Canad. J. Math. Vol. **00** (0), 2025 pp. 1–35
http://dx.doi.org/10.4153/xxxx

Subspaces spanned by eigenforms with nonvanishing twisted central L-values

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Abstract. **In this paper, we construct explicit spanning sets for two spaces of modular forms. One is the subspace generated by integral-weight Hecke eigenforms with nonvanishing quadratic twisted central L-values. The other is a subspace generated by half-integral weight Hecke eigenforms with certain nonvanishing Fourier coefficients. Along the way, we show that these subspaces are isomorphic via the Shimura lift.**

1 Introduction

Let $\ell \geq 2$ be an integer. For $N \geq 1$ and a Dirichlet character χ modulo N, let $M_\ell(N,\chi)$ and $S_\ell(N,\chi)$ be the space of modular forms and cuspforms of weight ℓ, level N and nebentypus χ, respectively. When χ is trivial, we simply write $M_\ell(N)$ and $S_\ell(N)$. Let $M_{\ell+1/2}(4N)$ and $S_{\ell+1/2}(4N)$ be the space of modular forms and the space of cuspforms of weight $\ell+1/2$ for $\Gamma_0(4N)$, respectively. For $N=1$ we recall the Kohnen [10] plus space as the subspace

$$M^+_{\ell+1/2}(4) := \{f = \sum_{n\geq 0} c_f(n)q^n \in M_{\ell+1/2}(4) \mid c_f(n) = 0 \text{ if } (-1)^\ell n \equiv 2,3 \pmod 4\},$$

and put $S^+_{\ell+1/2}(4) := M^+_{\ell+1/2}(4) \cap S_{\ell+1/2}(4)$. Let D be a fundamental discriminant (i.e. $D=1$ or is the discriminant of a quadratic field) such that $(-1)^\ell D > 0$. Following Kohnen [10, p. 251], for $f(z) = \sum_{n\geq 0} c_f(n)q^n \in M^+_{\ell+1/2}(4)$, we define its D-th Shimura lift as

$$\mathcal{S}_D\left(\sum_{n\geq 0} c_f(n)q^n\right) := \frac{c_f(0)}{2}L_D(1-\ell) + \sum_{n\geq 1}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{\ell-1}c_f\left(|D|\frac{n^2}{d^2}\right)\right)q^n \quad (1.1)$$

where $\left(\frac{D}{\cdot}\right)$ is the Kronecker symbol. It is known that $\mathcal{S}_D$ maps $M^+_{\ell+1/2}(4)$ to $M_{2\ell}(1)$ and $S^+_{\ell+1/2}(4)$ to $S_{2\ell}(1)$, and commutes with the action of Hecke operators; see Kohnen [10, Theorem 1] and Shimura [19].

Now we recall the Selberg identity on the Shimura lift. Let $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} \in M_{1/2}(4)$ be the Jacobi theta function. Selberg observed that for a normalized Hecke

AMS subject classification: 11F37; 11F67.
Keywords: Rankin-Cohen brackets, Shimura lift, nonvanishing of twisted central L-values.

eigenform $f(z) \in M_\ell(1)$ with $a_f(1) = 1$, the first Shimura lift provides the identity

$$\mathcal{S}_1(f(4z)\theta(z)) = f(z)^2 \in M_{2\ell}(1). \tag{1.2}$$

For a fundamental discriminant D with $(-1)^k D > 0$ with $k \geq 4$ an integer, if one defines

$$\mathcal{F}_D(z) := \mathrm{Tr}_1^D(G_{k,D}(z)^2) \in M_{2k}(1)$$
$$\mathcal{G}_D(z) := \frac{3}{2}\left(1 - \left(\frac{D}{2}\right)2^{-k}\right)^{-1} \mathrm{pr}^+ \, \mathrm{Tr}_4^{4D}(G_{k,4D}(4z)\theta(|D|z)) \in M^+_{k+1/2}(4),$$

then Kohnen-Zagier [9, Proposition 3] proved the following generalization of (1.2):

$$\mathcal{S}_D(\mathcal{G}_D(z)) = \mathcal{F}_D(z). \tag{1.3}$$

We must make several definitions for the above to make sense. The Eisenstein series $G_{k,D}$ and $G_{k,4D}$ are given by [9, p. 185]

$$G_{k,D}(z) := \frac{L_D(1-k)}{2} + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\right)q^n \in M_k\left(|D|, \left(\frac{D}{\cdot}\right)\right), \tag{1.4}$$

$$G_{k,4D}(z) := G_{k,D}(4z) - 2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z) \in M_k\left(4|D|, \left(\frac{D}{\cdot}\right)\right), \tag{1.5}$$

where $L_D(s) = \sum_{n\geq 1}\left(\frac{D}{n}\right)n^{-s}$. The operator pr^+ is the projection from $M_{\ell+1/2}(4)$ to $M^+_{\ell+1/2}(4)$ given by [9, p. 195]

$$(\mathrm{pr}^+ g)(z) = \frac{1-(-1)^\ell i}{6}(\mathrm{Tr}_4^{16} Vg)(z) + \frac{1}{3}g(z), \tag{1.6}$$

where $V(g)(z) = g(z + \frac{1}{4}) = g(z)|_{k+1/2}\left[\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right]$, using the notation of (1.7) and (1.8). Additionally, for $N \mid M$, Tr_N^M is the trace map

$$\mathrm{Tr}_N^M : M_m(M) \longrightarrow M_m(N), \quad g \mapsto \sum_{\gamma \in \Gamma_0(M)\backslash\Gamma_0(N)} g|_m\gamma, \tag{1.7}$$

where for any real number m and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{R})$ we define the slash operator [2, Theorem 7.1]

$$(g|_m\gamma)(z) = \det(\gamma)^{m/2}(cz+d)^{-m}g\left(\frac{az+b}{cz+d}\right). \tag{1.8}$$

On the other hand, the Selberg identity (1.2) for the first Shimura lift has been generalized to the setting of Rankin-Cohen brackets. Let us first introduce the definition of Rankin-Cohen brackets for modular forms.

***Definition* 1.1** Let $f(z) \in M_a(\Gamma)$ and $g(z) \in M_b(\Gamma)$ be modular forms for some congruence subgroup Γ of weights a and b, respectively. For an nonnegative integer e,

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we define the e-th Rankin-Cohen bracket as

$$[f(z), g(z)]_e := \sum_{r=0}^{e} (-1)^r \binom{e+a-1}{e-r} \binom{e+b-1}{r} f(z)^{(r)} g(z)^{(e-r)}, \tag{1.9}$$

where $f(z)^{(r)}$ is the r-th normalized derivative $f(z)^{(r)} := \frac{1}{(2\pi i)^r} \frac{d^r f(z)}{dz^r}$ of f. Here a, b can be in $\frac{1}{2}\mathbb{Z}$ and the binomial coefficients are defined through gamma functions. Moreover, $[f, g]_e \in M_{a+b+2e}(\Gamma)$ and $[f, g]_e \in S_{a+b+2e}(\Gamma)$ for $e > 1$; see [2, Theorem 7.1]. We remark that the Rankin-Cohen bracket defined in Zagier [25, (73)] is related to (1.9) through $F_e^{(a,b)}(f(z), g(z)) = (-2\pi i)^e e! [f(z), g(z)]_e$; see [12, (1.1)].

Choie-Kohnen-Zhang [1] and Xue [24] independently showed that if $k \geq 4$ is an even integer, $f(z) \in M_k(1)$ is a normalized Hecke eigenform, and e is a nonnegative integer, then

$$\mathcal{S}_1([f(4z), \theta(z)]_e) = \frac{\binom{k+e-1}{e}}{\binom{k+2e-1}{2e}} [f(z), f(z)]_{2e}. \tag{1.10}$$

Note that (1.10) was also proved in [17, Proposition B1] when f is an Eisenstein series. Recently, Wang [21] generalized (1.10) to higher-level forms. Let $k \geq 4$ and $e > 0$ be integers with $\ell = k + 2e$ and let D be a fundamental discriminant such that $(-1)^\ell D > 0$. We introduce functions

$$\mathcal{F}_{D,k,e}(z) := \mathrm{Tr}_1^D([G_{k,D}(z), G_{k,D}(z)]_{2e}) \in S_{2\ell}(1), \tag{1.11}$$

$$\mathcal{G}_{D,k,e}(z) := \frac{3}{2}\left(1 - \left(\frac{D}{2}\right) 2^{-k}\right)^{-1} \mathrm{pr}^+ \, \mathrm{Tr}_4^{4D} \left[G_{k,4D}(z), \theta(|D|z)\right]_e \in S^+_{\ell+1/2}(4) \tag{1.12}$$

Note that both $\mathcal{F}_{D,k,e}(z)$ are $\mathcal{G}_{D,k,e}(z)$ are cuspforms, since $e > 0$. Now, we state our first main result, which can be viewed as a combination of (1.3) and (1.10).

***Theorem* 1.1** *Let D be an odd fundamental discriminant such that $(-1)^\ell D > 0$ and let $k \geq 4$ and $e > 0$ be integers such that $k + 2e = \ell$. Then we have the identity*

$$\mathcal{S}_D\left(\mathcal{G}_{D,k,e}\right) = |D|^e \frac{\binom{k+e-1}{e}}{\binom{k+2e-1}{2e}} \mathcal{F}_{D,k,e}. \tag{1.13}$$

We have required that $e > 0$ because the case $e = 0$ is exactly (1.3). Our next main result concerns the nonvanishing of twisted central values of L-functions associated to Hecke eigenforms. Before stating the precise result, let us first introduce some notation.

***Definition* 1.2** Let D be a fundamental discriminant such that $(-1)^\ell D > 0$.

(1) Let $S_{2\ell}^{0,D}(1)$ denote the subspace of $S_{2\ell}(1)$ generated by normalized Hecke eigenforms f with nonzero central twisted L-values $L(f, D, \ell)$, where $L(f, D, s) = \sum_{n \geq 1} \left(\frac{D}{n}\right) a_f(n) n^{-s}$ is the L-function of f twisted by $\left(\frac{D}{\cdot}\right)$. We write $S_{2\ell}^{-,D}(1)$ for the orthogonal complement of $S_{2\ell}^{0,D}(1)$, which is spanned by Hecke eigenforms with vanishing central twisted L-values.

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(2) Let $S^{0,D}_{\ell+1/2}(4)$ be the subspace of $S^{+}_{\ell+1/2}(4)$ generated by Hecke eigenforms $g = \sum_{n\geq 1} c_g(n)q^n$ with $c_g(|D|) \neq 0$. We write $S^{-,D}_{\ell+1/2}(4)$ for the orthogonal complement of $S^{0,D}_{\ell+1/2}(4)$, which is spanned by Hecke eigenforms $g = \sum_{n\geq 1} c_g(n)q^n$ with $c_g(|D|) = 0$.

The twisted L-function $L(f, D, s)$, originally defined for $\mathrm{Re}(s) \gg 0$, can be analytically continued to the whole complex plane, and for a Hecke eigenform $f \in S_{2\ell}(1)$ satisfies [16, Lemma 9.2]:

$$\Lambda(f, D, s) = (-1)^{\ell} \left(\frac{D}{-1}\right) \Lambda(f, D, 2\ell - s),$$

where $\Lambda(f, D, s) = (2\pi)^{-s}\Gamma(s)L(f, D, s)$ is the completed twisted L-function of f. Since $\left(\frac{D}{-1}\right)$ is the sign of D, the assumption $(-1)^{\ell}D > 0$ implies that the functional equation for $L(f, D, s)$ has a positive sign. Therefore, the subspace $S^{0,D}_{2\ell}(1)$ in Definition 1.2 (1) is not trivially zero. It is speculated that the central L-value $L(f, D, \ell)$ is nonvanishing for every Hecke eigenform $f \in S_{2\ell}(1)$. Thus, it is believed that $S_{2\ell}(1) = S^{0,D}_{2\ell}(1)$ for every fundamental discriminant D. For further discussion, see Section 7.

Our second main result gives an explicit construction of a set of generators for the subspaces $S^{0,D}_{2\ell}(1)$ and $S^{0,D}_{\ell+1/2}(4)$. We hope this result would help investigate the aforementioned speculation on the nonvanishing of twisted central L-values. Furthermore, we prove that the D-th Shimura lift $\mathcal{S}_D$ gives an isomorphism between $S^{0,D}_{\ell+1/2}(4)$ and $S^{0,D}_{2\ell}(1)$, which generalizes Kohnen's results [10, Theorem 2] and [24, Proposition 3.3].

***Theorem* 1.2** *Let D be an odd fundamental discriminant with $(-1)^{\ell}D > 0$. Then*

$$S^{0,D}_{\ell+1/2}(4) = \mathrm{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell}, \quad \textit{and} \quad S^{0,D}_{2\ell}(1) = \mathrm{Span}\{\mathcal{F}_{D,k,e}\}_{2k+4e=2\ell},$$

where $k \geq 4$ and $e > 0$. Additionally, the restricted D-th Shimura lift

$$\mathcal{S}_D : S^{0,D}_{\ell+1/2}(4) \to S^{0,D}_{2\ell}(1)$$

is an isomorphism.

We assume D to be odd throughout the paper in order to avoid the technical complications caused by even D, although we believe our results continue to hold in this case.

This paper is organized as follows. Section 2 discusses the main results of this paper. The proof of Theorem 1.1 is based on the same idea as the proof of (1.10) (see [1] and [24]), but requires explicit computations of the Fourier coefficients of both sides of (1.13). Most of the technical details required for the proof of Theorem 1.1 are presented in Section 6. Based on the Petersson inner product formalae for $\mathcal{F}_{D,k,e}$ and $\mathcal{G}_{D,k,e}$ derived in Section 5, we explicitly construct a spanning set for $S^{0,D}_{2\ell}(1)$ (Proposition 2.4). We then show that the D-th Shimura lift is an isomorphism from $S^{0,D}_{\ell+1/2}(4)$ to $S^{0,D}_{2\ell}(1)$ (Proposition 2.2). Finally, using these results we prove Proposition 2.6, explicitly constructing a spanning set for $S^{0,D}_{\ell+1/2}(4)$ and finishing the proof of Theorem 1.2.

The remaining sections are dedicated to proofs of the results needed in Section 2. Section 3 proves an alternate formula for $\mathcal{G}_{D,k,e}$ which we use to compute its Fourier coefficients in Section 6. Section 4 recalls the theory of Eisenstein series, which will be useful to the Fourier development of $\mathcal{F}_{D,k,e}$ and $\mathcal{G}_{D,k,e}$ in Section 6. Assuming those two sections, Section 5 derives Petersson inner product formulae for $\mathcal{F}_{D,k,e}$ and $\mathcal{G}_{D,k,e}$ via the Rankin-Selberg convolution. In Section 6, we carry out the computations of Fourier coefficients for Theorem 1.1. Section 7 discusses the relationship between these results and their potential applications to the nonvanishing of twisted central L-values of Hecke eigenforms in $S_{2\ell}(1)$.

2 Selberg identity and spanning sets of subspaces

This section proves our main results, assuming the necessary results to be proved later. We begin by proving Theorem 1.1, a generalization of the Selberg identity.

Proof Recall that $\mathcal{G}_{D,k,e}$ (1.12) and $\mathcal{F}_{D,k,e}$ (1.11) are cuspforms. Write

$$\mathcal{S}_D(\mathcal{G}_{D,k,e}(z)) = \sum_{n\geq 1} g_{D,k,e}(n)q^n \quad \text{and} \quad \mathcal{F}_{D,k,e}(z) = \sum_{n\geq 1} f_{D,k,e}(n)q^n.$$

Comparing the Fourier coefficients $f_{D,k,e}(n)$ and $g_{D,k,e}(n)$ that are respectively given by Lemma 6.5 and Lemma 6.6, it suffices to show for each nonnegative integer pair (a_1, a_2) with $a_1 + a_2 = n|D_1|$ that

$$\begin{aligned} &\binom{k+e-1}{e}\sum_{r=0}^{2e}(-1)^r a_1^r a_2^{2e-r}\binom{2e+k-1}{2e-r}\binom{2e+k-1}{r} \\ &= \binom{k+2e-1}{2e}\sum_{r+s=e}(-1)^r\binom{k+e-1}{s}\binom{e-1/2}{r}4^r(a_2-a_1)^{2s}(a_1a_2)^r. \end{aligned} \tag{2.1}$$

Without loss of generality we may assume that $R \leq S$ and compare the coefficients of the monomial $a_1^R a_2^S$ of the two sides of (2.1). The $a_1^R a_2^S$-coefficient on the left hand side of (2.1) is

$$(-1)^R\binom{k+e-1}{e}\binom{2e+k-1}{2e-R}\binom{2e+k-1}{R},$$

and the right hand side of (2.1) has $a_1^R a_2^S$-coefficient

$$\begin{aligned} &\binom{k+2e-1}{2e}\sum_{r=0}^{R}(-1)^r\binom{k+e-1}{e-r}\binom{e-1/2}{r}4^r\binom{2e-2r}{R-r}(-1)^{R-r} \\ =&(-1)^R\binom{k+2e-1}{2e}\sum_{r=0}^{R}\binom{k+e-1}{e-r}\binom{e-1/2}{r}4^r\binom{2e-2r}{R-r}. \end{aligned}$$

Using Lemma 2.1, we finish the proof of Theorem 1.1. ■

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Lemma 2.1 *Let $R \le e$ be nonnegative and $k \ge 4$. Then we have the following identity*

$$\binom{k+e-1}{e}\binom{k+2e-1}{2e-R}\binom{k+2e-1}{R} = \binom{k+2e-1}{2e}\sum_{r=0}^{R} 4^r \binom{k+e-1}{e-r}\binom{e-1/2}{r}\binom{2e-2r}{R-r},$$

where fractional binomial coefficients are defined by the Γ function.

Proof We reproduce the proof of [24, Proposition 2.1]. By definition we have

$$\binom{e-1/2}{r} = \frac{\Gamma(e+1/2)}{\Gamma(r+1)\Gamma(e+1/2-r)}.$$

By Legendre's duplication formulas we have

$$\Gamma(e+1/2) = \frac{(2e)!}{4^e e!}\sqrt{\pi}, \qquad \Gamma(e-r+1/2) = \frac{(2(e-r))!}{4^{e-r}(e-r)!}\sqrt{\pi}.$$

These together yield

$$\binom{e-1/2}{r} = \frac{(2e)!4^{e-r}(e-r)!\sqrt{\pi}}{r!4^e e!(2(e-r))!\sqrt{\pi}} = \frac{(2e)!4^{-r}(e-r)!}{r!e!(2(e-r))!},$$

which yields the following formula for each term on the right hand side

$$4^r\binom{k+2e-1}{2e}\binom{k+e-1}{e-r}\binom{e-1/2}{r}\binom{2e-2r}{R-r} = \frac{(k+2e-1)!(k+e-1)!}{(k-1)!(k+r-1)!(R-r)!(2e-R-r)!e!r!}.$$

The left hand side expands in to

$$\binom{k+e-1}{e}\binom{k+2e-1}{2e-R}\binom{k+2e-1}{R} = \frac{(k+e-1)!(k+2e-1)!(k+2e-1)!}{e!(k-1)!(2e-R)!(k+R-1)!R!(k+2e-1-R)!}.$$

If we cancel $(k+e-1)!(k+2e-1)!$ from both sides, and multiply by $R!(k+2e-R-1)$, we see that it suffices to show

$$\sum_{r=0}^{R}\binom{R}{R-r}\binom{k+2e-R-1}{k+r-1} = \binom{k+2e-1}{k+R-1}.$$

After applying the involution $r \mapsto R-r$, this is then the Vandermonde's identity [18, p.11]

$$\sum_{j=0}^{t}\binom{n}{j}\binom{m}{t-j} = \binom{n+m}{t}$$

for the case of $n = R$, $m = k+2e-R-1$, and $t = k+R-1$. ■

We now build toward the proof of Theorem 1.2. We begin by showing that the D-th Shimura lift gives rise to an isomorphism between $S^{0,D}_{\ell+1/2}(4)$ and $S^{0,D}_{2\ell}(1)$, which is a generalization of [10, Theorem 2] for $D = 1$.

Proposition 2.2. *Let D be an odd fundamental discriminant with $(-1)^\ell D > 0$. Then the D-th Shimura lift $\mathcal{S}_D$ restricts to an isomorphism $S^{0,D}_{\ell+1/2}(4) \to S^{0,D}_{2\ell}(1)$ for all $\ell \ge 6$.*

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Proof Recall that by [10, Theorem 1] or [9, p. 182], if $g = \sum_{n\geq 1} c_g(n)q^n \in S^+_{\ell+1/2}(4)$ is a Hecke eigenform and $f \in S_{2\ell}(1)$ is the normalized Hecke eigenform corresponding to g, then $\mathcal{S}_D(g) = c_g(|D|)f$. This means that $\mathcal{S}_D$ is a monomorphism when restricted to $S^{0,D}_{\ell+1/2}(4)$. Thus, in order to show $\mathcal{S}_D$ restricts to an isomorphism from $S^{0,D}_{\ell+1/2}(4)$ to $S^{0,D}_{2\ell}(1)$ it suffices to show that $\dim S^{0,D}_{\ell+1/2}(4) = \dim S^{0,D}_{2\ell}(1)$ since $c_g(|D|) = 0$ if and only if $L_D(f,\ell) = 0$ by Theorem [9, Theorem 1].

Note that $\dim S^{0,D}_{2\ell}(1)$ is the number of Hecke eigenforms in $S_{2\ell}(1)$ with nonzero central twisted L-value, and $\dim S^{0,D}_{\ell+1/2}(4)$ is the number of Hecke eigenforms in $S^+_{\ell+1/2}(4)$ with nonzero $|D|$-th Fourier coefficient. According to [9, Theorem 1], these two nonvanishing conditions are the same under the Shimura correspondence, thus we conclude that $\dim S^{0,D}_{\ell+1/2}(4) = \dim S^{0,D}_{2\ell}(1)$. ■

Remark 2.3 In the $\ell = 5, 7$ case, the space of cuspforms $S_{2\ell}(1)$ is zero, and so is the space $S^+_{\ell+1/2}(4)$. So this proposition is trivially true.

We now construct an explicit spanning set for $S^{0,D}_{2\ell}(1)$. Before doing so, we need to introduce the period of a modular form. For $f \in S_{2\ell}(1)$ and $0 \leq t \leq 2\ell - 2$, the t-th period of f is given by

$$r_t(f) := \frac{t!}{(-2\pi i)^{t+1}} L(f, t+1). \tag{2.2}$$

Here the L-series of $f(z) = \sum_{n\geq 1} a_n q^n$ is $L(f,s) = \sum_{n\geq 1} a_n n^{-s}$, which converges for $\mathrm{Re}(s) \gg 0$ and can be extended analytically to the whole complex plane; for details, see [14].

Proposition 2.4. *The set $\{\mathcal{F}_{D,k,e}\}_{2k+4e=2\ell}$ for $1 \leq e \leq \lfloor \frac{\ell-4}{2} \rfloor$ spans $S^{0,D}_{2\ell}(1)$, for all $\ell \geq 6$.*

Proof By Proposition 5.6, we know that if $g \in S^{-,D}_{2\ell}(1)$ then g is orthogonal to the subspace of $S^D_{2\ell}(1)$ spanned by $\{\mathcal{F}_{D,k,e}\}_{2k+4e=2\ell}$. So it suffices to show that the orthogonal complement of the span of $\{\mathcal{F}_{D,k,e}\}_{2k+4e=2\ell}$ is contained in $S^{-,D}_{2\ell}(1)$.

We will show that any modular form $G = \sum_j c_j g_j$ which is a linear combination of normalized Hecke eigenforms in $g_j \in S^{0,D}_{2\ell}(1)$ such that $\langle G, \mathcal{F}_{D,k,e}\rangle = 0$ for all $\mathcal{F}_{D,k,e}$ must be zero.

Note that Proposition 5.6 and (2.2) imply that

$$\langle \mathcal{F}_{D,k,e}, g_j\rangle = \frac{1}{2}\frac{\Gamma(2k+4e-1)\Gamma(k+2e)}{(2e)!(4\pi)^{2k+4e-1}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}\frac{(-2\pi i)^{2k+2e-1}}{(2k+2e-2)!}L(g_j, D, k+2e)r_{2k+2e-2}(g_j).$$

Thus, the orthogonality condition $\langle G, \mathcal{F}_{D,k,e}\rangle = 0$ is equivalent to

$$\sum_j c_j L(g_j, D, k+2e) r_{2k+2e-2}(g_j) = 0. \tag{2.3}$$

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Following an idea from the proof of [6, Theorem 1], we define another form in $S_{2\ell}(1)$ by

$$F = \sum_j c_j L(g_j, D, k+2e) g_j.$$

Hence (2.3) implies that

$$r_{2k+2e-2}(F) = \sum_j c_j L(g_j, D, k+2e) r_{2k+2e-2}(g_j) = 0.$$

As $1 \le e \le \lfloor \frac{\ell-4}{2} \rfloor$ and $k + 2e = \ell$, we have $\ell - 2 \ge k \ge 4$. Then $t = 2k + 2e - 2$ ranges through all even values $\ell + 2 \le t \le 2\ell - 4$, so $r_t(F) = 0$ for all even $\ell + 2 \le t \le 2\ell - 4$. As a result of the following lemma, we have $F = 0$. Since $L(g_j, D, k + 2e) \ne 0$ as $g_j \in S^{0,D}_{2\ell}(1)$, we must have $c_j = 0$ for all j, and thus $G = 0$. ■

Lemma 2.5 *Let $F \in S_{2\ell}(1)$ and $\ell \ge 6$, and let $r_t(F)$ be the t-th period of F. If $r_t(F) = 0$ for all even t such that $\ell + 2 \le t \le 2\ell - 4$, then $F = 0$.*

Proof We follow the idea of [23]. By the Eichler-Shimura theory [14, Proposition 2.3 (b)] and [23, Remark 2.4], we know that $F = 0$ if and only if $r_t(F) = 0$ for all even $2 \le t \le 2\ell - 4$. By the Eichler-Shimura relation

$$r_t(F) + (-1)^t r_{2\ell-2-t}(F) = 0, \tag{2.4}$$

and the assumption that $r_t(F) = 0$ for all even $\ell + 2 \le t \le 2\ell - 4$, we know that $r_t(F) = 0$ also for all even $2 \le t \le \ell - 4$. To show that the periods $\ell - 4 < t < \ell + 2$ are zero, we split into cases based on the parity of ℓ.

(1) If ℓ is even, it suffices to show that $r_\ell(F) = r_{\ell-2}(F) = 0$. Since ℓ is even, by (2.4)

$$r_\ell(F) + r_{\ell-2}(F) = 0. \tag{2.5}$$

Substituting $t = \ell - 2$ into the Eichler-Shimura relation

$$(-1)^t r_t(F) + \sum_{\substack{0 \le m \le t \\ m \equiv 0 \pmod 2}} \binom{t}{m} r_{2\ell-2-t+m}(F) + \sum_{\substack{0 \le m \le 2\ell-2-t \\ m \equiv t \pmod 2}} \binom{2\ell-2-t}{m} r_m(F) = 0 \tag{2.6}$$

and noting that $r_0(F) + r_{2\ell-2}(F) = 0$, we obtain

$$\left(\binom{\ell}{2} + 1\right) r_{\ell-2}(F) + 2 r_\ell(F) = 0.$$

This equation, along with (2.5) implies that $r_\ell(F) = r_{\ell-2}(F) = 0$ for $\ell \ge 6$.

(2) If ℓ is odd, it suffices to show that $r_{\ell-3}(F) = r_{\ell-1}(F) = r_{\ell+1}(F) = 0$. Substituting $t = \ell - 1$ into (2.6), we get

$$3 r_{\ell-1}(F) + \binom{\ell-1}{2} r_{\ell+1}(F) + \binom{\ell-1}{\ell-3} r_{\ell-3}(F) = 0.$$

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Since $\binom{\ell-1}{2} = \binom{\ell-1}{\ell-3}$, and we know by (2.4) that $r_{\ell-3}(F) + r_{\ell+1}(F) = 0$, we conclude that $r_{\ell-1}(F) = 0$. Substituting $t = \ell + 1$ into (2.6) yields

$$2r_{\ell-3}(F) + \binom{\ell+1}{2} r_{\ell-1}(F) + \left(1 + \binom{\ell+1}{4}\right) r_{\ell+1}(F) = 0,$$

and noting that $r_{\ell-1}(F) = 0$, we conclude by (2.4) that

$$r_{\ell-3}(F) = r_{\ell+1}(F) = 0.$$

This finishes the proof. ∎

Finally, we construct a spanning set for $S^{0,D}_{\ell+1/2}(4)$ and finish the proof of Theorem 1.2.

Proposition 2.6. *The set $\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell}$ for $1 \leq e \leq \lfloor \frac{\ell-4}{2} \rfloor$ spans the subspace $S^{0,D}_{\ell+1/2}(4)$, for all $\ell \geq 6$.*

Proof For a Hecke eigenform $g \in S^{-,D}_{\ell+1/2}(4)$, we have $\langle g, \mathcal{G}_{D,k,e} \rangle = 0$ by Proposition 5.7. So g is orthogonal to $\operatorname{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell}$ and thus

$$\operatorname{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell} \subseteq S^{0,D}_{\ell+1/2}(4). \tag{2.7}$$

Note that Theorem 1.1 implies that

$$\dim \operatorname{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell} \geq \dim \operatorname{Span}\{\mathcal{F}_{D,k,e}\}_{k+2e=\ell}. \tag{2.8}$$

By Propositions 2.2 and 2.4, we have

$$\operatorname{Span}\{\mathcal{F}_{D,k,e}\}_{k+2e=\ell} = S^{0,D}_{2\ell}(1) \quad \text{and } \dim S^{0,D}_{2\ell}(1) = \dim S^{0,D}_{\ell+1/2}(4). \tag{2.9}$$

Now (2.7), (2.8) and (2.9) together imply that $\dim \operatorname{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell} \geq \dim S^{0,D}_{\ell+1/2}(4)$. So we conclude that $\operatorname{Span}\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell} = S^{0,D}_{\ell+1/2}(4)$. ∎

Combining Propositions 2.2, 2.4 and 2.6, we complete the proof of Theorem 1.2.

3 Projection

In this section we prove an alternate formula for $\mathcal{G}_{D,k,e}$ (1.12):

$$\mathcal{G}_{D,k,e}(z) = \operatorname{Tr}^{4D}_{4}[G_{k,D}(4z), \theta(|D|z)]_e.$$

A similar formula is implicit in equations (6) and (7) in [9]. This formula allows us to compute the Fourier coefficients (Proposition 6.4).

We need to introduce some notation and facts needed for the proof of Lemma 3.2. Let

$$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) = \{(a:b) : a, b \in \mathbb{Z}/N\mathbb{Z}, \gcd(a,b,N) = 1\}/\sim$$

be the projective line over $\mathbb{Z}/N\mathbb{Z}$, where $(a:b) \sim (a':b')$ if there exists $u \in (\mathbb{Z}/N\mathbb{Z})^*$ such that $a = ua'$, $b = ub'$. It is known that there is a bijection between $\Gamma_0(N)\backslash \operatorname{SL}_2(\mathbb{Z})$ and $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, which sends a coset representative $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to the class $(c : d)$ in

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$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, see [20, Proposition 3.10]. For future reference, we prove a result on coset representatives of certain quotients of congruence subgroups.

***Lemma* 3.1** *Let $N \in \mathbb{N}$ and let $S \in \mathbb{N}$ be squarefree with $(N, S) = 1$. Then*

$$\left\{ \begin{bmatrix} 1 & 0 \\ NS_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \quad : \quad S_1 \mid S, \quad \mu \bmod \frac{S}{S_1} \right\}$$

is a set of coset representatives for $\Gamma_0(NS)\backslash\Gamma_0(N)$.

Proof The statement follows easily fom the description of the cosets given in ([5], p. 276). ∎

***Lemma* 3.2** *Let $\ell \geq 1$ be an integer and D be odd. We have $V\,\mathrm{Tr}_4^{4D}\, g = \mathrm{Tr}_{16}^{16D}\, Vg$ for all $g \in M_{\ell+1/2}(4|D|)$.*

Proof The statement was mentioned in [9, p. 195], we only sketch it. We first remark that by direct calculation, $Vg \in M_{\ell+\frac{1}{2}}(16|D|)$, so $\mathrm{Tr}_{16}^{16D}\, Vg$ is well-defined. Note that applying the fixed set of cosets for $\Gamma_0(4D)\backslash\Gamma_0(4)$ and $\Gamma_0(16D)\backslash\Gamma_0(16)$ given by Lemma 3.1 to $N = 4, 16$ and $S = |D|$, we have the following explicit formulas (see (1.8) for the definition of slash operators)

$$V\,\mathrm{Tr}_4^{4D}\, g(z) = \sum_{D_1 D_2 = D} \sum_{\mu \bmod |D_2|} g(z)|_\ell \gamma_{D_1,\mu} \left[\begin{smallmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{smallmatrix}\right],$$
$$\mathrm{Tr}_{16}^{16D}\, Vg(z) = \sum_{D_1 D_2 = D} \sum_{\mu \bmod |D_2|} g(z)|_\ell \left[\begin{smallmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{smallmatrix}\right] \gamma'_{D_1,\mu},$$

where

$$\gamma_{D_1,\mu} = \begin{bmatrix} 1 & 0 \\ 4|D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \gamma'_{D_1,\mu} = \begin{bmatrix} 1 & 0 \\ 16|D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}.$$

And the outer sums are over all factorizations of D into a product of fundamental discriminants D_1, D_2. Therefore, to prove the desired equality it suffices to show that the set of cosets

$$\left\{ \Gamma_0(4|D|) \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \gamma'_{D_1,\mu} \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}^{-1} : D_1 D_2 = D, \quad \mu \bmod |D_2| \right\}$$

is a system of representatives of $\Gamma_0(4|D|)\backslash\Gamma_0(4)$, which can be easily checked. ∎

***Definition* 3.1** For $m \in \mathbb{N}$ and $f(z) = \sum_{n \geq 0} a_f(n) q^n \in S_k(N, \chi)$ we define $U_m f$ by

$$U_m f(z) = \frac{1}{m} \sum_{v \bmod m} f\left(\frac{z+v}{m}\right) = \sum_{n \geq 0} a_f(mn) q^n. \tag{3.1}$$

Equivalently, we may write via (1.8)

$$U_m f(z) = m^{k/2-1} \sum_{v \bmod m} f(z)\Big|_k \begin{bmatrix} 1 & v \\ 0 & m \end{bmatrix}. \tag{3.2}$$

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We need the following two simple observations. Note that Lemma 3.3 follows from (3.1) and it implies Lemma 3.4.

***Lemma* 3.3** *Let U_2 be the operator defined in (3.1). Then*

$$U_2 G_{k,D}(z) = \left(1 + 2^{k-1}\left(\frac{D}{2}\right)\right) G_{k,D}(z) - 2^{k-1}\left(\frac{D}{2}\right) G_{k,D}(2z).$$

***Lemma* 3.4** *The following identity holds:*

$$G_{k,D}(4z) - G_{k,D}(8z) - 2^{-k}\left(\frac{D}{2}\right)\left(G_{k,D}\left(2z+\frac{1}{2}\right) + G_{k,D}(2z)\right) = -\left(\frac{D}{2}\right) 2^{-k+1} G_{k,D}(4z).$$

Note that $\gamma_v = \left[\begin{smallmatrix} 1 & 0 \\ 4|D|v & 1 \end{smallmatrix}\right]$ for $v = 0, 1, 2, 3$ form a system of representatives of $\Gamma_0(16|D|)\backslash\Gamma_0(4|D|)$ [9, p. 195]. The following lemma explicitly computes each term in $\mathrm{Tr}_{4D}^{16D}(VG_{k,D}(2z))$.

***Lemma* 3.5** *For $\gamma_v = \left[\begin{smallmatrix} 1 & 0 \\ 4|D|v & 1 \end{smallmatrix}\right]$, we have*

$$V(G_{k,D}(2z))\Big|_k \gamma_v = \begin{cases} G_{k,D}\left(2z + \frac{1}{2}\right) & v \equiv 0, 2 \pmod 4, \\ \left(\frac{D}{2}\right) 2^k G_{k,D}(8z) & v \equiv 1, 3 \pmod 4. \end{cases}$$

Proof First,

$$\begin{aligned} V(G_{k,D}(2z))\Big|_k \begin{bmatrix} 1 & 0 \\ 4|D|v & 1 \end{bmatrix} &= 2^{-k/2} G_{k,D}(z)\Big|_k \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4|D|v & 1 \end{bmatrix} \\ &= 2^{-k/2} G_{k,D}(z)\Big|_k \begin{bmatrix} 8(|D|v+1) & 2 \\ 16|D|v & 4 \end{bmatrix}. \end{aligned}$$

Now we do some casework.

(1) $v = 0$: We have

$$V(G_{k,D}(2z))\Big|_k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = V(G_{k,D}(2z)) = G_{k,D}\left(2\left(z + \frac{1}{4}\right)\right) = G_{k,D}\left(2z + \frac{1}{2}\right).$$

(2) $v = 1, 3$: Since v and $|D|$ are odd, $v|D| + 1$ must be even, $\gcd(\frac{|D|v+1}{2}, |D|v) = 1$, and there exist some $x, y \in \mathbb{Z}$ such that $\frac{|D|v+1}{2}x + |D|vy = 1$. Note also that $x \equiv 2 \pmod D$ and $\left(\frac{D}{x}\right) = \left(\frac{D}{2}\right)$. Thus

$$\begin{aligned} V(G_{k,D}(2z))\Big|_k \begin{bmatrix} 1 & 0 \\ 4|D|v & 1 \end{bmatrix} &= 2^{-k/2} G_{k,D}(z)\Big|_k \begin{bmatrix} 8(|D|v+1) & 2 \\ 16|D|v & 4 \end{bmatrix} \\ &= 2^{-k/2} G_{k,D}(z)\Big|_k \begin{bmatrix} \frac{|D|v+1}{2} & -y \\ |D|v & x \end{bmatrix}\begin{bmatrix} 16 & 2x+4y \\ 0 & 2 \end{bmatrix} \end{aligned}$$

2025/09/30 13:43

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$$= 2^{-k/2}\left(\frac{D}{x}\right) G_{k,D}(z)\bigg|_k \begin{bmatrix} 16 & 2x+4y \\ 0 & 2 \end{bmatrix}$$
$$= 2^k\left(\frac{D}{x}\right) G_{k,D}(8z+x+2y)$$
$$= \left(\frac{D}{2}\right) 2^k G_{k,D}(8z).$$

(3) $v = 2$: Since $\gcd(2|D|+1, 4|D|) = 1$, we can pick $x, y \in \mathbb{Z}$ such that $(2|D|+1)x + 4|D|y = 1$. As $4|D|y$ is even, x must be odd, so $G_{k,D}(2z + \frac{x}{2}) = G_{k,D}(2z + \frac{1}{2})$, and further $\left(\frac{D}{x}\right) = 1$ since $x \equiv 1 \pmod{D}$. Hence

$$V(G_{k,D}(2z))\bigg|_k \begin{bmatrix} 1 & 0 \\ 8|D| & 1 \end{bmatrix} = 2^{-k/2} G_{k,D}(z)\bigg|_k \begin{bmatrix} 8(2|D|+1) & 2 \\ 32|D| & 4 \end{bmatrix}$$
$$= 2^{-k/2} G_{k,D}(z)\bigg|_k \begin{bmatrix} 2|D|+1 & -y \\ 4|D| & x \end{bmatrix}\begin{bmatrix} 8 & 2x+4y \\ 0 & 4 \end{bmatrix}$$
$$= \left(\frac{D}{x}\right) 2^{-k/2} G_{k,D}(z)\bigg|_k \begin{bmatrix} 8 & 2x+4y \\ 0 & 4 \end{bmatrix}$$
$$= G_{k,D}\left(2z + \frac{x}{2}\right)$$
$$= G_{k,D}\left(2z + \frac{1}{2}\right).$$

Thus, the proof is complete. ■

The following lemma explicitly computes each term in $\mathrm{Tr}_{4D}^{16D}(V\theta(|D|z))$.

Lemma 3.6 *Let D be an odd fundamental discriminant. Then*

$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\gamma_0 = \theta\left(|D|z + \frac{|D|}{4}\right),$$
$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\gamma_1 = \begin{cases} (2i)^{1/2}(\theta(|D|z) - \theta(4|D|z)) & D > 0, \\ -i(2i)^{1/2}\theta(4|D|z) & D < 0, \end{cases}$$
$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\gamma_2 = \mathrm{sgn}(D) i\theta\left(|D|z - \frac{|D|}{4}\right),$$
$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\gamma_3 = \begin{cases} (2i)^{1/2}\theta(4|D|z) & D > 0, \\ -i(2i)^{1/2}(\theta(|D|z) - \theta(4|D|z)) & D < 0, \end{cases}$$

taking the principal branch of every square root.

Proof Recall that for $W_4 := \left[\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right]$, we have

$$\theta(z)|_{\frac{1}{2}} W_4 = i^{-1/2}\theta(z),$$

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see e.g [4, Proposition 15.1.1]. Note that

$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\begin{bmatrix}1 & 0\\ 4|D|v & 1\end{bmatrix} = |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}}\begin{bmatrix}|D| & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 4|D|v & 1\end{bmatrix}$$
$$= |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}}\begin{bmatrix}|D|v+1 & |D|\\ 4v & 4\end{bmatrix}\begin{bmatrix}4|D| & 0\\ 0 & 1\end{bmatrix}.$$

In the following we only give detailed proofs for $v = 0, 1, 2$, and leave out details for $v = 3$ because it follows a similar argument to $v = 1$.

(1) $v = 0$:

$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \theta\left(|D|z + \frac{|D|}{4}\right).$$

(2) $v = 1$: We have

$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\begin{bmatrix}1 & 0\\ 4|D| & 1\end{bmatrix} = |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}}\begin{bmatrix}|D|+1 & |D|\\ 4 & 4\end{bmatrix}\begin{bmatrix}4|D| & 0\\ 0 & 1\end{bmatrix}$$
$$= |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}}\begin{bmatrix}-|D| & |D|+1\\ -4 & 4\end{bmatrix}W_4\begin{bmatrix}|D| & 0\\ 0 & 1\end{bmatrix}.$$

Since $|D|$ is odd, we can choose $x, y \in \mathbb{Z}$ such that $-|D|x - 4y = 1$. This gives us

$$\begin{bmatrix}-|D| & |D|+1\\ -4 & 4\end{bmatrix} = \begin{bmatrix}-|D| & -y\\ -4 & x\end{bmatrix}\begin{bmatrix}1 & x-1\\ 0 & 4\end{bmatrix}.$$

Note that $\begin{bmatrix}-|D| & -y\\ -4 & x\end{bmatrix} \in \Gamma_0(4)$, so we have

$$V(\theta(|D|z))\Big|_{\frac{1}{2}}\begin{bmatrix}1 & 0\\ 4|D| & 1\end{bmatrix} = |D|^{-1/4}\left(\frac{-4}{x}\right)\varepsilon_x^{-1}\theta(z)\Big|_{\frac{1}{2}}\begin{bmatrix}1 & x-1\\ 0 & 4\end{bmatrix}W_4\begin{bmatrix}|D| & 0\\ 0 & 1\end{bmatrix}$$
$$= 2^{-1/2}|D|^{-1/4}\left(\frac{-4}{x}\right)\varepsilon_x^{-1}\theta\left(\frac{z+x-1}{4}\right)\Big|_{\frac{1}{2}}W_4\begin{bmatrix}|D| & 0\\ 0 & 1\end{bmatrix},$$

where

$$\varepsilon_x = \begin{cases}1 & x \equiv 1 \pmod 4,\\ i & x \equiv 3 \pmod 4.\end{cases}$$

Now we have two cases since $-|D|x - 4y = 1$ and the sign of D determines ε_x.

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(a) If $D > 0$, then $|D| \equiv 1 \pmod 4$, so $x \equiv 3 \pmod 4$, $\theta\left(\frac{z+x-1}{4}\right) = \theta\left(\frac{z}{4} + \frac{1}{2}\right) = 2\theta(z) - \theta(\frac{z}{4})$, $\varepsilon_x = i$, and $\left(\frac{-4}{x}\right) = -1$. So we have

$$V(\theta(|D|z))\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 4|D| & 1 \end{bmatrix} = 2^{-1/2}|D|^{-1/4}\left(\frac{-4}{x}\right)\varepsilon_x^{-1}\theta\left(\frac{z+x-1}{4}\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix}$$

$$= i2^{-1/2}|D|^{-1/4}\left(2\theta(z) - \theta\left(\frac{z}{4}\right)\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix}.$$

Explicitly computing these, we get

$$\theta(z)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} = i^{-1/2}|D|^{1/4}\theta(|D|z),$$

$$\begin{aligned}\theta\left(\frac{z}{4}\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} &= 2^{1/2}\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2^{1/2}\theta(z)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2^{1/2}i^{-1/2}\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2i^{-1/2}|D|^{1/4}\theta(4|D|z).\end{aligned}$$

So our expression simplifies to

$$i2^{-1/2}|D|^{-1/4}\left(2\theta(z) - \theta\left(\frac{z}{4}\right)\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} = (2i)^{1/2}(\theta(|D|z) - \theta(4|D|z)).$$

(b) If $D < 0$, then $|D| \equiv 3 \pmod 4$, $x \equiv 1 \pmod 4$, $\theta\left(\frac{z+x-1}{4}\right) = \theta(\frac{z}{4})$, $\varepsilon_x = 1$, and $\left(\frac{-4}{x}\right) = 1$. So we have

$$V(\theta(|D|z))\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 4|D| & 1 \end{bmatrix} = 2^{-1/2}|D|^{-1/4}\left(\frac{-4}{x}\right)\varepsilon_x^{-1}\theta\left(\frac{z+x-1}{4}\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}&= 2^{-1/2}|D|^{-1/4}\theta\left(\frac{z}{4}\right)\Big|_{\frac{1}{2}} W_4 \begin{bmatrix} |D| & 0 \\ 0 & 1 \end{bmatrix} \\ &= -i(2i)^{1/2}\theta(4|D|z).\end{aligned}$$

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(3) $v = 2$: Since $2|D|+1$ is coprime to 8, we can find $x, y \in \mathbb{Z}$ such that $(2|D|+1)x+8y = 1$.

$$V(\theta(|D|z))\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 8|D| & 1 \end{bmatrix} = |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} 2|D|+1 & |D| \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 4|D| & 0 \\ 0 & 1 \end{bmatrix}$$
$$= |D|^{-1/4}\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} 2|D|+1 & -y \\ 8 & x \end{bmatrix} \begin{bmatrix} 1 & |D|x+4y \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4|D| & 0 \\ 0 & 1 \end{bmatrix}.$$

Now that $\begin{bmatrix} 2|D|+1 & -y \\ 8 & x \end{bmatrix}$ is in $\Gamma_0(4)$, we get

$$V(\theta(|D|z))\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 8|D| & 1 \end{bmatrix} = |D|^{-1/4}\varepsilon_x^{-1}\left(\frac{8}{x}\right)\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & \frac{1-x}{2} \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4|D| & 0 \\ 0 & 1 \end{bmatrix}$$
$$= |D|^{-1/4}\varepsilon_x^{-1}\left(\frac{8}{x}\right)2^{-1/2}\theta\left(\frac{z}{4}+\frac{1-x}{8}\right)\Big|_{\frac{1}{2}} \begin{bmatrix} 4|D| & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \varepsilon_x^{-1}\left(\frac{8}{x}\right)\theta\left(|D|z+\frac{1-x}{8}\right)$$

As $(2|D|+1)x+8y = 1$ and the sign of D determines ε_x, we do casework again.

(a) If $D > 0$, then $|D| \equiv 1 \pmod 4$, which implies that $3x \equiv 1 \pmod 8$, $x \equiv 3 \pmod 8$, $\varepsilon_x = i$ and $\left(\frac{8}{x}\right) = \left(\frac{8}{3}\right) = -1$. Note also that $\theta(|D|z+\frac{1-x}{8}) = \theta(|D|z-\frac{1}{4})$.

(b) If $D < 0$, then $|D| \equiv 3 \pmod 4$, which gives $7x \equiv 1 \pmod 8$, $x \equiv 7 \pmod 8$, $\left(\frac{8}{x}\right) = \left(\frac{8}{7}\right) = 1$, $\varepsilon_x = i$ and $\theta(|D|z+\frac{1-x}{8}) = \theta(|D|z-\frac{3}{4})$.

Combining these two cases, we can write

$$V(\theta(|D|z))\Big|_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 4|D|v & 1 \end{bmatrix} = \operatorname{sgn}(D)i\theta\left(|D|z-\frac{|D|}{4}\right).$$

(4) $v = 3$: The argument in this case is similar to that of $v = 1$, and is omitted.

The above arguments complete the proof. ■

We also need the following two lemmas.

Lemma 3.7 *We have that*

$$V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_1 + V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_3 = \varepsilon_{|D|}^{-1}(2i)^{1/2}\theta(|D|z).$$

Proof It is a trivial consequence of Lemma 3.6. ■

Lemma 3.8 *We have that*

$$V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_0 + V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_2 = (1+i\operatorname{sgn}(D))\theta(|D|z).$$

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Proof By Lemma 3.6, we have

$$V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_0 + V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_2 = \theta\left(|D|z + \frac{|D|}{4}\right) + i\operatorname{sgn}(D)\theta\left(|D|z - \frac{|D|}{4}\right)$$

Note that

$$\theta\left(|D|z + \frac{|D|}{4}\right) = \sum_{n\in\mathbb{Z}} e^{2\pi i\frac{n^2|D|}{4}} e^{2\pi i n^2|D|z} = \sum_{n\in\mathbb{Z}} a(n)e^{2\pi i n^2|D|z}$$

where $a(n) = i\operatorname{sgn}(D)$ if n is odd and $a(n) = 1$ if n is even. On the other hand,

$$\theta\left(|D|z - \frac{|D|}{4}\right) = \sum_{n\in\mathbb{Z}} e^{2\pi i\frac{-n^2|D|}{4}} e^{2\pi i n^2|D|z} = \sum_{n\in\mathbb{Z}} b(n)e^{2\pi i n^2|D|z}$$

where $b(n) = -i\operatorname{sgn}(D)$ if n is odd and $b(n) = 1$ if n is even. Hence

$$\begin{aligned}\theta\left(|D|z + \frac{|D|}{4}\right) + \operatorname{sgn}(D)i\theta\left(|D|z - \frac{|D|}{4}\right) &= \sum_{n\in\mathbb{Z}}(a(n) + i\operatorname{sgn}(D)(b(n))e^{2\pi i n^2|D|z}\\ &= (1 + i\operatorname{sgn}(D))\theta(|D|z),\end{aligned}$$

as desired. ■

Now we are ready to prove the alternate formula (1.12) for $\mathcal{G}_{D,k,e}$ promised at the beginning of this section.

Proposition 3.9. *Let $k \geq 4$ and $e > 0$ be integers such that $k + 2e = \ell$ and let D be an odd fundamental discriminant such that $(-1)^{\ell}D > 0$. Then*

$$\mathcal{G}_{D,k,e}(z) = \operatorname{Tr}_4^{4D}[G_{k,D}(4z), \theta(|D|z)]_e.$$

Proof We closely follow [9, p. 195], where a similar result is implicit in the proof of formulas [9, (6), (7)]. Write $h = [G_{k,4D}(z), \theta(|D|z)]_e$. By Lemma 3.2 we get

$$\begin{aligned}\mathcal{G}_{D,k,e} &= \frac{3}{2}\left(1 - \left(\frac{D}{2}\right)2^{-k}\right)^{-1} \operatorname{pr}^+ \operatorname{Tr}_4^{4D}(h)\\ &= \frac{3}{2}\left(1 - \left(\frac{D}{2}\right)2^{-k}\right)^{-1}\left(\frac{1-(-1)^{\ell}i}{6}\operatorname{Tr}_4^{16} V(\operatorname{Tr}_4^{4D}(h)) + \frac{1}{3}\operatorname{Tr}_4^{4D}(h)\right)\\ &= \frac{3}{2}\left(1 - \left(\frac{D}{2}\right)2^{-k}\right)^{-1}\operatorname{Tr}_4^{4D}\left(\frac{1-(-1)^{\ell}i}{6}\operatorname{Tr}_{4D}^{16D}(V(h)) + \frac{1}{3}h\right)\\ &= \frac{3}{2}\left(1 - \left(\frac{D}{2}\right)2^{-k}\right)^{-1}\operatorname{Tr}_4^{4D} g_D,\end{aligned} \tag{3.3}$$

with

$$g_D = \frac{1-(-1)^k i}{6}\operatorname{Tr}_{4D}^{16D}(V(h)) + \frac{1}{3}h.$$

Note that $k \equiv \ell \bmod 2$, so we can substitute in $(-1)^k$ for $(-1)^{\ell}$ above. We now compute g_D. The matrices $\gamma_v = \left[\begin{smallmatrix}1 & 0\\ 4|D|v & 1\end{smallmatrix}\right]$, where $v = 0, 1, 2, 3$, form a set of coset representatives

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for $\Gamma_0(16|D|)\backslash\Gamma_0(4|D|)$ [9, p. 195]. Then we have

$$\begin{aligned}\mathrm{Tr}_{4D}^{16D}(V(h)) &= \mathrm{Tr}_{4D}^{16D}(V[G_{k,4D}(z),\theta(|D|z)]_e)\\ &= \mathrm{Tr}_{4D}^{16D}\left[VG_{k,4D}(z),V\theta(|D|z)\right]_e\\ &= \mathrm{Tr}_{4D}^{16D}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)V(G_{k,D}(2z)),V\theta(|D|z)\right]_e\\ &= \sum_{\gamma_v}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)V(G_{k,D}(2z)),V\theta(|D|z)\right]_e\Big|_{k+\frac{1}{2}+2e}\gamma_v.\end{aligned}$$

Since $\gamma_v \in \Gamma_0(4|D|)$, $G_{k,D}(4z)\Big|_k\gamma_v = G_{k,D}(4z)$. By Lemma 3.5, we get

$$\begin{aligned}\mathrm{Tr}_{4D}^{16D}(V(h)) &= \sum_{\gamma_v}\left[G_{k,D}(4z)\Big|_k\gamma_v - 2^{-k}\left(\frac{D}{2}\right)V(G_{k,D}(2z))\Big|_k\gamma_v, V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_v\right]_e\\ &= \left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}\left(2z+\frac{1}{2}\right),V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_0+V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_2\right]_e\\ &\quad+\left[G_{k,D}(4z)-G_{k,D}(8z),V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_1+V\theta(|D|z)\Big|_{\frac{1}{2}}\gamma_3\right]_e.\end{aligned}$$

Using Lemmas 3.7 and 3.8 and noting that $\mathrm{sgn}(D)=(-1)^k$ by our assumption, we can simplify this to

$$\begin{aligned}\mathrm{Tr}_{4D}^{16D}(V(h)) &= \left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}\left(2z+\frac{1}{2}\right),(1+i(-1)^k)\theta(|D|z)\right]_e\\ &\quad+\left[G_{k,D}(4z)-G_{k,D}(8z),\varepsilon_{|D|}^{-1}(2i)^{1/2}\theta(|D|z)\right]_e.\end{aligned}$$

Now we can finally compute the projection.

$$\begin{aligned}g_D(z) &= \frac{1-i(-1)^k}{6}\,\mathrm{Tr}_{4D}^{16D}(V(h(z)))+\frac{1}{3}h(z)\\ &= \frac{1-i(-1)^k}{6}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}\left(2z+\frac{1}{2}\right),(1+i(-1)^k)\theta(|D|z)\right]_e\\ &\quad+\frac{1-i(-1)^k}{6}\left[G_{k,D}(4z)-G_{k,D}(8z),\varepsilon_{|D|}^{-1}(2i)^{1/2}\theta(|D|z)\right]_e\\ &\quad+\frac{1}{3}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z),\theta(|D|z)\right]_e\\ &= \frac{1}{3}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}\left(2z+\frac{1}{2}\right),\theta(|D|z)\right]_e\\ &\quad+\frac{1}{3}\left[G_{k,D}(4z)-G_{k,D}(8z),\theta(|D|z)\right]_e\end{aligned}$$

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$$+\frac{1}{3}\left[G_{k,D}(4z)-2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z),\theta(|D|z)\right]_e$$
$$=\frac{1}{3}\left[G_{k,D}(4z)-G_{k,D}(8z)-2^{-k}\left(\frac{D}{2}\right)\left(G_{k,D}\left(2z+\frac{1}{2}\right)+G_{k,D}(2z)\right),\theta(|D|z)\right]_e$$
$$+\frac{2}{3}[G_{k,D}(4z),\theta(|D|z)]_e.$$

Using Lemma 3.4, we can simplify the first term to get

$$g_D(z)=\frac{1}{3}\left[-\left(\frac{D}{2}\right)2^{-k+1}G_{k,D}(4z),\theta(|D|z)\right]_e+\frac{2}{3}[G_{k,D}(4z),\theta(|D|z)]_e$$
$$=\frac{2}{3}\left(1-\left(\frac{D}{2}\right)2^{-k}\right)\left[G_{k,D}(4z),\theta(|D|z)\right]_e. \tag{3.4}$$

Plugging (3.4) into (3.3) gives the desired result. ∎

4 Eisenstein Series

In this section, we define various Eisenstein series and show that $G_{k,4D}(z)$ (1.5) is an Eisenstein series for the cusp at infinity of level $4|D|$. We recall the theory of Eisenstein series as developed in Miyake [15, §7]. Let χ and ψ be Dirichlet characters mod L and mod M, respectively. For $k \geq 3$, we put

$$E_k(z;\chi,\psi)=\sum_{m,n\in\mathbb{Z}}{}' \chi(m)\psi(n)(mz+n)^{-k}.$$

Here $\sum'$ is the summation over all pairs of integers (m,n) except $(0,0)$. In particular, $E_k(Mz;\chi,\psi)$ is a modular form in $M_k(LM,\chi\overline{\psi})$, see [15, pp. 269-271] for details.

***Lemma* 4.1** *([15, Theorem 7.1.3])* *Assume $k \geq 3$. Let χ and ψ be Dirichlet characters mod L and mod M, respectively, satisfying $\chi(-1)\psi(-1)=(-1)^k$. Let m_ψ be the conductor of ψ, and ψ^0 be the primitive character associated with ψ. Then*

$$E_k(z;\chi,\psi)=C+A\sum_{n=1}^{\infty}a(n)e^{2\pi inz/M},$$

where

$$A=2(-2\pi i)^k G(\psi^0)/M^k(k-1)!,$$
$$C=\begin{cases}2L_M(k,\psi) & \chi:\text{ the principal character},\\ 0 & \text{otherwise},\end{cases}$$
$$a(n)=\sum_{0<c|n}\chi(n/c)c^{k-1}\sum_{0<d|(l,c)}d\mu(l/d)\psi^0(l/d)\overline{\psi^0}(c/d).$$

Here $l=M/m_\psi$, μ is the Möbius function, $L_M(k,\psi)=\sum_{n=1}^{\infty}\psi(n)n^{-k}$ is the Dirichlet series, and $G(\psi^0)$ is the Gauss sum of ψ^0.

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For a fundamental discriminant D, we write $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ and $L_D(k) = \sum_{n=1}^{\infty} \chi_D(n) n^{-k}$.

Example 4.2 Let D be a fundamental discriminant and $\mathbf{1}$ be the principal character. Then

$$E_k(z; \mathbf{1}, \chi_D) = 2L_D(k) + \frac{2(-2\pi i)^k G(\chi_D)}{(k-1)!|D|^k} \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{D}{d}\right) d^{k-1} \right) q^{2\pi i n z/|D|}.$$

Example 4.3 If $D = D_1 D_2$ is a product of relatively prime fundamental discriminants then

$$E_k(z; \chi_{D_2}, \chi_{D_1}) := C + \frac{2(-2\pi i)^k G(\chi_{D_1})}{|D_1|^k (k-1)!} \sum_{n=1}^{\infty} \left(\sum_{\substack{d_1, d_2 > 0 \\ d_1 d_2 = n}} \left(\frac{D_1}{d_1}\right) \left(\frac{D_2}{d_2}\right) d_1^{k-1} \right) e^{2\pi i n z/|D_1|},$$

where C is zero unless $D_2 = 1$.

We shall compare our Eisenstein series $G_{k,D}(z)$ (1.4) and $G_{k,D_1,D_2}(z)$, defined below in (4.2) [9, p. 193] with the ones above given in Miyake [15]. Comparing the Fourier coefficients of $G_{k,D}(z)$ and $E_k(z; \mathbf{1}, \chi_D)$ gives

$$G_{k,D}(z) = \frac{(k-1)!|D|^k}{2(-2\pi i)^k G(\chi_D)} E_k(|D|z, \mathbf{1}, \chi_D) \in M_k(|D|, \chi_D). \tag{4.1}$$

Recall that [9, p. 193] for D_1, D_2 relatively prime fundamental discriminants with $(-1)^k D_1 D_2 > 0$:

$$G_{k,D_1,D_2}(z) = \sum_{n \geq 0} \sigma_{k-1,D_1,D_2}(n) q^n, \tag{4.2}$$

$$\sigma_{k-1,D_1,D_2}(n) = \begin{cases} -L_{D_1}(1-k) L_{D_2}(0) & n = 0, \\ \sum\limits_{\substack{d_1, d_2 > 0 \\ d_1 d_2 = n}} \left(\frac{D_1}{d_1}\right) \left(\frac{D_2}{d_2}\right) d_1^{k-1} & n > 0. \end{cases}$$

where the constant term is zero unless $D_2 = 1$. Hence by comparing the Fourier coefficients of $G_{k,D_1,D_2}(z)$ and $E_k(z; \chi_{D_2}, \chi_{D_1})$, we get

$$G_{k,D_1,D_2}(z) = \frac{|D_1|^k (k-1)!}{2(-2\pi i)^k G(\chi_{D_1})} E_k(|D_1| z; \chi_{D_2}, \chi_{D_1}) \in M_k(|D_1 D_2|, \chi_{D_1}\chi_{D_2}). \tag{4.3}$$

The following expression of $G_{k,D_1,D_2}(z)$ is useful for Lemma 6.1.

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Lemma 4.4 *Let $k \geq 3$ and $D = D_1 D_2$ be a product of coprime fundamental discriminants. Then*

$$G_{k,D_1,D_2}(z) = \frac{|D_1|^k (k-1)!}{2(-2\pi i)^k G(\chi_{D_1})} \chi_{D_2}(|D_1|) \sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z} \\ D_1|m}} \frac{\chi_{D_2}(m)\chi_{D_1}(n)}{(mz+n)^k}. \tag{4.4}$$

Proof Note that

$$\begin{aligned} E_k(|D_1|z; \chi_{D_2}, \chi_{D_1}) &= \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \chi_{D_2}(m)\chi_{D_1}(n)(m|D_1|z+n)^{-k} \\ &= \chi_{D_2}(|D_1|) \sideset{}{'}\sum_{\substack{\ell,n\in\mathbb{Z} \\ D_1|\ell}} \chi_{D_2}(\ell)\chi_{D_1}(n)(\ell z+n)^{-k}. \end{aligned}$$

Thus the result follows from (4.3). ∎

Let $k \geq 3$ and χ be a Dirichlet character mod N. We define the Eisenstein series for the cusp at infinity [15, p. 272] as

$$E^*_{k,N}(z;\chi) = \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\chi(d)}{(cz+d)^k},$$

where $\Gamma_\infty = \{\pm \left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\}$.

Now, we are ready to prove that $G_{k,4D}$ is an Eisenstein series for the cusp at infinity of level $4|D|$.

Lemma 4.5 *[15, (7.1.30)] Let* **1** *denote the principal Dirichlet character. Then*

$$2L_N(k,\chi)E^*_{k,N}(z;\chi) = E_k(Nz; \mathbf{1}, \chi).$$

From (4.1) and Lemma 4.5, we know that $G_{k,D}(z)$ is an Eisenstein series at infinity. We have

$$G_{k,D}(z) = \frac{L_D(1-k)}{2} E^*_{k,|D|}(z;\chi_D). \tag{4.5}$$

Note also that (4.1) and the proof of Lemma 4.5 imply that

$$G_{k,D}(z) = \frac{(k-1)!|D|^k}{2(-2\pi i)^k G(\chi_D)} \sideset{}{'}\sum_{\substack{c,d\in\mathbb{Z} \\ D|c}} \frac{\chi_D(d)}{(cz+d)^k}. \tag{4.6}$$

In fact, equation (4.6) will be more convenient for us to compute the Fourier expansion of $G_{k,D}(z)$ at different cusps. We need the following lemma; see also [5, p. 271].

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***Lemma* 4.6** *Let $L_D^{(4)}(k) = \sum_{\substack{(n,4)=1\\ n\geq 1}} \chi_D(n) n^{-k}$. Then*

$$E^*_{k,4|D|}(z;\chi_D) = \frac{L_D(k)}{L_D^{(4)}(k)} \left(E^*_{k,|D|}(4z;\chi_D) - 2^{-k}\left(\frac{D}{2}\right) E^*_{k,|D|}(2z;\chi_D)\right).$$

Proof Observe that

$$\begin{aligned} 2L_D^{(4)}(k)E^*_{k,4|D|}(z;\chi_D) &= 2\sum_{\substack{n\geq 1\\ (4,n)=1}} \frac{\chi_D(n)}{n^k}\left(\frac{1}{2}\sum_{\substack{(c,d)=1\\ 4D|c}} \frac{\chi_D(d)}{(cz+d)^k}\right) \\ &= \sum_{\substack{n\geq 1\\ (4,n)=1}} \sum_{\substack{(c,d)=1\\ 4D|c}} \frac{\chi_D(nd)}{(ncz+nd)^k} \\ &= \sum_{\substack{(d',4D)=1\\ 4D|c'}} \frac{\chi_D(d')}{(c'z+d')^k}, \end{aligned}$$

where $nc = c'$ and $nd = d'$. Note that we can replace $(d', 4D) = 1$ by $(d', 4)$ since $\chi_D(d') = 0$ otherwise. It follows that

$$\begin{aligned} 2L_D^{(4)}(k)E^*_{k,4|D|}(z;\chi_D) &= \sum_{\substack{(d,4)=1\\ 4D|c}} \frac{\chi_D(d)}{(cz+d)^k} \\ &= \sum_{\substack{c,d\in\mathbb{Z}\\ 4D|c}}{}' \left(\sum_{e|(d,4),e>0} \mu(e)\right) \frac{\chi_D(d)}{(cz+d)^k} \\ &= \sum_{e|4,e>0} \mu(e) \sum_{\substack{c,d\in\mathbb{Z}\\ 4D|c, e|d}}{}' \frac{\chi_D(d)}{(cz+d)^k}, \end{aligned}$$

where we used $\sum_{e|(d,4)} \mu(e) = 0$ for $(d,4) > 1$ in the second equality. Substituting $d = ey$ and $c = 4x$,

$$\begin{aligned} 2L_D^{(4)}(k)E^*_{k,4|D|}(z;\chi_D) &= \sum_{e|4,e>0} \mu(e) \sum_{\substack{x,y\in\mathbb{Z}\\ D|x}}{}' \frac{\chi_D(ey)}{(4xz+ey)^k} \\ &= \sum_{e|4,e>0} \mu(e)e^{-k}\chi_D(e) \sum_{\substack{x,y\in\mathbb{Z}\\ D|x}}{}' \frac{\chi_D(y)}{(x4z/e+y)^k} \\ &= \sum_{e|4,e>0} \mu(e)e^{-k}\chi_D(e) 2L_D(k)E^*_{k,D}(4z/e,\chi_D) \end{aligned}$$

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$$= 2L_D(k)\left(E^*_{k,|D|}(4z;\chi_D) - 2^{-k}\left(\frac{D}{2}\right)E^*_{k,|D|}(2z;\chi_D)\right),$$

where the second to last equality is from the proof of Lemma 4.5. ■

From Lemma 4.6 and (4.5), we know that $G_{k,4D}(z)$ is an Eisenstein series for the cusp at infinity in $M_k(4|D|,\chi_D)$. We have

$$G_{k,4D}(z) = \frac{L_D(1-k)}{2}\left(1 - 2^{-k}\left(\frac{D}{2}\right)\right)E^*_{k,4D}(z;\chi_D). \tag{4.7}$$

5 The Rankin-Selberg convolution

The purpose of this section is to prove Propositions 5.6 and 5.7. For two elements f and g of $M_k(N)$ such that fg is a cuspform, the Petersson inner product is given by

$$\langle f, g\rangle_{\Gamma_0(N)} = \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)^k d\mu, \tag{5.1}$$

where $z = x + iy$ and $d\mu = dxdy/y^2$. We use $\langle\cdot,\cdot\rangle$ to denote $\langle\cdot,\cdot\rangle_{\Gamma_0(N)}$ if the level is clear from the context. For $f(z) = \sum_{n\geq 1} a_f(n)q^n \in S_k(N,\chi)$, we put $f_\rho(z) := \sum_{n\geq 1}\overline{a_f(n)}q^n$. Note that $f_\rho(z) = f(z)$ if f is a newform and χ is trivial.

We now review the classical result on the Rankin-Selberg convolution, which was reformulated and generalized in Zagier [25], keeping in mind the difference between our definition of the Rankin-Cohen bracket and the one used therein.

***Lemma* 5.1** *([25, Propsition 6]) Let k_1 and k_2 be real numbers with $k_2 \geq k_1 + 2 > 2$. Let $f(z) = \sum_{n=1}^\infty a(n)q^n$ and $g(z) = \sum_{n=0}^\infty b(n)q^n$ be modular forms in $S_k(N,\chi)$ and $M_{k_1}(N,\chi_1)$, where $k = k_1 + k_2 + 2e$, $e \geq 0$ and $\chi = \chi_1\chi_2$. Then*

$$\langle f, [g, E^*_{k_2,N}(\cdot;\chi_2)]_e\rangle = \frac{(-1)^e}{e!}\frac{\Gamma(k-1)\Gamma(k_2+e)}{(4\pi)^{k-1}\Gamma(k_2)}\sum_{n=1}^\infty \frac{a(n)\overline{b(n)}}{n^{k_1+k_2+e-1}}.$$

To obtain Proposition 5.6, we need to deal with the case $k_1 = k_2$, which can be done by following Shimura [19] and Lanphier's work [11]. For $f(z) = \sum_{n=1}^\infty a(n)q^n \in S_k(N,\chi)$ and $g(z) = \sum_{n=0}^\infty b(n)q^n \in M_\ell(N,\psi)$, we put

$$D(s,f,g) = \sum_{n=1}^\infty a(n)b(n)n^{-s}, \quad \mathrm{Re}(s) \gg 0.$$

We are particularly interested in the following case.

***Lemma* 5.2** *Let $f = \sum_{n=1}^\infty a(n)q^n \in S_{2\ell}(1)$ be a normalized eigenform with $\ell = k + 2e$, $e > 0$ and $k \geq 4$ integers, and let D be an odd fundamental discriminant. Then*

$$D(s,f,G_{k,D}) = \frac{L(f,s)L(f,D,s-k+1))}{L_D(2s-3k-4e+2)}, \quad \mathrm{Re}(s) \gg 0.$$

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Proof Note that for $\mathrm{Re}(s) \gg 0$, we have

$$D(s, f, G_{k,D}) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1,\mathbf{1},\chi_D}(n)a(n)}{n^s},$$

where $\sigma_{k-1,\mathbf{1},\chi_D}(n) = \sum_{d|n} \chi_D(d)d^{k-1}$. A standard computation (see [22, Proposition 4.1]) gives

$$\sum_{n=1}^{\infty} \frac{\sigma_{k-1,\mathbf{1},\chi_D}(n)a(n)}{n^s} = \frac{L(f,s)L(f,D,s-(k-1))}{L_D(2s-(k-1)+1-(2k+4e))},$$

as desired. ∎

From Shimura [19, p. 786-789], $D(s, f, G_{k,D})$ has a meromorphic continuation to the whole complex plane and $D(s, f, G_{k,D})$ is holomorphic at $s = 2k + 2e - 1$, see [19, p. 789].

The Maass-Shimura operators [19, p. 788 (2.8)] are defined by

$$\delta_\lambda = \frac{1}{2\pi i}\left(\frac{\lambda}{2iy} + \frac{\partial}{\partial z}\right), \quad 0 < \lambda \in \mathbb{Z},$$
$$\delta_\lambda^{(r)} = \delta_{\lambda+2r-2} \cdots \delta_{\lambda+2}\delta_\lambda, \quad 0 \le r \in \mathbb{Z},$$

where we understand that $\delta_\lambda^{(0)}$ is the identity operator. A relation between Maass-Shimura operators and the Rankin-Cohen bracket is given by

$$\left(\delta_k^{(n)} f(z)\right) g(z) = \sum_{j=0}^{n} \frac{(-1)^j \binom{n}{j}\binom{k+n-1}{n-j}}{\binom{k+\ell+2j-2}{j}\binom{k+\ell+n+j-1}{n-j}} \delta_{k+\ell+2j}^{(n-j)}[f,g]_j(z), \tag{5.2}$$

where $f \in M_k(\Gamma)$ and $g \in M_\ell(\Gamma)$ for any congruence subgroup Γ; see [11, Theorem 1].

We recall the following two results.

***Lemma* 5.3** *([19, Lemma 6]) Suppose $f \in S_k(N,\chi)$, $g \in M_l(N,\overline{\chi})$ and $k = l + 2r$ with a positive integer r. Then $\langle \delta^{(r)} g, f_\rho \rangle = 0$.*

***Lemma* 5.4** *([19, Theorem 2]) Suppose $f \in S_{2\ell}(|D|)$ with $\ell = k + 2e$, $e > 0$ and $k \ge 4$, and D is a fundamental discriminant. Then*

$$D(2k + 4e - 1 - 2e, f, G_{k,D}) = c\pi^{2k+4e-1}\langle G_{k,D}\delta_k^{(2e)} E^*_{k,|D|}(z;\chi_D), f_\rho\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the non-normalized Petersson inner product on defined in (5.1) and

$$c = \frac{\Gamma(2k+4e-k-2(2e))}{\Gamma(2k+4e-1-2e)\Gamma(2k+4e-k-2e)}(-1)^{2e}4^{2k+4e-1}.$$
$$= \frac{\Gamma(k)}{\Gamma(2k-1+2e)\Gamma(k+2e)}4^{2k+4e-1}$$

We apply these two results in our situation to obtain the following.

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Proposition 5.5. *Let $f \in S_{2\ell}(1)$ be a normalized eigenform with $\ell = k + 2e, e > 0$ and $k \geq 4$. Then*

$$\langle [G_{k,D}, G_{k,D}]_{2e}, f\rangle_{\Gamma_0(|D|)} = \frac{1}{2}\frac{\Gamma(2k+4e-1)\Gamma(k+2e)}{(2e)!(4\pi)^{2k+4e-1}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}L(f, 2k+2e-1)L(f, D, k+2e).$$

Proof Note that $f_\rho = f$ since f is a normalized eigenform. Lemma 5.4 gives

$$\langle G_{k,D}\delta_k^{(2e)}E^*_{k,|D|}(z;\chi_D), f\rangle_{\Gamma_0(|D|)} = \frac{\Gamma(2k+2e-1)\Gamma(k+2e)}{(4\pi)^{2k+4e-1}\Gamma(k)}D(2k+2e-1, f, G_{k,D}).$$

By Lemma 5.3 and (5.2),

$$\langle G_{k,D}\delta_k^{(2e)}E^*_{k,|D|}(z;\chi_D), f\rangle_{\Gamma_0(|D|)} = \frac{1}{\binom{2k+4e-2}{2e}}\langle [E^*_{k,|D|}(z;\chi_D), G_{k,D}]_{2e}, f\rangle_{\Gamma_0(|D|)},$$

which implies that

$$\langle [E^*_{k,|D|}(z;\chi_D), G_{k,D}]_{2e}, f\rangle_{\Gamma_0(|D|)} = \frac{\binom{2k+4e-2}{2e}\Gamma(2k+2e-1)\Gamma(k+2e)}{(4\pi)^{2k+4e-1}\Gamma(k)}D(2k+2e-1, f, G_{k,D}).$$

Since $G_{k,D}(z) = \frac{L_D(1-k)}{2}E^*_{k,|D|}(z;\chi_D)$ (4.5) and by Lemma 5.2, we have

$$\begin{aligned}\langle [G_{k,D}, G_{k,D}]_{2e}, f\rangle_{\Gamma_0(|D|)} &= \frac{L_D(1-k)}{2}\frac{\binom{2k+4e-2}{2e}\Gamma(2k+2e-1)\Gamma(k+2e)}{(4\pi)^{2k+4e-1}\Gamma(k)}D(2k+2e-1, f, G_{k,D})\\ &= \frac{1}{2}\frac{\Gamma(2k+4e-1)\Gamma(k+2e)}{(2e)!(4\pi)^{2k+4e-1}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}L(f, 2k+2e-1)L(f, D, k+2e),\end{aligned}$$

as desired. ∎

Now we prove Propositions 5.6 and 5.7, which generalize [9, Proposition 1] and [9, Proposition 2], respectively.

Proposition 5.6. *Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized eigenform in $S_{2\ell}(1)$ with $\ell = k + 2e, e > 0$ and $k \geq 4$, and let D be an odd fundamental discriminant with $(-1)^\ell D > 0$. Then*

$$\langle \mathcal{F}_{D,k,e}, f\rangle = \frac{1}{2}\frac{\Gamma(2k+4e-1)\Gamma(k+2e)}{(2e)!(4\pi)^{2k+4e-1}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}L(f, 2k+2e-1)L(f, D, k+2e).$$

Proof Recall that (1.11)

$$\mathcal{F}_{D,k,e}(z) = \mathrm{Tr}_1^D[G_{k,D}(z), G_{k,D}(z)]_{2e}.$$

As $\langle f, g\rangle_{\Gamma_0(M)} = \langle f, \mathrm{Tr}_N^M g\rangle_{\Gamma_0(N)}$ for $N \mid M$, for $f \in S_k(N), g \in M_k(M)$ (see [5, p. 271]), we get

$$\langle \mathcal{F}_{D,k,e}, f\rangle = \langle [G_{k,D}(z), G_{k,D}(z)]_{2e}, f\rangle_{\Gamma_0(|D|)}.$$

Then the result follows from Proposition 5.5. ∎

Proposition 5.7. *Let $g = \sum c_g(n)q^n \in S^+_{\ell+1/2}(4)$ be a Hecke eigenform and $f \in S_{2\ell}(1)$ be the normalized Hecke eigenform corresponding to it by the Shimura correspondence, where $\ell = k+2e$, $e > 0$ and $k \geq 4$. Let D be an odd fundamental discriminant with $(-1)^\ell D > 0$. Then*

$$\langle g, \mathcal{G}_{D,k,e}\rangle = \frac{3}{2}\frac{\Gamma(k+2e-\frac{1}{2})\Gamma(k+e)}{e!(4\pi)^{k+2e-1/2}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}|D|^{-k-e+1/2}L(f, 2k+2e-1)c_g(|D|),$$

where the Petersson inner product is $\langle g, \mathcal{G}_{D,k,e}\rangle := \int_{\Gamma_0(4)\backslash\mathbb{H}} g(z)\overline{\mathcal{G}_{D,k,e}(z)}\,\mathrm{Im}(z)^{k+2e+1/2}d\mu$.

Proof Recall that $\mathcal{G}_{D,k,e}$ is given in (1.12):

$$\mathcal{G}_{D,k,e}(z) = \frac{3}{2}\left(1 - 2^{-k}\left(\frac{D}{2}\right)\right)^{-1}\mathrm{pr}^+\,\mathrm{Tr}_4^{4D}[G_{k,4D}(z), \theta(|D|z)]_e.$$

Since pr^+ (1.6) is the projection from $M_{\ell+1/2}(4)$ to $M^+_{\ell+1/2}(4)$, we have

$$\begin{aligned}
\langle g, \mathcal{G}_{D,k,e}\rangle &= \frac{3}{2}\left(1 - 2^{-k}\left(\frac{D}{2}\right)\right)^{-1}\langle \mathrm{pr}^+ g, \mathrm{Tr}_4^{4D}([G_{k,4D}(z), \theta(|D|z)]_e\rangle\\
&= \frac{3}{2}\left(1 - 2^{-k}\left(\frac{D}{2}\right)\right)^{-1}\langle g, \mathrm{Tr}_4^{4D}([G_{k,4D}(z), \theta(|D|z)]_e\rangle\\
&= \frac{3}{2}\left(1 - 2^{-k}\left(\frac{D}{2}\right)\right)^{-1}\langle g, ([G_{k,4D}(z), \theta(|D|z)]_e\rangle_{\Gamma_0(4|D|)}\\
&= \frac{3}{4}L_D(1-k)\langle g(z), [E^*_{k,4D}(z;\chi_D), \theta(|D|z)]_e\rangle_{\Gamma_0(4|D|)}\\
&= \frac{3(-1)^e}{4}L_D(1-k)\langle g(z), [\theta(|D|z), E^*_{k,4D}(z;\chi_D)]_e\rangle_{\Gamma_0(4|D|)},
\end{aligned}$$

where we used (4.7) in the second to last equality. Now Lemma 5.1 gives

$$\langle g, \mathcal{G}_{D,k,e}\rangle = \frac{3(-1)^e}{4}L_D(1-k)\frac{(-1)^e}{e!}\frac{\Gamma(k+2e-\frac{1}{2})\Gamma(k+e)}{(4\pi)^{k+2e-1/2}\Gamma(k)}\sum_{n=1}^{\infty}\frac{2c_g(n^2|D|)}{(|D|n^2)^{k+e+1/2-1}}$$

$$= \frac{3}{2}\frac{\Gamma(k+2e-\frac{1}{2})\Gamma(k+e)}{e!(4\pi)^{k+2e-1/2}\Gamma(k)}L_D(1-k)|D|^{-(k+e-1/2)}\sum_{n=1}^{\infty}\frac{c_g(n^2|D|)}{n^{2k+2e-1}}.$$

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By [10, Theorem 1 (ii)], we get

$$L_D(s-(k+2e)+1)\sum_{n=1}^{\infty}\frac{c_g(n^2|D|)}{n^{2k+2e-1}}=c_g(|D|)L(f,s),$$

which implies that

$$\langle g,\mathcal{G}_{D,k,e}\rangle=\frac{3}{2}\frac{\Gamma(k+2e-\frac{1}{2})\Gamma(k+e)}{e!(4\pi)^{k+2e-1/2}\Gamma(k)}\frac{L_D(1-k)}{L_D(k)}|D|^{-k-e+1/2}L(f,2k+2e-1)c_g(|D|),$$

as desired. ■

6 Fourier expansions

In this section, we compute the Fourier coefficients needed for the proof of Theorem 1.1. It is convenient to have explicit formulas for $G_{k,D}$ and θ under the action of certain matrices in $\mathrm{SL}_2(\mathbb{Z})$, which we do in Lemmas 6.1 and 6.2. Propositions 6.3 and 6.4 then give formulas for $\mathcal{F}_{D,k,e}$ and $\mathcal{G}_{D,k,e}$, which we use in the final computation of the Fourier coefficients carried out in Lemmas 6.5 and 6.6.

***Lemma* 6.1** *Let $k\geq 3$. Suppose D is an odd fundamental discriminant and $D=D_1D_2$ is a product of two fundamental discriminants. Let $\gamma=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\in\mathrm{SL}_2(\mathbb{Z})$ with $\gcd(c,D)=|D_1|$. Then*

$$G_{k,D}(z)\Big|_k\begin{bmatrix}a&b\\c&d\end{bmatrix}=\left(\frac{D_2}{c}\right)\left(\frac{D_1}{d|D_2|}\right)\left(\frac{D_2}{|D_1|}\right)\frac{\varepsilon_{|D_1|}}{\varepsilon_{|D|}}|D_2|^{-1/2}G_{k,D_1,D_2}\left(\frac{z+c^*d}{|D_2|}\right),$$

where c^* is an integer with $cc^*\equiv 1 \pmod{|D_2|}$, and ε_n is given by

$$\varepsilon_n=\begin{cases}1 & n\equiv 1 \pmod 4,\\ i & n\equiv 3 \pmod 4.\end{cases}\tag{6.1}$$

Proof We follow the idea in Gross-Zagier [5, pp. 273-275]. By equation (4.6), we have

$$\frac{2(-2\pi i)^kG(\chi_D)}{(k-1)!|D|^k}G_{k,D}(z)\Big|_k\begin{bmatrix}a&b\\c&d\end{bmatrix}=\sideset{}{'}\sum_{\substack{l,r\in\mathbb{Z}\\ D|l}}\frac{\chi_D(r)}{(l(az+b)+r(cz+d))^k}$$
$$=\sideset{}{'}\sum_{\substack{m,n\in\mathbb{Z}\\ md\equiv nc \bmod |D|}}\frac{\chi_D(an-bm)}{(mz+n)^k},$$

where $(m,n)=(l,r)\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$. Since $md\equiv nc \pmod{|D|}$, we have

$$d(an-bm)=adn-bmd\equiv adn-bcn\equiv n \pmod{|D|},\tag{6.2}$$
$$c(an-bm)=anc-bcm\equiv adm-bcm\equiv m \pmod{|D|}.\tag{6.3}$$

Note also that $\gcd(D_1,D_2)=1$. Then (6.2) and (6.3) imply that

$$\chi_D(an-bm)=\chi_{D_1}(an-bm)\chi_{D_2}(an-bm)$$

$$= \chi_{D_1}(d)\chi_{D_1}(n)\chi_{D_2}(c)\chi_{D_2}(m).$$

Since $D_1, D_2 \mid (md - nc)$, $(d, D_1) = 1$, $(c, D_2) = 1$ and $(c, D) = |D_1|$, we must have $D_1 \mid m$; and $n \equiv c^* md \pmod{|D_2|}$. By the Chinese Remainder Theorem, we can choose c^* such that $D_1 \mid c^*$. Now we write $n = c^* md + l|D_2|$. It follows that

$$\frac{2(-2\pi i)^k G(\chi_D)}{(k-1)!|D|^k} G_{k,D}(z)\bigg|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \sum_{\substack{m,l \in \mathbb{Z} \\ D_1 \mid m}}' \frac{\chi_{D_1}(d)\chi_{D_1}(c^* md + l|D_2|)\chi_{D_2}(c)\chi_{D_2}(m)}{(mz + mc^* d + l|D_2|)^k}$$

$$= \chi_{D_2}(c)\chi_{D_1}(d|D_2|) \sum_{\substack{m,l \in \mathbb{Z} \\ D_1 \mid m}}' \frac{\chi_{D_2}(m)\chi_{D_1}(l)}{(mz + mc^* d + l|D_2|)^k}$$

$$= \chi_{D_2}(c)\chi_{D_1}(d|D_2|)|D_2|^{-k} \sum_{\substack{m,l \in \mathbb{Z} \\ D_1 \mid m}}' \frac{\chi_{D_2}(m)\chi_{D_1}(l)}{\left(m\frac{z+c^* d}{|D_2|} + l\right)^k}. \tag{6.4}$$

Note that (4.4) implies that

$$\sum_{\substack{m,l \in \mathbb{Z} \\ D_1 \mid m}}' \frac{\chi_{D_2}(m)\chi_{D_1}(l)}{\left(m\frac{z+c^* d}{|D_2|} + l\right)^k} = \frac{2(-2\pi i)^k G(\chi_{D_1})}{|D_1|^k (k-1)!} \chi_{D_2}(|D_1|) G_{k,D_1,D_2}\left(\frac{z + c^* d}{|D_2|}\right). \tag{6.5}$$

Plugging (6.5) into (6.4) gives

$$G_{k,D}(z)\bigg|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \chi_{D_2}(c)\chi_{D_1}(d|D_2|)\chi_{D_2}(|D_1|)\frac{G(\chi_{D_1})}{G(\chi_D)} G_{k,D_1,D_2}\left(\frac{z + c^* d}{|D_2|}\right) \tag{6.6}$$

From [3, Proposition 2.2.24. p. 49] we know that

$$G(\chi_{D_1}) = \varepsilon_{|D_1|}|D_1|^{1/2} \quad \text{and} \quad G(\chi_D) = \varepsilon_{|D|}|D|^{1/2},$$

which implies that

$$G_{k,D}(z)\bigg|_k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(\frac{D_2}{c}\right)\left(\frac{D_1}{d|D_2|}\right)\left(\frac{D_2}{|D_1|}\right)\frac{\varepsilon_{|D_1|}}{\varepsilon_{|D|}}|D_2|^{-1/2} G_{k,D_1,D_2}\left(\frac{z + c^* d}{|D_2|}\right),$$

as desired. ∎

***Lemma* 6.2** *Let D be an odd fundamental discriminant and $D = D_1 D_2$ be a product of two fundamental discriminants. Then*

$$\theta(z)\bigg|_{\frac{1}{2}} \begin{bmatrix} |D| & 0 \\ 4|D_1| & 1 \end{bmatrix} = \varepsilon_{|D_2|}^{-1}|D|^{1/4}|D_2|^{-1/2}\theta\left(\frac{|D_1|z + 4^*}{|D_2|}\right),$$

where 4^* is an integer such that $44^* \equiv 1 \pmod{|D_2|}$.

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Proof Since $(4, D_2) = 1$, there exist $n, m \in \mathbb{Z}$ such that $n|D_2| + 4m = 1$ and

$$\begin{bmatrix} |D| & 0 \\ 4|D_1| & 1 \end{bmatrix} = \begin{bmatrix} |D_2| & -m \\ 4 & n \end{bmatrix} \begin{bmatrix} |D_1| & m \\ 0 & |D_2| \end{bmatrix}.$$

It follows that

$$\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} |D| & 0 \\ 4|D_1| & 1 \end{bmatrix} = \theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} |D_2| & -m \\ 4 & n \end{bmatrix} \begin{bmatrix} |D_1| & m \\ 0 & |D_2| \end{bmatrix}.$$

Recall that the transformation law for θ (see e.g [8, p. 148]) gives

$$\theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} |D_2| & -m \\ 4 & n \end{bmatrix} = \left(\frac{4}{n}\right) \varepsilon_n^{-1} \theta(z),$$

where ε_n is as in (6.1). Since $n|D_2| + 4m = 1$ and $D_2 \equiv 1 \pmod 4$, we have $\varepsilon_n = \varepsilon_{|D_2|}$. Hence

$$\begin{aligned} \theta(z)\Big|_{\frac{1}{2}} \begin{bmatrix} |D| & 0 \\ 4|D_1| & 1 \end{bmatrix} &= \varepsilon_{|D_2|} \theta(z)\Big|_{1/2} \begin{bmatrix} |D_1| & m \\ 0 & |D_2| \end{bmatrix} \\ &= \varepsilon_{|D_2|} |D|^{1/4} |D_2|^{-1/2} \theta\left(\frac{|D_1|z + 4^*}{|D_2|}\right), \end{aligned}$$

which gives the desired result. ■

Next, we give some computations generalizing the lemma in [9, p. 193].

Proposition 6.3. *Let $k \geq 4$ and $e > 0$ with $\ell = k + 2e$ and let D be an odd fundamental discriminant with $(-1)^\ell D > 0$. Then*

$$\mathcal{F}_{D,k,e}(z) = \sum_{D = D_1 D_2} \left(\frac{D_2}{-1}\right) |D_2|^{-2e} U_{|D_2|}([G_{k,D_1,D_2}(z), G_{k,D_1,D_2}(z)]_{2e}),$$

where the summation is over all decompositions of D as a product of two fundamental discriminants, and $U_{|D_2|}$ is the operator defined in (3.1).

Proof We consider the following system of representatives (Lemma 3.1) of $\Gamma_0(|D|)\backslash \mathrm{SL}_2(\mathbb{Z})$,

$$\left\{ \begin{bmatrix} 1 & 0 \\ |D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \qquad \text{where } D = D_1 D_2, \quad \mu \bmod |D_2| \right\}$$

and D_1, D_2 are fundamental discriminants. By (6.6) we have

$$G_{k,D}(z)\Big|_k \begin{bmatrix} 1 & 0 \\ |D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} = \left(\frac{D_2}{|D_1|}\right) \left(\frac{D_1}{|D_2|}\right) \left(\frac{D_2}{|D_1|}\right) \frac{G(\chi_{D_1})}{G(\chi_D)} G_{k,D_1,D_2}\left(\frac{z + \mu + |D_1|^*}{|D_2|}\right)$$

where $|D_1|^*|D_1| = 1 \mod |D_2|$. We then compute $\mathcal{F}_{D,k,e}(z)$, which is

$$\begin{aligned} &\mathrm{Tr}_1^D([G_{k,D}(z), G_{k,D}(z)]_{2e}) \\ &= \sum_{D_1 D_2 = D} \sum_{\mu \bmod |D_2|} \left[G_{k,D}(z), G_{k,D}(z)\right]_{2e} \Big|_{2k+4e} \begin{bmatrix} 1 & 0 \\ |D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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$$= \sum_{D_1D_2=D} \sum_{\mu \bmod |D_2|} \left[G_{k,D}(z)\Big|_k \begin{bmatrix} 1 & 0 \\ |D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}, G_{k,D}(z)\Big|_k \begin{bmatrix} 1 & 0 \\ |D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right]_{2e}$$

$$= \sum_{D_1D_2=D} \sum_{\mu \bmod |D_2|} \left(\frac{D_2}{-1}\right) |D_2|^{-1} \left[G_{k,D_1,D_2}\left(\frac{z+\mu+|D_1|^*}{|D_2|}\right), G_{k,D_1,D_2}\left(\frac{z+\mu+|D_1|^*}{|D_2|}\right) \right]_{2e},$$

where we used the well-known fact $G(\chi_{D_1})^2 = \left(\frac{D_1}{-1}\right)|D_1|$ and $G(\chi_D)^2 = \left(\frac{D}{-1}\right)|D|$ in the last equality (see e.g [3, Corollary 2.1.47 on p. 33]). On the other hand, we have by our equivalent definition (3.2) of the U operator that

$$U_{|D_2|}([G_{k,D_1,D_2}(z), G_{k,D_1,D_2}(z)]_{2e})$$

$$= \sum_{\nu \bmod |D_2|} |D_2|^{(2k+4e)/2-1} [G_{k,D_1,D_2}(z), G_{k,D_1,D_2}(z)]_{2e}\Big|_{2k+4e} \begin{bmatrix} 1 & \nu \\ 0 & |D_2| \end{bmatrix}$$

$$= \sum_{\nu \bmod |D_2|} |D_2|^{(2k+4e)/2-1} \left[|D_2|^{-k/2} G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right), |D_2|^{-k/2} G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right) \right]_{2e}$$

$$= \sum_{\nu \bmod |D_2|} |D_2|^{2e-1} \left[G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right), G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right) \right]_{2e}.$$

It follows that

$$\sum_{D=D_1D_2} \left(\frac{D_2}{-1}\right) |D_2|^{-2e} U_{|D_2|}([G_{k,D_1,D_2}(z), G_{k,D_1,D_2}(z)]_{2e})$$

$$= \sum_{D_1D_2=D} \sum_{\nu \bmod |D_2|} \left(\frac{D_2}{-1}\right) |D_2|^{-1} \left[G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right), G_{k,D_1,D_2}\left(\frac{z+\nu}{|D_2|}\right) \right]_{2e}$$

$$= \mathrm{Tr}_1^D([G_{k,D}(z), G_{k,D}(z)]_{2e}),$$

as desired. ■

Proposition 6.4. *Let $k \geq 4$ and $e > 0$ and let D be an odd fundamental discriminant with $(-1)^k D > 0$. Then*

$$\mathcal{G}_{D,k,e}(z) = \sum_{D=D_1D_2} \left(\frac{D_2}{-|D_1|}\right) |D_2|^{-e} U_{|D_2|}([G_{k,D_1,D_2}(4z), \theta(|D_1|z)]_e)$$

where the summation is over all decompositions of D as a product of two fundamental discriminants, and $U_{|D_2|}$ is the map defined in (3.1).

Proof The proof follows a similar outline to Proposition 6.3. From Proposition 3.9, we know that $\mathcal{G}_{D,k,e}(z) = \mathrm{Tr}_4^{4D}\left[G_{k,D}(4z), \theta(|D|z)\right]_e$. We use the coset representatives (Lemma 3.1) for $\Gamma_0(4|D|)\backslash\Gamma_0(4)$,

$$\left\{ \gamma_{D_1,\mu} = \begin{bmatrix} 1 & 0 \\ 4|D_1| & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} : \quad \text{where } D = D_1D_2, \quad \mu \pmod{|D_2|} \right\},$$

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where $D = D_1D_2$ is a product of fundamental discriminants. By a simple casework we have

$$\frac{\varepsilon_{|D_1|}}{\varepsilon_{|D|}\cdot\varepsilon_{|D_2|}}\left(\frac{D_1}{|D_2|}\right) = \left(\frac{D_2}{-|D_1|}\right). \tag{6.7}$$

Now, Lemmas 6.1 and 6.2 together with (6.7) imply that

$$\begin{aligned}
&\sum_{D_1D_2=D}\sum_{\mu \bmod |D_2|}[G_{k,D}(4z),\theta(|D|z)]_e\big|_{k+2e+\frac{1}{2}}\gamma_{D_1,\mu}\\
&=\sum_{D_1D_2=D}\sum_{\mu \bmod |D_2|}\frac{\varepsilon_{|D_1|}}{\varepsilon_{|D|}\cdot\varepsilon_{|D_2|}}\left(\frac{D_1}{|D_2|}\right)|D_2|^{-1}\\
&\quad\times\left[G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*+4\mu}{|D_2|}\right),\theta\left(\frac{|D_1|z+4^*+|D_1|\mu}{|D_2|}\right)\right]_e\\
&=\sum_{D_1D_2=D}\sum_{\mu \bmod |D_2|}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}\\
&\quad\times\left[G_{k,D_1,D_2}\left(\frac{4(z+4^*|D_1|^*+\mu)}{|D_2|}\right),\theta\left(\frac{|D_1|(z+4^*|D_1|^*+\mu)}{|D_2|}\right)\right]_e\\
&=\sum_{D_1D_2=D}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-e}U_{D_2}[G_{k,D_1,D_2}(4z),\theta(|D_1|z)]_e,
\end{aligned}$$

as desired. ∎

We are now ready to compute the Fourier expansions of $\mathcal{F}_{D,k,e}$ and $\mathcal{S}_D(\mathcal{G}_{D,k,e})$.

***Lemma* 6.5** *Let $k \geq 4$, $e > 0$ and let D be an odd fundamental discriminant with $(-1)^kD > 0$. Then we have the Fourier expansion*

$$\mathcal{F}_{D,k,e}(z) = \sum_{n\geq 1} f_{D,k,e}(n)q^n,$$

where

$$f_{D,k,e}(n) = \sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)|D_2|^{-2e}\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}}\sum_{d|(a_1,a_2)}\left(\frac{D}{d}\right)d^{k-1}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)C_{e,a_1,a_2},$$

$$C_{e,a_1,a_2} = \sum_{r=0}^{2e}(-1)^r a_1^r a_2^{2e-r}\binom{2e+k-1}{2e-r}\binom{2e+k-1}{r}.$$

Proof By Proposition 6.3, we have

$$f_{D,k,e}(n) = \sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)|D_2|^{-2e}F_{D_1,D_2,e}(n),$$

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where $F_{D_1,D_2,e}(n)$ is the $n|D_2|$-th Fourier coefficient of $[G_{k,D_1,D_2}(z), G_{k,D_1,D_2}(z)]_{2e}$. Note that

$$G_{k,D_1,D_2}(z)^{(r)} = \sum_{n\geq 0} n^r \sigma_{k-1,D_1,D_2}(n) q^n,$$

which implies that the $n|D_2|$-th Fourier coefficient of $G^{(r)}_{k,D_1,D_2}(z) G^{(2e-r)}_{k,D_1,D_2}(z)$ is

$$\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} a_1^r \sigma_{k-1,D_1,D_2}(a_1) a_2^{2e-r} \sigma_{k-1,D_1,D_2}(a_2).$$

It follows that $F_{D_1,D_2,e}(n) =$

$$\sum_{r=0}^{2e} (-1)^r \binom{2e+k-1}{2e-r}\binom{2e+k-1}{r} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} a_1^r \sigma_{k-1,D_1,D_2}(a_1) a_2^{2e-r} \sigma_{k-1,D_1,D_2}(a_2)$$

$$= \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sigma_{k-1,D_1,D_2}(a_1)\sigma_{k-1,D_1,D_2}(a_2) \sum_{r=0}^{2e} a_1^r a_2^{2e-r} (-1)^r \binom{2e+k-1}{2e-r}\binom{2e+k-1}{r}$$

$$= \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sigma_{k-1,D_1,D_2}(a_1)\sigma_{k-1,D_1,D_2}(a_2) C_{e,a_1,a_2}$$

$$= \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k-1} \sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right) C_{e,a_1,a_2}.$$

where the last equality is given by the Hecke multiplicative relation [9, p. 194]

$$\sigma_{k-1,D_1,D_2}(a_1)\sigma_{k-1,D_1,D_2}(a_2) = \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k-1} \sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right).$$

This finishes the proof. ∎

***Lemma* 6.6** *Let $k \geq 4$, $e > 0$ and let D be an odd fundamental discriminant with $(-1)^k D > 0$. Then we have the Fourier expansion*

$$\mathcal{S}_D\left(\mathcal{G}_{D,k,e}(z)\right) = \sum_{n\geq 1} g_{D,k,e}(n) q^n,$$

where

$$g_{D,k,e}(n) = |D|^e \sum_{D=D_1D_2} \left(\frac{D_2}{-1}\right) |D_2|^{-2e} \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k-1} \sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right) E(a_1,a_2),$$

$$E(a_1,a_2) = \sum_{r=0}^{e} (-1)^r \binom{e+k-1}{e-r}\binom{e-1/2}{r} 4^r \left(a_1a_2\right)^r \left(a_2-a_1\right)^{2(e-r)}.$$

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Proof By Proposition 6.4 and (1.9), we have $\mathcal{S}_D(\mathcal{G}_{D,k,e}(z)) =$

$$\sum_{D_1D_2=D}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-e}\sum_{r=0}^{e}(-1)^r\binom{e+k-1}{e-r}\binom{e-1/2}{r}\mathcal{S}_D\left[U_{|D_2|}(G_{k,D_1,D_2}(4z)^{(r)}\theta(|D_1|z)^{(e-r)})\right],$$

where we abuse notation to move the Shimura operator $\mathcal{S}_D$ into the sums. Note that

$$G_{k,D_1,D_2}(4z)^{(r)} = \sum_{n\geq 0}(4n)^r\sigma_{k-1,D_1,D_2}(n)q^{4n},$$

$$\theta(|D_1|z)^{(e-r)} = \sum_{n\in\mathbb{Z}}(n^2|D_1|)^{e-r}q^{n^2|D_1|}.$$

This allows us to rewrite the product

$$G_{k,D_1,D_2}(4z)^{(r)}\theta(|D_1|z)^{(e-r)} = \sum_{n\geq 0}c_r(n)q^n,$$

$$c_r(n) := \sum_{m\equiv n \mod 2}\left(n-m^2|D_1|\right)^r\sigma_{k-1,D_1,D_2}\left(\frac{n-m^2|D_1|}{4}\right)(m^2|D_1|)^{e-r},$$

where we take the convention that $\sigma_{k-1,D_1,D_2}(x) = 0$ if $x \notin \mathbb{Z}$ or $x < 0$. It follows that

$$U_{|D_2|}\left(G_{k,D_1,D_2}(4z)^{(r)}\theta(|D_1|z)^{(e-r)}\right) = U_{|D_2|}\left(\sum_{n\geq 1}c_r(n)q^n\right) = \sum_{n\geq 1}c_r(n|D_2|)q^n. \tag{6.8}$$

Now we compute the D-th Shimura lift of (6.8). If we write

$$\mathcal{S}_D\left(\sum_{n\geq 1}c_r(n|D_2|)q^n\right) = \sum_{n\geq 1}a_{r,D_2}(n)q^n$$

for some $a_{r,D_2}(n)$, then by the definition of $\mathcal{S}_D$ (1.1), we have $a_{r,D_2}(n) =$

$$\sum_{d|n}\left(\frac{D}{d}\right)d^{k+2e-1}\sum_{m\in\mathbb{Z}}\left(|D_2||D|\frac{n^2}{d^2}-|D_1|m^2\right)^r\left(m^2|D_1|\right)^{e-r}\sigma_{k-1,D_1,D_2}\left(\frac{|D_2||D|\frac{n^2}{d^2}-m^2|D_1|}{4}\right).$$

Note that we can write

$$\frac{|D_2||D|\frac{n^2}{d^2}-m^2|D_1|}{4} = |D_1|a_1a_2, \quad \text{where } a_1 = \frac{|D_2|\frac{n}{d}+m}{2} \text{ and } a_2 = \frac{|D_2|\frac{n}{d}-m}{2}.$$

It follows that $a_{r,D_2}(n) =$

$$\sum_{d|n}\left(\frac{D}{d}\right)d^{k+2e-1}\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=\frac{n}{d}|D_2|}}(4|D_1|a_1a_2)^r\,(a_2-a_1)^{2(e-r)}|D_1|^{e-r}\sigma_{k-1,D_1,D_2}(|D_1|a_1a_2)$$

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$$= \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k+2e-1} \left(4|D_1|\frac{a_1a_2}{d^2}\right)^r \left(\frac{a_2-a_1}{d}\right)^{2(e-r)} |D_1|^{e-r}\sigma_{k-1,D_1,D_2}\left(|D_1|\frac{a_1a_2}{d^2}\right)$$

$$= \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right)\left(\frac{D_2}{|D_1|}\right) d^{k-1}|D_1|^e (4a_1a_2)^r (a_2-a_1)^{2(e-r)}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right). \tag{6.9}$$

Now we substitute (6.9) back into our equation for $\mathcal{S}_D(\mathcal{G}_{D,k,e}(z))$. Let

$$\mathcal{S}_D(\mathcal{G}_{D,k,e}(z)) = \sum_{n\geq 1} g_{D,k,e}(n)q^n.$$

Then $g_{D,k,e}(n) =$

$$\sum_{D=D_1D_2}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-e}\sum_{r=0}^{e}(-1)^r\binom{e+k-1}{e-r}\binom{e-1/2}{r}a_{r,D_2}(n)$$

$$= \sum_{D=D_1D_2}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-e}\sum_{r=0}^{e}(-1)^r\binom{e+k-1}{e-r}\binom{e-1/2}{r}$$
$$\times \sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right)\left(\frac{D_2}{|D_1|}\right) d^{k-1}|D_1|^e (4a_1a_2)^r (a_2-a_1)^{2(e-r)}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)$$

$$=|D|^e\sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)|D_2|^{-2e}\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k-1}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)$$
$$\times\sum_{r=0}^{e}(-1)^r\binom{e+k-1}{e-r}\binom{e-1/2}{r}4^r(a_1a_2)^r(a_2-a_1)^{2(e-r)}$$

$$=|D|^e\sum_{D=D_1D_2}\left(\frac{D_2}{-1}\right)|D_2|^{-2e}\left(\sum_{\substack{a_1,a_2\geq 0\\ a_1+a_2=n|D_2|}} \sum_{d|(a_1,a_2)} \left(\frac{D}{d}\right) d^{k-1}\sigma_{k-1,D_1,D_2}\left(\frac{a_1a_2}{d^2}\right)E(a_1,a_2)\right),$$

as desired. ∎

7 Discussion

It is a folklore conjecture that $S_{2\ell}^{0,D}(1) = S_{2\ell}(1)$. Luo [13] showed that for ℓ sufficiently large one has $\dim S_{2\ell}^{0,1}(1) \gg \ell$. Our Theorem 1.2 (the case $D = 1$ was proved earlier by Xue [24, Proposition 3.5]) provides a possible different approach to the conjecture. By studying the linear independence of $\mathcal{G}_{D,k,e}$ or $\mathcal{F}_{D,k,e}$, one could obtain lower bounds on the dimension of $S_{2\ell}^{0,D}(1)$.

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Conjecture 7.1. *For ℓ even, D a positive fundamental discriminant, the set $\{\mathcal{G}_{D,k,e} \mid k + 2e = \ell, 1 \le e \le \lfloor \frac{\ell}{6} \rfloor\}$ is linearly independent.*

We checked this conjecture computationally in the $D = 1$ case up to $\ell = 1000$ and for prime D less than 50 up to $\ell = 100$, using code written in Pari/GP [7]. In particular, we computationally verified that the matrix

$$\begin{bmatrix} g_{D,\ell-2,1}(4) & g_{D,\ell-2,1}(8) & \cdots & g_{D,\ell-2,1}(4\lfloor \frac{\ell}{6} \rfloor) \\ g_{D,\ell-4,2}(4) & g_{D,\ell-4,2}(8) & \cdots & g_{D,\ell-4,2}(4\lfloor \frac{\ell}{6} \rfloor) \\ \vdots & \vdots & \ddots & \vdots \\ g_{D,\ell-2\lfloor \frac{\ell}{6} \rfloor,\lfloor \frac{\ell}{6} \rfloor}(4) & g_{D,\ell-2\lfloor \frac{\ell}{6} \rfloor,\lfloor \frac{\ell}{6} \rfloor}(8) & \cdots & g_{D,\ell-2\lfloor \frac{\ell}{6} \rfloor,\lfloor \frac{\ell}{6} \rfloor}(4\lfloor \frac{\ell}{6} \rfloor) \end{bmatrix},$$

where $\mathcal{G}_{D,k,e} = \sum_{n \ge 1} g_{D,k,e}(n) q^n$ for $1 \le e \le \lfloor \frac{\ell}{6} \rfloor$, has nonzero determinant. Further work in this area should try to prove that this determinant is nonzero in general.

The conjecture would have several interesting consequences. Using the isomorphism between $S^{0,D}_{\ell+1/2}(4)$ and $S^{0,D}_{2\ell}(1)$ given by the D-th Shimura lift, we find that the dimension of $S^{0,D}_{2\ell}$ would be at least $\lfloor \frac{\ell}{6} \rfloor$. Since the dimension of $S_{2\ell}(1)$ for even ℓ is $\lfloor \frac{2\ell}{12} \rfloor = \lfloor \frac{\ell}{6} \rfloor$ and $S^{0,D}_{2\ell}(1) \subseteq S_{2\ell}(1)$, we would conclude that $S^{0,D}_{2\ell}(1) = S_{2\ell}(1)$, settling the conjecture on the non-vanishing of twisted central L-values for Hecke eigenforms.

This would then imply that $S^{0,D}_{\ell+1/2}(4) = S^{+}_{\ell+1/2}(4)$, so the Kohnen plus space for k even is generated by Hecke eigenforms whose D-th coefficients are nonzero for all fundamental discriminants D. Further, we would conclude that $\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell, 1 \le e \le \lfloor \frac{\ell}{6} \rfloor}$ is a basis for $S^{+}_{\ell+1/2}(4)$, and the set $\{\mathcal{G}_{D,k,e}\}_{k+2e=\ell, 0 \le e \le \lfloor \frac{\ell}{6} \rfloor}$ is a basis for $M^{+}_{\ell+1/2}(4)$ (since the 0-th Rankin-Cohen bracket produces a modular form which is non-cuspidal but still in the Kohnen plus space). To the best of our knowledge, a similar basis was first mentioned by Henri Cohen in a MathOverflow post.

Acknowledgements

This research was supported by NSA MSP grant H98230-24-1-0033. We thank the anonymous referees for the detailed comments and insightful advice that have greatly improved the exposition of this paper.

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