AN ALTERNATIVE PROOF OF DIESTEL’S THEOREM

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(Received 28 July, 1982)

Introduction. We show that Diestel’s theorem on weak compactness of subsets of \( L_1(\mu, X) \) can be derived as a simple corollary of James’s theorem. It is a pleasure to acknowledge several stimulating conversations with Dave Emmons and the remarks of an anonymous referee. Errors are, of course, solely mine.

Let \((T, \mathcal{F}, \mu)\) be a finite measure space and \(X\) a Banach space. Denote by \(L_1(\mu, X)\) the Banach space of (equivalence classes of) \(\mu\)-strongly measurable \(X\)-valued Bochner integrable functions \(f : T \to X\) normed by \(\|f\| = \int_T \|f(t)\| \, d\mu(t)\). In [3] Diestel has proved through the use of the factorization method in [2] the following result.

**Theorem.** Let \(K\) be a weakly compact convex subset of \(X\) and

\[ \tilde{K} = \{ f \in L_1(\mu, X) : f(t) \in K \quad \text{for almost all } t \in T \}; \]
then \(\tilde{K}\) is weakly compact in \(L_1(\mu, X)\).

In this note, we offer an alternative proof of Diestel’s theorem which relies instead on James’s theorem [5] and on Brooks’s extension of the classical Vitali’s theorem [1].

Before presenting our proof, we recall that \(V_\infty(\mu, X^*)\), where \(X^*\) is the continuous dual of \(X\), is isometrically isomorphic to \(L_1(\mu, X^*)\) with the correspondence between \(F \in V_\infty(\mu, X^*)\) and \(\phi \in L_1(\mu, X^*)\) given by \(\phi(f) = \int dF\). (For an explanation and properties of \(V_\infty(\mu, X^*)\) see [3] and his references.)

**Proof.** Pick an arbitrary \(\phi \in (L_1(\mu, X))^*\). If we can show that \(\phi\) attains its supremum on \(\tilde{K}\), James’s theorem [5, Theorem 5] assures us that \(\tilde{K}\) is relatively weakly compact. Since \(\tilde{K}\) is convex and closed (hence by Mazur’s theorem weakly closed) in \(L_1(\mu, X)\), the proof is then finished.

Let \(F \in V_\infty(\mu, X^*)\) correspond to \(\phi\). Towards showing that \(\phi\) attains its supremum on \(\tilde{K}\), select a pairwise disjoint sequence of elements \(T_i \in \mathcal{F}\) such that each \(T_i\) has positive measure and \(\bigcup_{i=1}^{\infty} T_i = T\). Consider the finite partition \(\pi_n = \{T_1, T_2, \ldots, T_{n-1}, \bigcup_{i=n}^{\infty} T_i\}\) in which \(T_n\) will denote \(\bigcup_{i=n}^{\infty} T_i\). Let \(\pi = \{\pi_n\}_{n=1}^{\infty}\). It is clear that for all integers \(n\), \(\pi_{n+1}\) is a refinement of \(\pi_n\).

For any partition \(\pi_n\) construct the function \(f_n \in L_1(\mu, X)\) such that

\[ f_n(t) = x_i \quad \text{for all } t \in T_i, \quad (i = 1, \ldots, n), \]

where \(x_i\) is characterized by the equality \(\langle x_i, F(T_i) \rangle = \sup_{y \in K} \langle y, F(T_i) \rangle\). Since \(K\) is nonempty and weakly compact, certainly \(x_i \in K\) for all \(i\). We are now going to take a suitable limit of a subsequence of these functions.

By our construction, for any $t$ in $T_1$, we can let $f_n(t) = f_n(T_1)$, where $f_n(T_1) \in K$ for all $n$. Since $K$ is weakly compact, the Eberlein–Šmulian Theorem guarantees a subsequence $f_n^{(1)}(T_1)$ which converges weakly to an element, say $f(T_1)$, in $K$. The procedure is now clear. We can now manufacture a function $f : T \rightarrow K$ such that $f$ is the almost everywhere limit of $f_n^{(n)}$, where for any $i = 1, \ldots, n - 1$, $f_n^{(i+1)}$ is a subsequence of $f_n^{(i)}$ such that for all $n$, $f_n^{(i)}(t) = f_n^{(i)}(T_{i+1})$ for all $t$ in $T_{i+1}$ and $f_n^{(i+1)}(T_{i+1})$ converges weakly to an element, say $f(T_{i+1})$, in $K$.

Since $K$ is weakly compact, for all $x \in K$ there exists $M > 0$ such that $\|x\| \leq M$. Using this fact it is now easy to show that the sequence $f_n^{(n)}$ is bounded and uniformly integrable. We can therefore apply Brooks's extension [1, Theorem 3] of Vitali's convergence theorem to claim that $f \in L_1(\mu, X)$ and hence $f \in \tilde{K}$ and that $\|f - f_n^{(n)}\|_1 \rightarrow 0$. Then, certainly $\int f_n^{(n)} dF \rightarrow \int f dF$.

We now claim that $\phi$ attains its supremum on $\tilde{K}$ at $f$. Suppose not; that is there exists $z \in \tilde{K}$ such that

$$\phi(z) = \int_T z dF > \int_T f dF = \phi(f).$$

(1)

For each partition $\pi$ in $\Pi$, define the linear operator $E_\pi : L_1(\mu, X) \rightarrow L_1(\mu, X)$ by

$$E_\pi(z) = \sum_{T_i \in \pi} \int_{T_i} z(t) d\mu(t) \chi_{T_i},$$

where $\chi_A$ is the characteristic function of $A$ and the $0/0 = 0$ convention is in force. By the mean value theorem for the Bochner integral [4, Corollary 8, p. 48] and the convexity of $K$, certainly $\frac{1}{\mu(T_i)} \int_{T_i} z(t) d\mu(t) \in K$. Thus $E_\pi(z) \in \tilde{K}$. We can now apply Lemma 1 in [3, p. 67] to assert that $\|E_\pi(z) - z\|_1 \rightarrow 0$. Then certainly $\int_T E_\pi(z) dF \rightarrow \int_T z dF$.

However by construction,

$$\int_T E_\pi(z) dF \leq \int_T f_n^{(n)} dF.$$  

By taking limits on both sides, we obtain our sought-after contradiction to (1).

REFERENCES

1. J. K. Brooks, Equicontinuous sets of measures and applications to Vitali's integral convergence theorem and control measures, Advances in Math. 10 (1973), 165-171.

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