
(ii) *no non-trivial element of the centre $\zeta(G)$ of G is a commutator;*
(iii) *$G/\zeta(G)$ has no non-trivial abelian normal subgroups; and*
(iv) *if G is non-abelian and has non-trivial centre then the second bounded cohomology group $H_b^2(G, \mathbb{R})$ is non-trivial and G is non-amenable.*

In fact it is not difficult to show that the centre of a finitely generated RO-simple group is cyclic, but we do not need this. The authors do not know whether Bergman's example (1) is RO-simple, but it certainly has an RO-simple quotient (by Lemma 1 below), and this gives an example of a non-abelian finitely generated RO-simple group with infinite cyclic centre. At the time of writing it is not known whether there exist examples with trivial centre, and this question remains of some interest.

It is also of some interest that bounded cohomology is involved in Theorem B(iv). For the reader's convenience, we briefly review the aspects of bounded cohomology which are relevant. Further information can be found in [2]. Bounded cohomology with coefficients in $\mathbb{R}$ can be defined in the same way as ordinary cohomology, by considering $\mathbb{R}$-valued cochains, cocycles and coboundaries which arise from the bar resolution, but for the bounded theory one insists that all these functions are bounded. One important source of bounded 2-cocycles is given by considering boundaries of (unbounded) 1-cochains. Let G be a group and let $\phi: G \longrightarrow \mathbb{R}$ be any function. Then

$$\delta\phi(g, h) = \phi(gh) - \phi(g) - \phi(h)$$

is automatically a 2-cocycle and it is possible for it to be bounded even if ϕ is unbounded. If $\delta\phi$ is bounded then it represents an element of $H_b^2(G, \mathbb{R})$, and to say that this bounded cohomology class is zero is to say that there is a *bounded* function $\theta: G \longrightarrow \mathbb{R}$ such that $\delta\theta = \delta\phi$. In turn, this implies that $\delta(\phi - \theta) = 0$ or equivalently that $\phi - \theta$ is a homomorphism. We can conclude that $\delta\phi$ represents zero in $H_b^2(G, \mathbb{R})$ if and only if ϕ differs from a homomorphism by a bounded function.

The relevance of bounded 2-cocycles to right orderable groups comes about in the following way. Suppose that G is a right ordered group and that $z > 1$ is a central element such that for all g there is an integer i with $z^i \leq g < z^{i+1}$. Then we can define a function $\phi: G \longrightarrow \mathbb{Z}$ by $\phi(g) = i$. It is easy to check, using the fact that z is central, that $\delta\phi(g, h) = 0$ or 1 for all g, h, and so is a bounded 2-cocycle.

In Bergman's example (1), the bounded 2-cocycles which arise in this way give rise to non-zero elements of $H_b^2(\Gamma, \mathbb{R})$. To see this, note that Γ has infinite cyclic centre, generated by $z = a^2$. Fix a right ordering of Γ in which $z > 1$. Since z is central, one can show that the set

$$\{g \in \Gamma\,;\, z^i \leq g < z^{i+1} \text{ for some } i\}$$

is a subgroup of Γ. Moreover, a, b, c all belong to this subgroup: indeed we have $1 < a < z$, $1 < b < z$ and $1 < c < z$ because $a^2 = b^3 = c^7 = z$. Thus the subgroup is in fact the whole of Γ and a function ϕ can be defined as above, so that $\delta\phi$ is a bounded 2-cocycle.

Since Γ is perfect and ϕ is unbounded on $\langle z\rangle$, it follows that there is no homomorphism $\Gamma \longrightarrow \mathbb{R}$ differing from ϕ by a bounded function. In summary, every right ordering on Γ gives rise to a non-zero element of $H_b^2(\Gamma, \mathbb{R})$.

2. **Deduction of Theorem A from Theorem B.** One Lemma is needed:

LEMMA 1. *Let G be a non-trivial finitely generated right orderable group. Then G has an* RO-*simple quotient.*

PROOF. Let $\mathfrak{X}$ be the set of proper normal subgroups N of G such that G/N is right orderable. Then the quotient of G by any maximal element of $\mathfrak{X}$ is RO-simple and so it suffices to prove that $\mathfrak{X}$ has maximal elements. We use Zorn's Lemma. The trivial subgroup belongs to $\mathfrak{X}$ so $\mathfrak{X}$ is non-empty. Moreover, if (N_λ) is any totally ordered chain in $\mathfrak{X}$ then $N = \bigcup N_\lambda$ belongs to $\mathfrak{X}$. To see this, note first that N must be a proper subgroup because each N_λ is proper and G is finitely generated. Now suppose that G/N is not right orderable. Then according to the semigroup criterion, Lemma 13.2.1 of [7], there exist finitely many non-identity elements $Ng_1, \ldots, Ng_n$ of G/N such that for all choices of sign $\epsilon_i = \pm 1$ the subsemigroup generated by $Ng_1^{\epsilon_1}, \ldots, Ng_n^{\epsilon_n}$ contains the identity element. Thus if F denotes the free semigroup on $X_1, \ldots, X_n$ then for each $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ there is a word w_ϵ in F such that $w_\epsilon(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$ belongs to N_λ for some $\lambda = \lambda(\epsilon)$. Since there are only finitely many choices of ϵ we can choose a fixed μ such that $w_\epsilon(g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n})$ belongs to N_μ for all ϵ. But now the elementary reverse direction of Lemma 13.2.1 of [7] implies that G/N_μ is not right orderable, a contradiction. This proves Lemma 1.

Theorem A amounts to the assertion that every non-trivial finitely generated soluble-by-finite right orderable group G has an infinite cyclic quotient. By Lemma 1, we may replace G by an RO-simple quotient, and hence we may assume that G is finitely generated and RO-simple. By Theorem B(iii) $G/\zeta(G)$ has no non-trivial normal abelian subgroups, but since it is soluble-by-finite it must be finite. Thus G is centre-by-finite, and being torsion-free it must be abelian. Thus Theorem A follows from Theorem B(iii). Theorem A can also be proved in a rather direct way using Theorem B(iv), because all soluble-by-finite groups are amenable, (see Theorem 10.4(a,b,e) of [9]).

In effect, if $\mathfrak{O}$ denotes the class of finitely generated RO-simple groups then a group is right orderable if and only if every non-trivial finitely generated subgroup has a quotient which is an $\mathfrak{O}$-group. Theorem B shows that soluble-by-finite $\mathfrak{O}$-groups are infinite cyclic. Bergman's example (1) shows that not all $\mathfrak{O}$-groups are infinite cyclic. It is of some interest to have a greater understanding of the class $\mathfrak{O}$.

3. **Some general techniques.** We refer the reader to [6] for general information about right orderable groups, and in particular for the following important characterisation (Theorem 7.1.2 of [6]).

LEMMA 2. *A group is right orderable if and only if it acts faithfully as a group of order-preserving permutations of some totally ordered set.*

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Let G be a right ordered group and let X be a subset. We write $\mathrm{hull}_G(X)$ for the convex hull of X:

$$\mathrm{hull}_G(X) = \{g \in G\,;\, x \leq g \leq x' \text{ for some } x, x' \in X\}.$$

We say that X is convex if $\mathrm{hull}_G(X) = X$. If H is a subgroup of G, we write $\mathrm{core}_G(H)$ for the intersection of the conjugates of H. The next two lemmas are undoubtedly well known.

LEMMA 3. *Let G be a right ordered group and let N be a normal subgroup. Then* $\mathrm{hull}_G(N)$ *is a subgroup.*

PROOF. Obviously $1 \in \mathrm{hull}_G(N)$. Let g, h be elements of $\mathrm{hull}_G(N)$, and choose $a, b, c, d \in N$ such that $a \leq g \leq b$ and $c \leq h \leq d$. Then we have

$$b^{-g^{-1}} \leq bg^{-1}.b^{-g^{-1}} = g^{-1} = ag^{-1}.a^{-g^{-1}} \leq a^{-g^{-1}},$$

which shows that g^{-1} belongs to $\mathrm{hull}_G(N)$, and

$$ca^h \leq ha^h = ah \leq gh \leq bh = hb^h \leq db^h,$$

so that gh belongs to $\mathrm{hull}_G(N)$.

LEMMA 4. *Let G be a right ordered group, and let H be a convex subgroup of G. Then the set $H \setminus G$ of cosets of H inherits a natural right ordering from G and G/K is right orderable, where* $K = \mathrm{core}_G(H)$.

PROOF. The ordering on $H \setminus G$ is defined by $Hg < Hg' \iff hg < h'g'$ for all $h, h' \in H$. It is easy to deduce that this ordering has the desired properties from the fact that each coset Hg is convex. Now G acts as order preserving automorphisms of $H \setminus G$ and this induces a faithful action of G/K, so that G/K is right orderable by Lemma 2.

We shall make much use of the following simple observation, (*cf.* § 7.2 of [6]).

LEMMA 5. *Let G be a right ordered group. Then the convex subgroups of G are totally ordered by inclusion, and the union of any non-empty collection of convex subgroups is again a convex subgroup.*

We shall need one technical Lemma which is essentially the same as Conrad's (3.1) of [3].

LEMMA 6. *Let a and g be elements of a right ordered group G. If $a > 1$ and $g^{-1} \in$* $\mathrm{hull}_G(\langle a \rangle)$ *then $a^g > 1$.*

PROOF. If $a > 1$ and $g^{-1} \in \mathrm{hull}_G(\langle a \rangle)$ then there is an integer i such that $a^i \leq g^{-1} < a^{i+1}$. This implies $g^{-1}a^{-i} \geq 1$ and $a^{i+1}g > 1$, whence $a^g = g^{-1}a^{-i}.a^{i+1}g > 1$.

LEMMA 7. *Let G be an* RO*-simple group. If N is a non-trivial normal subgroup of G then* $\mathrm{hull}_G(N) = G$.

PROOF. Let H be the convex hull of N. By Lemma 3, H is a subgroup, and so Lemma 4 shows that $G/\mathrm{core}_G(H)$ is right orderable. Now $\mathrm{core}_G(H)$ is a non-trivial subgroup of G because it contains N. Since G is RO-simple we thus have $\mathrm{core}_G(H) = G$, and hence $H = G$ as required.

Finally, the following is an easy variation on Lemma 13.2.1 of [7].

LEMMA 8. *Let G be a group and let H be a subgroup. Suppose that for any finite collection of elements $y_1, \ldots, y_n$ of $G - H$ there exist choices of sign $\epsilon_i = \pm 1$ such that the subsemigroup generated by $y_1^{\epsilon_1}, \ldots, y_n^{\epsilon_n}$ has empty intersection with H. Then there is a G-invariant ordering on the coset space $H \setminus G$, and $G/\mathrm{core}_G(H)$ is right orderable.*

PROOF. For each finite set $D \subseteq G - H$, let $\mathfrak{F}(D)$ be the set of all functions $\epsilon: G - H \longrightarrow \{\pm 1\}$ such that the subsemigroup generated by $\{g^{\epsilon(g)} ; g \in D\}$ has empty intersection with H. The sets $\mathfrak{F}(D)$ are non-empty by hypothesis, and have the finite intersection property because $\mathfrak{F}(D) \cap \mathfrak{F}(D') \supseteq \mathfrak{F}(D \cup D')$. By compactness (see Theorem 6.3.1 of [7]) there is a function ϵ in the intersection of all the $\mathfrak{F}(D)$. Now let $P = \{g \in G - H ; \epsilon(g) = 1\}$. It is clear now that G is the disjoint union of P, H and P^{-1}, and it is straightforward to check that P is a subsemigroup of G satisfying $P = HPH$. A right ordering on $H \setminus G$ can be defined by $Hg < Hg'$ if and only if $g'g^{-1} \in P$: the relation $<$ is well-defined because $P = HPH$, transitive because P is a subsemigroup, and defines a total ordering on $H \setminus G$. This ordering is plainly preserved by the group action. Thus $G/\mathrm{core}_G(H)$ is right orderable by Lemma 2.

4. **The Proof of Theorem B.** Let G be finitely generated and RO-simple, and let A be a non-trivial abelian normal subgroup of G. We shall say that a subgroup B of A is infinitesimal if and only if $\mathrm{hull}_A(B)$ is a proper subgroup of A. We shall make repeated use of the following:

LEMMA 9. *Let $B_1, \ldots, B_k$ be a finite collection of infinitesimal subgroups of A. Then the subgroup $\langle B_1, \ldots, B_k \rangle$ that they generate is also infinitesimal.*

PROOF. The convex hull in A of each B_i is a proper convex subgroup of A and since these are totally ordered (Lemma 5) and finite in number, there is a maximum and it contains the group generated by the B_i.

Returning to the proof of Theorem B, set

$$H = \{g \in G ; [A, g] \text{ is infinitesimal}\}.$$

Note that since A is normal and abelian, the set $[A, g] = \{a^{-1}g^{-1}ag ; a \in A, g \in G\}$ is always a subgroup. Moreover, since A is non-trivial, H obviously contains A, and if g, h are elements of H then $[A, gh^{-1}]$ is infinitesimal by Lemma 9, because it is contained in the subgroup generated by $[A, g]$ and $[A, h]$, so gh^{-1} belongs to H. Thus H is a subgroup of G which contains A. We shall use Lemma 8 to establish the following, from which Theorem B(i) will follow easily.

CLAIM. *There is a G-invariant ordering on the coset space $H \setminus G$, and $G/\mathrm{core}_G(H)$ is right orderable.*

By Lemma 8 it is sufficient to prove that if $y_1, \ldots, y_n$ is any finite collection of elements of $G-H$ then there is an n-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_i = \pm 1$ such that the subsemigroup generated by the $y_i^{\epsilon_i}$ does not meet H. Suppose this is false and that there is a choice of $y_1, \ldots, y_n$ such that for every choice of ϵ, the corresponding subsemigroup meets H. Let

F denote the free semigroup on $X_1, \ldots, X_n$, and for each ϵ let $w_\epsilon = w_\epsilon(X_1, \ldots, X_n)$ be an element of F such that $x_\epsilon = w_\epsilon(y_1^{\epsilon_1}, \ldots, y_n^{\epsilon_n})$ belongs to H. Set $Y := \{y_1^{\pm 1}, \ldots, y_n^{\pm 1}\}$. Choose a positive integer m such that every x_ϵ can be expressed as a word of length at most m in the elements of Y, and let X denote the set of all words of length at most m.

We now fix a proper convex subgroup A_0 of A and an element $a \in A$ with the following properties.

(2) $\quad A_0$ contains each of the infinitesimal subgroups $[A, x_\epsilon]$,

(3) $$\mathrm{hull}_G(\langle a \rangle) \text{ contains } X,$$

and

(4) $$a \notin A_0.$$

The idea is to choose A_0 so that (2) holds and so that every convex subgroup A_1 of A which properly contains A_0 satisfies $X \subseteq \mathrm{hull}_G(A_1)$: with such a proper convex subgroup A_0 chosen, (3) and (4) can be achieved by any choice of element a in $A - A_0$. Since the convex subgroups of A are totally ordered by inclusion, (4) implies

(4′) $$A_0 \subset \mathrm{hull}_G(\langle a \rangle).$$

We choose A_0 in the following way: for each $x \in X$, let D_x be the union of all the convex subgroups of A whose convex hulls in G do not contain x (or set $D_x = 1$ if $x = 1$). Then each D_x is a convex subgroup of A, by Lemma 5, and when $x \neq 1$ it is the unique maximal convex subgroup of A whose convex hull in G does not contain x. Moreover, since A is non-trivial and G is RO-simple, Lemma 7 shows that $G = \mathrm{hull}_G(A)$, and it follows that each D_x is a proper subgroup of A. We now have two finite families of proper convex subgroups of A: the D_x, for $x \in X$ and the $\mathrm{hull}_A([A, x_\epsilon])$, for each choice of ϵ. Again since the convex subgroups of A are totally ordered by inclusion, the collection made up of these two families has a maximum member which we denote by A_0. Fix any element a in $A - A_0$.

Now we claim that there is an element b in A such that for all $y \in Y$,

(5) $$a \in \mathrm{hull}_A(\langle [b, y] \rangle).$$

Suppose that there is no such b. For each y, let A_y be the set of all $b \in A$ such that

$$a \notin \mathrm{hull}_A(\langle [b, y] \rangle).$$

Then each A_y is a subgroup of A, and an element $b \in A$ satisfies (5) if and only if it does not belong to any of the A_y. Therefore if no b can be chosen we must have

$$A = \bigcup_{y \in Y} A_y.$$

This expresses A as a finite union of subgroups, and so for some y, A_y must have finite index in A. But it is easy to see that A/A_y is torsion-free for each y, and so there is a y such that $A_y = A$. This is a contradiction to the fact that y does not belong to H because it implies that, for this y, $\mathrm{hull}_A([A, y])$ does not contain a and hence that $[A, y]$ is infinitesimal. Thus b can be chosen to satisfy (5) for all y.

Now (5) shows that for each $y \in Y$, $\mathrm{hull}_A(\langle [b, y] \rangle)$ contains $\mathrm{hull}_A(\langle a \rangle)$, and therefore $\mathrm{hull}_G(\langle [b, y] \rangle)$ contains X, by (3). In particular, by Lemma 6,

$$(6) \qquad [b, y] > 1 \iff [b, y]^x > 1,$$

for any $x \in X$ and $y \in Y$. Now we claim that for all $y \in Y$,

$$(7) \qquad [b, y] > 1 \iff [b, y^{-1}] < 1.$$

In fact (7) follows from (6) together with the observation that $[b, y^{-1}]^{-1} = [b, y]^{y^{-1}}$. We now use (7) to fix a particular choice of ϵ: namely, for each i choose ϵ_i so that $[b, y_i^{\epsilon_i}] > 1$. Let $z_i = y_i^{\epsilon_i}$. Now x_ϵ is a positive word in the z_i, say $x_\epsilon = z_{i_1} \cdots z_{i_m}$. Now we consider the commutator $[b, x_\epsilon]$. In the first place this belongs to the proper convex subgroup A_0 of A. But secondly we can expand the commutator as follows:

$$[b, x_\epsilon] = [b, z_{i_m}][b, z_{i_{m-1}}]^{z_{i_m}} \cdots [b, z_{i_1}]^{z_{i_2} \cdots z_{i_m}}.$$

Here, every commutator lies in the positive cone by (6), and the first commutator, $[b, z_{i_m}]$, lies outside A_0, by (4′) and (5). But this implies that $[b, x_\epsilon]$ does not belong to A_0, a contradiction. This proves the Claim.

The proof of Theorem B(i) can be completed as follows. Since H contains A and A is normal we have $A \subseteq \mathrm{core}_G(H)$, so $\mathrm{core}_G(H)$ is a non-trivial subgroup of G. But G is RO-simple and $G/\mathrm{core}_G(H)$ is right orderable, so we must have $\mathrm{core}_G(H) = G$, and hence $G = H$.

Let $g_1, \ldots, g_m$ be a finite set of generators of G. Then, for each i, $[A, g_i]$ is infinitesimal. Since A is normal and the g_i generate G, so the subgroup generated by the $[A, g_i]$ is equal to $[A, G]$. Thus $[A, G]$ is infinitesimal. Therefore $[A, G]$ is trivial by Lemma 7, because it is normal and G is RO-simple. This shows that A is central in G, as asserted in Theorem B(i). The remaining parts of Theorem B are comparatively easy.

(ii) Suppose that g, h are elements of G such that $z = [g, h]$ is non-trivial and central. Interchanging g, h if necessary we may assume that $z > 1$. Now $\mathrm{hull}_G(\langle z \rangle) = G$ by Lemma 7, and therefore there are integers i and j such that $z^i \leq g < z^{i+1}$ and $z^j \leq h^2 < z^{j+1}$. From these inequalities it follows that $[g, h^2] < z^2$. However we also have $[g, h^2] = [g, h]^2 = z^2$, a contradiction.

(iii) Suppose that $H/\zeta(G)$ is a normal abelian subgroup of $G/\zeta(G)$. If g, h are any elements of H then $[g, h]$ is central and so $[g, h] = 1$ by part (ii) of the Theorem. Thus H is abelian. Since H is also normal, part (i) shows that H is central. Thus $H = \zeta(G)$.

(iv) Suppose now that G is non-abelian and has a central element $z > 1$. Since G is RO-simple, it is plain that there are no non-zero homomorphisms $G \longrightarrow \mathbb{R}$.

We have $\mathrm{hull}_G(\langle z \rangle) = G$, by Lemma 7, and so for any element g in G there is a unique integer i such that $z^i \leq g < z^{i+1}$. Let $\phi\colon G \longrightarrow \mathbb{Z}$ be the function defined by $\phi(g) = i$. Now it is easy to check that for all g, h in G,

$$\phi(gh) - \phi(g) - \phi(h) = 0 \text{ or } 1.$$

Thus $\delta\phi$ is a bounded 2-cocycle on G and it determines an element of $H_b^2(G, \mathbb{R})$. If the cohomology class it determines is the zero class then ϕ differs by a bounded amount from a homomorphism $\psi: G \longrightarrow \mathbb{R}$. But since ϕ is unbounded on the group generated by z, so also is ψ, a contradiction. Thus $H_b^2(G, \mathbb{R})$ is non-zero. In this case G must be non-amenable, because bounded cohomology vanishes for amenable groups by Trauber's Theorem (see § 3.0 of [4]).

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School of Mathematical Sciences
Queen Mary and Westfield College
London E1 4NS
United Kingdom