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Reducibility for SU_n and Generic Elliptic Representations

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Abstract. We study reducibility of representations parabolically induced from discrete series representations of $SU_n(F)$ for F a p-adic field of characteristic zero. We use the approach of studying the relation between R-groups when a reductive subgroup of a quasi-split group and the full group have the same derived group. We use restriction to show the quotient of R-groups is in natural bijection with a group of characters. Applying this to $SU_n(F) \subset U_n(F)$ we show the R group for SU_n is the semidirect product of an R-group for $U_n(F)$ and this group of characters. We derive results on non-abelian R-groups and generic elliptic representations as well.

Introduction

The problem of classifying the tempered spectrum of a connected reductive quasi-split group, defined over a local field, F, of characteristic zero, consists of three parts. The first is to classify the discrete series representations of any Levi subgroup. The second step is to understand the rank one Plancherel measures, which is equivalent to understanding the reducibility of those representations parabolically induced from a discrete series of maximal Levi subgroups. The third step is to understand the structure of representations parabolically induced from discrete series representations of an arbitrary parabolic subgroup, using the second step and the combinatorial theory of the Knapp-Stein R-group. We address this third step here for the case where the group in question is the quasi-split special unitary group.

This work builds on earlier results on R-groups due to several authors. For quasi-split unitary groups, $U_n(F)$, the R-groups attached to the principal series were computed by Keys [K1, K2] and the theory of restriction yielded R-groups for $SU_n(F)$. Keys gave a description of the R-groups in terms of the Langlands parameterization, which in that case is well understood [L]. Recent work by Ban and Zhang (oral communication) has shown that the construction of the R-group through L-group considerations, as described in [A1], will be valid for all quasi-split connected reductive groups, once the local parameterization conjecture is established. For $U_n(F)$ and an arbitrary parabolic subgroup, the R-groups were computed in [G2] and these were shown to be elementary two groups. The computation of the R-group via the L-group was carried out by D. Prasad [P], and one can see that this computation agrees with [G2], if you assume that the Langlands L-functions are the same as the Artin L-functions attached to the parameter. This has been established in some cases by Henniart [H].

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Following the methods of [K1, K2], and using the theory of [G-K, T], we work via restriction. As in [T] some of our results can be proved in a more general setting. Namely, we consider the case where $G \subset \widetilde{G}$ have the same derived group. We are able to give a rough description of the R-group in this setting. In the case of special unitary groups we make this explicit due to a stronger understanding of the restriction of irreducible smooth representations from Levi subgroups $\widetilde{M}$ of $\widetilde{G}$, to the subgroup $M = \widetilde{M} \cap G$. These results on restriction are given in the latter part of Section 2.

To be more precise we consider a p-adic field F of characteristic zero and residual characteristic q_F, and fix a quadratic extension E/F. Let $\widetilde{\mathbf{G}}$ be a quasi-split reductive group defined over F, with $\widetilde{\mathbf{G}}(F) = \widetilde{G}$. We assume that $\mathbf{G} \subset \widetilde{\mathbf{G}}$ is a reductive subgroup with the property that $[\widetilde{G}, \widetilde{G}] = [G, G]$, where these represent the derived groups. Then, for any Levi subgroup $\widetilde{\mathbf{M}}$ of $\widetilde{\mathbf{G}}$, we have that $MZ(\widetilde{M})$ is of finite index in $\widetilde{M}$. Here $M = \mathbf{M}(F)$, with $\mathbf{M} = \widetilde{\mathbf{M}} \cap \mathbf{G}$, and $Z(\widetilde{M})$ is the center of $\widetilde{M} = \widetilde{\mathbf{M}}(F)$. Thus, the theory of [T, §2] applies. In particular, if σ is a discrete series representation of M, then there is a discrete series representation π of $\widetilde{M}$ with σ a component of $\pi|_M$. Furthermore, any irreducible representation π' of $\widetilde{M}$ for which $\operatorname{Hom}_M(\sigma, \pi') \neq 0$ is of the form $\pi' \simeq \pi\chi$ for a character χ of $\widetilde{M}$ trivial on M.

The R-group, $R(\sigma)$, is a subgroup of the stabilizer, $W(\sigma)$, of σ in the Weyl group W, and by the considerations above,

$$W(\sigma) \subset \{w | \pi^w \simeq \pi\chi, \text{ for some } \chi\}.$$

The theory of the R-group dictates that the complement of $R(\sigma)$ in $W(\sigma)$ is the subgroup, $W'(\sigma)$, generated by the reflections w_α for which the rank one Plancherel measure $\mu_\alpha(\sigma) = 0$. We show that $\mu_\alpha(\sigma) = 0$ if and only if $\mu_\alpha(\pi) = 0$, and thus $W'(\sigma) = W'(\pi)$ (*cf.* Lemma 2.3). Let

$$\widehat{W(\sigma)} = \{\chi | \pi^w \simeq \pi\chi \text{ for some } w \in W(\sigma)\}.$$

We show that, for any $\chi \in \widehat{W(\sigma)}$, there is some $w_\chi \in R(\sigma)$ with $\pi^{w_\chi} \simeq \pi\chi$, and this element is unique up to multiplication by elements of $R(\pi) \cap W(\sigma)$. Thus, in this general setting we always have $R(\sigma)/\left(R(\pi) \cap W(\sigma)\right) \simeq \widehat{W(\sigma)}/X(\pi)$, where $X(\pi) = \{\chi | \pi\chi \simeq \pi\}$ (cf Proposition 3.2).

We then specialize to the case where $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. There we show $R(\pi) \triangleleft R(\sigma)$. Furthermore, we can split the sequence

$$1 \to R(\pi) \to R(\sigma) \to \widehat{W(\sigma)}/X(\pi) \to 1.$$

Thus, there is a subgroup Γ_σ of $R(\sigma)$ which is isomorphic to $\widehat{W(\sigma)}/X(\pi)$ for which $R(\sigma) = \Gamma_\sigma \ltimes R(\pi)$ (cf Theorem 3.7).

The structure of the (unitarily) induced representation $i_{G,M}(\sigma)$ is determined by the representation theory of a certain extension of $R(\sigma)$. More precisely, for $w \in R(\sigma)$, we choose an intertwining operator satisfying $T_w\sigma^w = \sigma T_w$. Then there is a 2-cocycle $\gamma \colon R(\sigma) \times R(\sigma) \to \mathbb{C}$ defined by $T_{w_1 w_2} = \gamma(w_1, w_2) T_{w_1} T_{w_2}$. The intertwining algebra $\mathcal{C}(\sigma)$ of $\operatorname{Ind}_P^G(\sigma)$ is then isomorphic to the twisted group algebra

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$\mathbb{C}[R(\sigma)]_\gamma$. For simplicity of this exposition, we assume that the cocycle splits (which is the case whenever σ is generic [K2], and is always the case for for the classical groups for which R-groups have been computed [Hr, G2, G4]). Then there is a correspondence $\rho \mapsto \pi(\rho)$ between irreducible representations of $R(\sigma)$ and classes of irreducible components of $i_{G,M}(\sigma)$ [A2, K2]. The multiplicity of $\pi(\rho)$ in $i_{G,M}(\sigma)$ is equal to $\dim\rho$. Further, the behavior of the character θ_ρ of ρ determines which components of $i_{G,M}(\sigma)$ are elliptic [A2]. More precisely, if $R(\sigma)_{\mathrm{reg}}$ is the set of elements of $R(\sigma)$ for which the fixed points $\mathfrak{a}_w$ of w in $\mathfrak{a}$, the real lie algebra of $\mathbf{A}$, is as small as possible, namely $\mathfrak{a}_G$, then π_ρ is elliptic if and only if θ_ρ is non-vanishing on the regular set $R(\sigma)_{\mathrm{reg}}$. We note a correction to the description of the elliptic tempered spectrum of $U(n)$ in [G2] (*cf.* remark 3.10). In the case of $G = SU_n(F)$, we examine the elliptic spectrum in the case where π is generic. Recall that, for $\widetilde{\mathbf{G}} = U_n$, the Weyl group is the semidirect product of a permutation group with an elementary two group (consisting of "sign changes"). We show that $i_{G,M}(\sigma)$ has an elliptic component, if and only if there is an element $w = sc$ of $R(\sigma)$ whose permutation component s is of maximal possible length, and c changes an odd number of signs. Finally we give an example of a phenomenon which we had not noted before. Namely, a case where $i_{G,M}(\sigma)$ has some elliptic components, but not all of the components are elliptic. In fact, induced representations which have this property were exhibited in [K2], but, as this predated Arthur's description of the elliptic spectrum, [A2], it was not noted there. We give a specific new example, and indicate how this can be generalized.

1 Notation and Preliminaries

Let F be a nonarchimedean local field of characteristic zero, and residual characteristic q_F. Fix a quadratic extension E/F. Let γ be the non-trivial Galois automorphism of E/F, which we also denote by $x \mapsto \bar{x}$. Fix an element $\beta \in E$ with $\gamma(\beta) = -\beta$. For $n \in \mathbb{Z}^+$, let

$$u_n = \begin{pmatrix} & & & & \cdot \\ & & & \cdot & \\ & & \cdot & & \\ & -1 & & & \\ 1 & & & & \\ -1 & & & & \end{pmatrix},$$

and fix a hermitian form $h_n \in M_n(E)$, by

$$h_n = \begin{cases} u_n & \text{if } n \text{ is odd,} \\ \beta u_n & \text{if } n \text{ is even.} \end{cases}$$

For $g = (g_{ij}) \in \mathrm{Res}_{E/F}\, GL_n$, we let $\bar{g} = (\bar{g}_{ij})$ and set $\varepsilon(g) = u_n {}^t\bar{g}^{-1} u_n^{-1}$. Denote by $\widetilde{\mathbf{G}} = \widetilde{\mathbf{G}}(n) = U_n$, the quasi-split unitary group defined with respect to E/F and h_n. Thus

$$\widetilde{\mathbf{G}} = \{g \in \mathrm{Res}_{E/F}\, GL_n \mid g h_n {}^t\bar{g} = h_n\},$$

We let $\mathbf{G} = \mathbf{G}(n) = SU_n = \widetilde{\mathbf{G}} \cap \mathrm{Res}_{E/F}\, SL_n$. If $\widetilde{\mathbf{H}} \subset \widetilde{\mathbf{G}}$, then we let $\mathbf{H} = \widetilde{\mathbf{H}} \cap \mathbf{G}$. We denote the F-points by $\widetilde{G} = \widetilde{\mathbf{G}}(F)$ and $G = \mathbf{G}(F)$, and similarly for other groups.

Let $\widetilde{\mathbf{T}}$ be the maximal torus in $\widetilde{\mathbf{G}}$ of diagonal elements, and let $\widetilde{\mathbf{A}}_0$ be the maximal split subtorus of $\widetilde{\mathbf{T}}$. Denote by $\Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{A}}_0)$ the roots of $\widetilde{\mathbf{A}}_0$ in $\widetilde{\mathbf{G}}$. We fix $\widetilde{\mathbf{B}} = \widetilde{\mathbf{T}}\widetilde{\mathbf{U}}$ to be the Borel subgroup of upper triangular elements of $\widetilde{\mathbf{G}}$. Let $\widetilde{\Delta}$ be the simple roots with respect to this fixed choice of Borel subgroup. If $\theta \subset \widetilde{\Delta}$, then we denote by $\widetilde{\mathbf{P}}_\theta = \widetilde{\mathbf{M}}_\theta\widetilde{\mathbf{N}}_\theta$ the associated standard parabolic subgroup. Since we will be working with a fixed θ, we drop the subscript and simply write $\widetilde{\mathbf{P}} = \widetilde{\mathbf{M}}\widetilde{\mathbf{N}}$. Let $\widetilde{\mathbf{A}}_{\widetilde{\mathbf{M}}}$ be the split component of $\mathbf{M}$. We write $H_{\widetilde{P}}$ (respectively, H_P) for the homomorphism from $\widetilde{M}$ (respectively M) to $\mathfrak{a}_{\widetilde{M}}$ (respectively, $\mathfrak{a}_M$) given in [H-C], where $\mathfrak{a}_{\widetilde{M}}$ and $\mathfrak{a}_M$ are the real Lie algebras of $\widetilde{A}_{\widetilde{M}}$ and A_M, respectively.

Note that, for some choice of partition $\{n_1, n_2, \dots, n_r, m'\}$ of $\left[\frac{n}{2}\right]$,

$$\widetilde{\mathbf{A}}_{\widetilde{\mathbf{M}}} = \left\{\operatorname{diag}\{x_1 I_{n_1}, x_2 I_{n_2}, \dots, x_r I_{n_r}, I_m, \bar{x}_r^{-1} I_{n_r}, \dots, \bar{x}_1^{-1} I_{n_1}\} | x_i \in \operatorname{Res}_{E/F} GL_1\right\},$$

where $m = 2m'$ or $2m' + 1$, depending on whether n is even or odd. Then

$$\begin{aligned}(1.1)\quad \widetilde{\mathbf{M}} &= \left\{\operatorname{diag}\{g_1, g_2, \dots, g_r, h, \varepsilon(g_r), \dots, \varepsilon(g_1)\} \ |g_i \in \operatorname{Res}_{E/F} GL_{n_i}, h \in U(m)\right\} \\ &\simeq \operatorname{Res}_{E/F} GL_{n_1} \times \operatorname{Res}_{E/F} GL_{n_2} \times \cdots \times \operatorname{Res}_{E/F} GL_{n_r} \times U(m).\end{aligned}$$

Let $\Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{A}})$ be the reduced roots of $\widetilde{\mathbf{A}}$ in $\widetilde{\mathbf{G}}$. The Weyl group,

$$W(\widetilde{\mathbf{M}}) = N_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{A}})/\widetilde{\mathbf{M}} \simeq \mathcal{S} \rtimes \mathcal{Z},$$

where $\mathcal{S} \subset S_r$ is generated by the transpositions $(i\,j)$ for which $n_i = n_j$, and $\mathcal{Z} \simeq \mathbb{Z}_2^r$ is the subgroup of $W(\widetilde{\mathbf{M}})$ is generated by "block sign changes" C_i given by

$$C_i(g_1, g_2, \dots, g_i, \dots, g_r, h) = (g_1, g_2, \dots, \varepsilon(g_i), \dots, g_r, h).$$

The realization of $s_{ij} \in \mathcal{S}$ with $s_{ij} \mapsto (i\,j)$ under the above isomorphism is given by

$$s_{ij}(g_1, \dots, g_i, \dots, g_j, \dots, g_r, h) = (g_1, \dots, g_j, \dots, g_i, \dots, g_r, h).$$

We use w to represent both a class in $W(\widetilde{M})$ and a representative of that class in $N_G(\widetilde{M})$. This should cause no confusion here. If $\mathbf{M} = \widetilde{\mathbf{M}} \cap \mathbf{G}$, and $\mathbf{A} = \widetilde{\mathbf{A}} \cap \mathbf{G}$, then $\Phi(\widetilde{\mathbf{G}}, \widetilde{\mathbf{A}}) = \Phi(\mathbf{G}, \mathbf{A})$, $W(\mathbf{M}) = W(\widetilde{\mathbf{M}})$, and we identify the action of $W(\mathbf{M})$ as the restriction of the action of $W(\widetilde{\mathbf{M}})$ to $\mathbf{M}$.

If π is an irreducible admissible representation of $\widetilde{M}$, then $\pi \simeq \pi_1 \otimes \cdots \otimes \pi_r \otimes \tau$, where each π_i is an irreducible admissible representation of $GL_{n_i}(E)$ and τ is one of $G(m)$. Let $W = W(\widetilde{\mathbf{M}})$ act on π by $\pi^w(m) = \pi(w^{-1}mw)$.

We use Harish–Chandra's notation $\mathcal{E}_c(\widetilde{G})$ to denote the (equivalence classes of) irreducible admissible representations of $\widetilde{G}$, and $\mathcal{E}_t(\widetilde{G})$, $\mathcal{E}_2(\widetilde{G})$, represent the tempered, and square integrable classes, respectively. Similar notation is used for $\widetilde{M}$, M, and G. We use the notation $i_{G,M}(\sigma)$ for the representation of G unitarily induced from the representation σ of M, of course extended trivially to the unipotent radical of a parabolic P with Levi component M.

Note that the center $Z(\widetilde{G})$ of $\widetilde{G}$ is isomorphic to $E^1 = \{x \in E | x\bar{x} = 1\}$. Since $G = [\widetilde{G}, \widetilde{G}]$, the theory of [T, §2] applies to restriction from $\widetilde{G}$ to G. Similarly we can apply these results to restriction from $\widetilde{M}$ to M, which is the subject of the next section.

We use $X(\widetilde{G})$ and $X(\widetilde{M})$ to denote the characters of $\widetilde{G}$ and $\widetilde{M}$, respectively. If $\chi \in X(\widetilde{G})$, then $\chi(g) = \chi'(\det g)$, for some character χ' of E^1. We will abuse notation and use χ to represent both the character of $\widetilde{G}$, and the character of E^1 through which it factors. If $\pi \in \mathcal{E}_2(\widetilde{M})$, and $\chi \in X(\widetilde{M})$ then we denote by $\pi\chi$ the representation $g \mapsto \pi(g)\chi(g)$. Then we let $X(\pi) = \{\chi \in X(\widetilde{M}) \mid \pi\chi \simeq \pi\}$.

In order to describe the generic elliptic spectrum we need to make some observations about the Lie algebra $\mathfrak{g}$ of G. We have

$$\mathfrak{g} \simeq \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid X u_n + u_n {}^t\bar{X} = 0\}.$$

If $\mathfrak{a}_M$ is the Lie algebra of A_M, then a straightforward calculation shows

$$\mathfrak{a}_M = \{\operatorname{diag}(a_1 I_{n_1}, a_2 I_{n_2}, \ldots, a_r I_{n_r}, 0_m, -\bar{a}_r I_{n_r}, \ldots, -\bar{a}_1 I_{n_1}) \mid \sum_i n_i(a_i - \bar{a}_i) = 0\}.$$

We will sometimes denote an element of $\mathfrak{a}_M$ by $Y_M(a_1, \ldots, a_r)$. For $w \in W(\mathbf{G}, \mathbf{A}_M)$, we let $\mathfrak{a}_w = \{X \in \mathfrak{a}_M | w \cdot X = X\}$, where $w \cdot X = ad(w)X$. Note that $\mathfrak{a}_G = \{0\}$.

2 Plancherel Measures and Restriction

In this section we establish some results on compatibility of Plancherel measures with restriction. Much of this follows immediately from earlier work to which we refer. Many of these results apply more generally to the situation where $G \subset \widetilde{G}$ and $[G, G] = [\widetilde{G}, \widetilde{G}]$. For the moment we work in that context, with $\widetilde{M}$ a Levi subgroup of $\widetilde{G}$ (possibly $\widetilde{G}$ itself) and $M = \widetilde{M} \cap G$.

Lemma 2.1 *Let $\pi \in \mathcal{E}_2(\widetilde{M})$.*

(a) *There is an integer m_0 so that $\pi|_M = m_0 \bigoplus_{i=1}^k \sigma_i$, with σ_i irreducible and inequivalent.* [G-K].

(b) *If $\operatorname{Hom}_M(\pi, \pi') \neq 0$, then there is a character χ of $\widetilde{M}$, so that $\pi' \simeq \pi\chi$.* [T, Corollary 2.5].

(c) *Every $\sigma \in \mathcal{E}_2(M)$ is a component of $\pi|_M$ for some $\pi \in \mathcal{E}_2(\widetilde{M})$* [T, Proposition 2.2].

Corollary 2.2 *Suppose $\sigma \in \mathcal{E}_2(M)$ and $\pi \in \mathcal{E}_2(\widetilde{M})$ with $\pi|_M$ containing σ. Suppose that for some $w \in W(\mathbf{M})$, $\sigma^w \simeq \sigma$. Then, there is a $\chi \in X(\widetilde{M})$ so that $\pi^w \simeq \pi\chi$.*

For a reduced root $\alpha \in \Phi(\widetilde{\mathbf{P}}, \widetilde{\mathbf{A}})$, we write $\mu_\alpha(\pi)$ for the rank one Plancherel measure attached to α (see [H-C]). We know that $\mu_\alpha(\pi) = 0$ if and only if the standard intertwining operator $\nu \mapsto A(\nu, \pi, w_\alpha)$ has a pole at $\nu = 0$. Here $\nu \in \widetilde{\mathfrak{a}}^*_{\mathbb{C}}$, the complexified dual of the Lie algebra of $\widetilde{\mathbf{A}}$, and w_α is the reflection associated to α. Similarly, for $\sigma \in \mathcal{E}_2(M)$ we have the Plancherel measure $\mu_\alpha(\sigma)$. This is given by the pole of the standard intertwining operator $\nu_0 \mapsto A(\nu_0, \sigma, \mu_\alpha)$, where $\nu_0 \in$

$\mathfrak{a}^*_{\mathbb{C}}$, the complexified dual of the Lie algebra of $\mathbf{A}$. Note that, since $\widetilde{N} = N$, the intertwining operators are given by the same formula. That is, for ν and ν_0 in the region of convergence,

$$A(\nu, \pi, w_\alpha)\widetilde{f}(\widetilde{g}) = \int_{^*N_\alpha} \widetilde{f}(w_\alpha^{-1} n\widetilde{g})\, dn, \tag{2.1}$$

$$A(\nu_0, \sigma, w_\alpha) f(g) = \int_{^*N_\alpha} f(w_\alpha^{-1} ng)\, dn, \tag{2.2}$$

where $\widetilde{f}$ is in the space of $i_{\widetilde{M}_\alpha,\widetilde{M}}(\pi \otimes q_F^{\langle \nu, H_{\widetilde{P}}()\rangle})$, f is in the space of $i_{M_\alpha,M}(\sigma \otimes q_F^{\langle \nu_0, H_P()\rangle})$, $\widetilde{g} \in \widetilde{M}_\alpha$, and $g \in M_\alpha$ (see [H-C] for definitions of M_α, $^*N_\alpha$). The operators are then defined for all ν by meromorphic continuation.

Lemma 2.3 *Suppose $[\widetilde{G}, \widetilde{G}] = [G, G]$ and $ZG\backslash\widetilde{G}$ is finite abelian. Let $\widetilde{M}$ be a Levi subgroup of $\widetilde{G}$ and set $M = \widetilde{M} \cap G$. Let $\pi \in \mathcal{E}_2(\widetilde{M})$ and suppose $\sigma \in \mathcal{E}_2(M)$ is a component of $\pi|_M$. Then, for any $\alpha \in \Phi(\widetilde{\mathbf{P}}\widetilde{\mathbf{A}}) = \Phi(\mathbf{P}, \mathbf{A})$, we have $\mu_\alpha(\pi) = 0$ if and only if $\mu_\alpha(\sigma) = 0$.*

Proof We note that one can adapt the proof of a similar result in [Sh3], but we choose a slightly different approach. Let $\pi|_M = m_0 \bigoplus_{i=1}^k \sigma_i$, and assume $\sigma = \sigma_1$. If $\Pi = i_{\widetilde{G},\widetilde{M}}(\pi)$, then $\Pi|_G \simeq i_{G,M}(\pi|_M) = m_0 \bigoplus_{i=1}^k i_{G,M}(\sigma_i)$. Let $w = w_\alpha$. Then the intertwining operators for $\widetilde{G}$ satisfy

$$A(w\nu, \pi^w, w^{-1})A(\nu, \pi, w) = \mu_\alpha(\nu, \pi)^{-1} \cdot \gamma_\alpha(\widetilde{G}/\widetilde{P})^2, \tag{2.3}$$

where

$$\gamma_\alpha(\widetilde{G}/\widetilde{P}) = \int_{^*N_\alpha} q_F^{-2\langle \rho_\alpha, H_{P_\alpha}(n)\rangle}\, dn,$$

is the constant given in [H-C]. The meromorphic function $\mu_\alpha(\nu, \pi)$ is the Plancherel density and $\mu_\alpha(0, \pi) = \mu_\alpha(\pi)$. Restricting the relation (2.3) to G, (and restricting to the $i_{G,M}(\sigma \otimes q_F^{\langle \nu|_{\mathfrak{a}}, -\rangle})$-isotypic subspace) gives

$$A(w\nu|_{\mathfrak{a}}, w\sigma, w^{-1})A(\nu|_{\mathfrak{a}}, \sigma, w) = \mu_\alpha(\nu|_{\mathfrak{a}}, \pi)^{-1} \cdot \gamma_\alpha(\widetilde{G}/\widetilde{P})^2. \tag{2.4}$$

On the other hand, we have

$$A(w\nu|_{\mathfrak{a}}, w\sigma, w^{-1})A(\nu|_{\mathfrak{a}}, \sigma, w) = \mu_\alpha(\nu|_{\mathfrak{a}}, \sigma)^{-1} \cdot \gamma_\alpha(G/P)^2, \tag{2.5}$$

and the result follows by letting ν go to zero. ■

Let $\Delta'(\pi) = \{\alpha \in \Phi(\widetilde{\mathbf{P}}, \widetilde{\mathbf{A}}) | \mu_\alpha(\pi) = 0\}$. Lemma 2.3 shows $\Delta'(\pi) = \Delta'(\sigma)$, where $\Delta'(\sigma)$ is similarly defined. We let $W'(\pi) = \langle w_\alpha | \alpha \in \Delta'(\pi)\rangle$. Then $W'(\pi) = W'(\sigma)$, and since $\mu_\alpha(\sigma) = 0$ implies $\sigma^{w_\alpha} \simeq \sigma$, $W'(\pi) \subset W(\sigma) = \{w \in W(M) \mid \sigma^w \simeq \sigma\}$.

We now give several partial converses to Corollary 2.2 in the case where $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. Some of these will be crucial in describing the R-group explicitly in Section 3, while others we include to show the extent to which we can establish the converse at this time. We have yet to determine whether the converse holds in general.

Lemma 2.4 *Let $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. Suppose $\widetilde{M}$ is maximal, $\pi \in \mathcal{E}_c(\widetilde{M})$ is generic, and $\pi^w \simeq \pi$. Then for each component σ of $\pi|_M$ we have $\sigma^w \simeq \sigma$.*

Proof Let $\eta = \eta_{E/F}$ be the quadratic character of $F^\times$ attached to the extension E/F by local class field theory. Fix any character χ_η of $E^\times$ so that $\chi_\eta|_{F^\times} = \eta$. Let m be as in equation (1.1). First suppose $m = 0$, which implies $n = 2k$ is even. Note that

$$M = \left\{ \begin{pmatrix} g & \\ & \varepsilon(g) \end{pmatrix} \;\middle|\; \det g \det \varepsilon(g) = 1 \right\}.$$

Since $\det \varepsilon(g) = \overline{\det g^{-1}}$, we have $M \simeq \{g \in GL_k(E) | \det g \in F^\times\}$. Thus, $\chi_\eta|_M = \eta$. Since $\widetilde{M} \simeq GL_k(E)$ and $\pi^w \simeq \pi^\varepsilon$, we have $\pi^\varepsilon \simeq \pi$. By [G3] exactly one of $L(s, \pi, A)$ or $L(s, \pi\chi_\eta, A)$ has a pole at $s = 0$, where A is the Asai representation of $GL_k(\mathbb{C})$. Thus, either $\mu(\pi) = 0$ or $\mu(\pi\chi_\eta) = 0$, where μ is the Plancherel measure [Sh2]. Let σ be an irreducible component of $\pi|M$. Then, by Lemma 2.3 either $\mu(\sigma) = 0$ or $\mu(\sigma\eta) = 0$. The first of these requires $\sigma^\varepsilon \simeq \sigma$. The second of these requires $(\sigma\eta)^\varepsilon \simeq \sigma\eta$. But since $\eta^\varepsilon = \eta$, we will again have $\sigma^\varepsilon \simeq \sigma$ in this case.

Now suppose that $m > 0$, so $\widetilde{M} \simeq GL_k(E) \times SU_m(F)$. Then $\pi = \pi_1 \otimes \tau$, where π_1 is an irreducible discrete series representation of $GL_k(E)$ and τ is an irreducible generic discrete series representation of $SU_m(F)$. Note that $\pi^w \simeq \pi$ implies $\pi_1^\varepsilon \simeq \pi_1$. Consider π and $\pi\chi_\eta$. Then $\mu(\pi) = 0$ if and only if $L(s, \pi_1, A)L(2s, \pi_1 \times \tau)$ has a pole at $s = 0$ (see [Sh2, G-S]). Similarly $\mu(\pi\chi_\eta) = 0$ if and only if $L(s, \pi_1\chi_\eta, A)L(2s, \pi_1\eta \times \tau\eta)$ has a pole. As in the case $m = 0$, precisely one of $L(s, \pi_1, A)$ or $L(s, \pi_1\chi_\eta, A)$ has a pole at $s = 0$, and thus, at least one of $\mu(\pi)$ and $\mu(\pi\chi_\eta)$ is zero. But, as σ is a component of both π and $\pi\chi_\eta$, we must have $\mu(\sigma) = 0$, which therefore requires $\sigma^w \simeq \sigma$. ■

Lemma 2.5 *Suppose $m = 1$, where m is as in* (1.1)*. Then, for any $\pi \in \mathcal{E}_c(\widetilde{M})$ the representation $\sigma = \pi|_M$ is irreducible. Hence when $\pi^w \simeq \pi\chi$, we have $\sigma^w \simeq \sigma$.*

Proof Since $m = 1$, we have

$$\widetilde{M} \simeq GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_r}(E) \times E^1.$$

Note that if $(g_1, g_2, \ldots, g_r, h) \in M$, then

$$h = \Bigl(\prod_{i=1}^{r} \det(g_i \varepsilon(g_i))^{-1} \Bigr).$$

Thus,

$$M \simeq GL_{n_1}(E) \times \cdots \times GL_{n_r}(E).$$

Now it is clear that if $\pi \simeq \pi_1 \otimes \cdots \otimes \pi_r \otimes \xi \in \mathcal{E}_c(\widetilde{M})$, then $\pi|_M \simeq (\pi_1 \otimes \cdots \otimes \pi_r)\, \xi\xi^{\varepsilon}$ is irreducible. ■

Lemma 2.6 *Let m be as in equation* (1.1). *Suppose $m = 0$, or $\pi \simeq \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r \otimes \tau$, with $\tau|_{SU(m)}$ of multiplicity* 1.

(a) *If $s \in \mathcal{S}$ satisfies $\pi^s \simeq \pi\chi$ for some $\chi \in X(\widetilde{M})$, then $\sigma^s \simeq \sigma$ for any irreducible component σ of $\pi|_M$.*

(b) *Suppose $w = sc$, where $s = s_1 s_2 \cdots s_k$ is the disjoint cycle decomposition, and c changes an even number of signs in each cycle s_i. If $\pi^w \simeq \pi\chi$, then $\sigma^w \simeq \sigma$, for each component σ of $\pi|_M$.*

Proof (a) The argument is essentially that of [G1, Lemma 2.3]. We give the proof in the case $m = 0$, and the proof when $\rho|_{SU(m)}$ is multiplicity one is identical. Let

$$M_0 = [M, M] = [\widetilde{M}, \widetilde{M}] \simeq SL_{n_1}(E) \times SL_{n_2}(E) \times \cdots \times SL_{n_r}(E).$$

Since each $\pi_i|_{SL_{n_i}(E)}$ is multiplicity one [T], so is $\pi|_{M_0}$. If ρ is a component of $\sigma|_{M_0}$, then $\rho \simeq \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_r$, for some choice of components ρ_i of $\pi_i|_{SL_{n_i}(E)}$. Suppose $s = s_1 s_2 \cdots s_k$ is the disjoint cycle decomposition of s and, without loss of generality, assume that $s_1 = (1\,2 \cdots j_1)\; s_2 = (j_1 + 1 \cdots j_2), \dots, s_k = (j_{k-1} + 1 \cdots j_k)$. Let $j = j_1$. Since $\pi^s \simeq \pi\chi$, we then have $\pi_{i+1} \simeq \pi_i\chi \simeq \pi_1\chi^i$, for $i = 1, 2, \dots, j-1$, and $\pi_1 \simeq \pi_j\chi$, *i.e.*, $\pi_1 \simeq \pi_1\chi^j$. Thus, for each $1 \le i \le j$, ρ_i is an irreducible component of π_1. By [G-K], for each $1 \le i \le j-1$, there is an $a_i \in E^{\times}$, so that $\rho_{i+1} = \rho_i^{\delta(a_i)}$, where

$$\delta(a) = \begin{pmatrix} a & \\ & I_{n_1-1} \end{pmatrix}.$$

Let $a_j = (a_1 a_2 \cdots a_{j-1})^{-1}$. Then $\rho_j^{\delta(a_j)} = \rho_1$. Set $g_1 = \operatorname{diag}\{\delta(a_1), \delta(a_2), \dots, \delta(a_j)\}$. Then $\det g_1 = 1$, and $(\rho_1 \otimes \cdots \otimes \rho_j)^{g_1} = \rho_2 \otimes \cdots \otimes \rho_j \otimes \rho_1 = (\rho_1 \otimes \cdots \otimes \rho_j)^{s_1}$. Similarly, for $i = 2, 3, \dots, k$, we can find such a g_i, with determinant 1. Setting $g = \operatorname{diag}\{g_1, g_2, \dots, g_k, \varepsilon(g_k), \dots, \varepsilon(g_1)\}$, we have $g \in M$ and $\rho \simeq \rho^g \simeq \rho^s$. Therefore, ρ^s is a component of both $\sigma|_{M_0}$ and $\sigma^s|_{M_0}$, and thus by multiplicity one, $\sigma^s \simeq \sigma$.

(b) Now suppose $w = sc$, with $c \neq 1$. We again assume $s = s_1 s_2 \cdots s_k$. Suppose $s_1 = (1\,2 \cdots j)$, and for some d with $0 \le d \le j-1$, that $c = C_{d+1}C_{d+2} \cdots C_j c'$, where c' acts trivially on $\{1, 2, \dots, j\}$. Then, $\pi^w \simeq \pi\chi$ implies $\pi_{i+1}^{\varepsilon^{b_i}} \simeq \pi_i\chi$ for $i = 1, 2, \dots, j-1$, and $\pi_1^{\varepsilon^{b_j}} \simeq \pi_j\chi$. Here each $b_i \in \{0, 1\}$. Let $\rho = \rho_1 \otimes \cdots \otimes \rho_r \otimes \tau$, be a component of $\pi|_{M_0}$, where, again, $M_0 = [\widetilde{M}, \widetilde{M}]$. Then, for $i = 1, \dots, j$, we have $(\rho_{i+1})^{\varepsilon^{b_i}} = \rho_i^{\delta(a_i)}$, for some a_i. Since $j - d$ is even, we let

$$b = \varepsilon\big(\delta(a_1)^{-1}\delta(a_2)^{-1} \cdots \delta(a_d)^{-1}\big)$$
$$(\delta(a_{d+1}))^{-1}\varepsilon(\delta(a_{d+2}))^{-1}(\delta(a_{d+3}))^{-1} \cdots \varepsilon(\delta(a_{j-1}))^{-1}.$$

Then $\rho_j^b \simeq \rho_1^\varepsilon$. Therefore, we set

$$g_1 = \operatorname{diag}\{\delta(a_1), \dots, \delta(a_{j-1}), b\}.$$

Note that $\det(g_1)\det(\varepsilon(g_1)) = 1$, and thus, choosing $g_2, \dots, g_k$ in a similar manner, we have

$$g = \operatorname{diag}\{g_1, g_2, \dots, g_k, \varepsilon(g_k), \dots, \varepsilon(g_1)\} \in M,$$

with $\rho^g \simeq \rho^w$. Therefore, we again see that $\rho^w \simeq \rho$. Then, as $\sigma|_{M_0}$ and $\sigma^w|_{M_0}$ have a common component, multiplicity one implies $\sigma \simeq \sigma^w$. ∎

Lemma 2.7 *Suppose $\pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r \otimes \tau$, and that $\tau|_{SU(m)}$ is of multiplicity one. If $c \in \mathcal{Z}$ satisfies $\pi^c \simeq \pi$, then $\sigma^c \simeq \sigma$, for each component σ of $\pi|_M$.*

Proof Let $M_0 = M_1 \times M_2 \times \cdots \times M_r \times SU(m) \subset M$, with $M_i = \{g \in GL_{n_i}(E) | \det g \in F^\times\}$. Then, $\pi|_{M_0}$ is of multiplicity one. If $c = \prod_{i \in J} C_i$ for some $J \subset \{1, 2, \dots, r\}$, then $\pi^c \simeq \bigotimes_{i \in J} \pi_i^\varepsilon \otimes \bigotimes_{i \notin J} \pi_i \otimes \tau$. Let σ be a component of $\pi|_M$, and suppose σ_0 is a component of $\sigma|_{M_0}$. Then $\sigma_0 = \rho_1 \otimes \cdots \otimes \rho_r \otimes \tau_0$, where ρ_i is a component of $\pi_i|_{M_i}$ and τ_0 is a component of $\tau|_{SU(m)}$. Therefore,

$$\sigma_0^c \simeq \bigotimes_{i \in J} \rho_i^\varepsilon \otimes \bigotimes_{i \notin J} \rho_i \otimes \tau_0 \simeq \sigma_0,$$

by Lemma 2.4. Thus, σ_0 is a component of both σ and σ^c upon restriction to M_0, and hence by multiplicity one $\sigma \simeq \sigma^c$. ∎

Remark If τ is generic, then the representation π satisfies the hypotheses of Lemmas 2.6 and 2.6.

Now assume $m \geq 2$. Recall that for $h \in U_m(F)$, we have $\det h \in E^1$, and thus, by Hilbert's Theorem 90, $\det h = a\bar{a}^{-1}$, for some $a \in E^\times$. For $a \in E^\times$ we let

$$\alpha_m(a) = \begin{bmatrix} a & & \\ & I_{m-2} & \\ & & \bar{a}^{-1} \end{bmatrix}.$$

Then $\det(\alpha_m(a)) = \det h$. Note $\alpha_m(ab) = \alpha_m(ba) = \alpha_m(a)\alpha_m(b)$. If $g \in GL_k(E)$, we abuse notation and write $\alpha_m(g)$ for $\alpha_m(\det(g))$.

Lemma 2.8 *If $\widetilde{M} \cong GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_r}(E) \times U_m(F)$, and $m \geq 2$, then $M \simeq GL_{n_1}(E) \times \cdots \times GL_{n_r}(E) \rtimes SU_m(F)$.*

Proof Let $g = (g_1, g_2, \dots, g_r) \in GL_{n_1}(E) \times \cdots \times GL_{n_r}(E)$. Then the map

$$(g, h_0) \mapsto \begin{pmatrix} g & & \\ & \alpha_m(g)^{-1}h_0 & \\ & & \varepsilon(g) \end{pmatrix}$$

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is a set bijection from $GL_{n_1} \times \cdots \times GL_n(E) \rtimes SU_m(F)$ to M. The action of $GL_{n_1}(E) \times \cdots \times GL_{n_r}(E)$ on $SU_m(F)$ is by $g \circ h = \alpha_m(g)h\alpha_m(g)^{-1}$. It is now easy to see that our map is an isomorphism. ■

If $x = (g_1, \ldots, g_r, h) = (g, h) \in \widetilde{M}$, and $w \in W$, then we denote wxw^{-1} by (g^w, h). Restricting to M, we see that under the above isomorphism W acts on $(GL_{n_1}(E) \times \cdots \times GL_{n_r}(E)) \ltimes SU_m(F)$ by $w \cdot (g, h_0) = (g^w, (\alpha_m(g)^{-1}\alpha_m(g^w)h_0)$. (Note that $\alpha_m(g^{-1})\alpha(g^w) \in SU_m(F)$.)

Now suppose $\pi \simeq \pi_1 \otimes \pi_2 \otimes \pi_2 \cdots \otimes \pi_r \otimes \tau$ and that $V = V_1 \otimes V_2 \otimes \cdots \otimes V_r \otimes V_\tau$ is the space of π. We may write $g = (g_1, \ldots, g_r) \in GL_{n_1}(E) \times \cdots \times GL_{n_r}(E)$, and $\pi_0(g) = \otimes_i \pi_i(g_i)$ acting on $V_0' = \otimes_{i=1}^r V_i$. Let (τ_0, V_0) be a component of $\tau|_{SU_m(F)}$. Then, with respect to the semidirect product decomposition in Lemma 2.8, the map

$$(g, h_0) \mapsto \left(v_0' \otimes v_0 \mapsto \pi_0(g)v_0' \otimes \tau(\alpha_m(g)^{-1})\tau_0(h_0)v_0\right), \tag{2.6}$$

is an irreducible component of $\pi|_M$. We now prove another partial converse to Corollary 2.2. Note that we do not assume multiplicity one upon restriction. This result is crucial to the R-group computations of Section 3.

Proposition 2.9 *Let $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. Suppose $\widetilde{M} \simeq GL_{n_1}(E) \times \cdots \times GL_{n_r}(E) \times U_m(F)$ and $m \geq 2$. Let $\pi \in \mathcal{E}_2(\widetilde{M})$ and suppose σ is an irreducible component of $\pi|_M$. Then $W(\pi) \subset W(\sigma)$.*

Proof Let $w \in W(\pi)$ and suppose that $w = sc$, with $s \in \mathcal{S}$ and $c \in \mathcal{Z}$. We note that if $\pi \simeq \pi_1 \otimes \cdots \otimes \pi_r \otimes \tau$, then

$$\pi^w \simeq \pi_{s(1)}^{\varepsilon_1} \otimes \pi_{s(2)}^{\varepsilon_2} \otimes \cdots \otimes \pi_{s(r)}^{\varepsilon_r} \otimes \tau,$$

where each ε_i is either ε or trivial. Since $\pi^w \simeq \pi$, we have $\pi_i \simeq \pi_{s(i)}^{\varepsilon_i}$ for each i, so fix an intertwining, $T\colon V_{s(i)} \to V_i$ with $\pi_i T_i = T_i \pi_{s(i)}^{\varepsilon_i}$. Then $T = \bigotimes_i T_i \otimes \mathrm{id}_{V_\tau}$ satisfies $\pi T = T\pi^w$. Now suppose that σ is given by (2.6). Then

$$\begin{aligned} T\sigma^w(g, h_0)(v_0' \otimes v_0) &= \bigotimes_{i=1}^r T_i \pi_{s(i)}^{\varepsilon_i}(g_i)v_i \ \otimes \tau(\alpha_m(g^w)^{-1})\tau_0(\alpha_m(g)^{-1}\alpha_m(g^w)h_0)v_0 \\ &= \bigotimes_{i=1}^r \pi_i(g_i)T_i(v_i) \ \otimes \tau(\alpha_m(g)^{-1})\tau_0(h_0)v_0 \\ &= \sigma(g, h_0)T(v_0' \otimes_0). \end{aligned}$$

Thus T is an equivalence between σ^w and σ. ■

Remarks

(a) The proof of Proposition 2.9 will apply to to the case $\pi^w \simeq \pi\chi$ whenever the identity is an equivalence between τ and $\tau\chi$.

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(b) There is a similar result when $m = 0$, using an identification $M \simeq (GL_{n_1}(E) \times GL_{n_2}(E) \times \cdots \times GL_{n_{r-1}}(E)) \ltimes SL_{n_r}(E)$. To the extent we need this result in Section 3, however, Lemma 2.7 will suffice.

Corollary 2.10 *Let $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. For any $\pi \in \mathcal{E}_2(\widetilde{M})$, and any irreducible component σ of $\pi|_M$, we have $W(\pi) \subset W(\sigma)$.*

Proof This will follow immediately from Lemmas 2.4, 2.6, 2.7, and Proposition 2.9, unless $m = 0$ and w does not satisfy the hypotheses of either part of Lemma 2.6. This case we now resolve. All that is left is to consider the case where $m = 0$ and $w = sc$, with $c \neq 1$ changing an odd number of signs in some cycle of s. So, suppose $s = s_1 s_2 \cdots s_k$ is the disjoint cycle decomposition for s, with $s_1 = (12 \cdots j)$. Further suppose that for some $1 \leq d \leq j$, we have $c = C_d C_{d+1} \cdots C_j c'$, with c' acting trivially on $\{1, \ldots, j\}$ and $j - d + 1$ odd. Then $\pi^w \simeq \pi$ implies

$$\pi_1 \simeq \pi_2 \simeq \cdots \simeq \pi_d \simeq \pi_{d+1}^{\varepsilon} \simeq \pi_{d+2} \simeq \cdots \simeq \pi_j \simeq \pi_1^{\varepsilon}.$$

Therefore, $\pi_i \simeq \pi_i^{\varepsilon}$ for $1 \leq i \leq j$. Hence, $C_j \in W(\pi)$. Taking $w_1 = wC_j$, we see that w_1 changes an even number of signs among $\{1, \ldots, j\}$. Proceeding in the same manner on the cycles $s_2, \ldots, s_k$, if necessary, we see we can write $w = w_0 c_0$, with w_0 satisfying the hypotheses of Lemma 2.6(b) and both w_0 and c_0 in $W(\pi)$. Since $m = 0$, Lemma 2.7 applies to c_0. Thus, we know $\sigma^{w_0} \simeq \sigma$ and $\sigma^{c_0} \simeq \sigma$, and hence $\sigma^w \simeq \sigma$. ■

3 *R*-Groups

For the moment we work in the more general setting where $G \subset \widetilde{G}$ have the same derived group. Let $R_\pi(\sigma) = R(\pi) \cap W(\sigma)$. Note that if $w \in R_\pi(\sigma)$, then $w\Delta'(\pi) = \Delta'(\pi)$, and thus by Lemma 2.3, $w\Delta'(\sigma) = \Delta'(\sigma)$, which, combined with the fact that $w \in W(\sigma)$, shows that $w \in R(\sigma)$. Note that if $w \in R(\sigma) \cap W(\pi)$, then as $w\Delta'(\sigma) = \Delta'(\sigma)$, we have $w \in R_\pi(\sigma)$, so we certainly have $R_\pi(\sigma) = R(\sigma) \cap W(\pi)$. It is clear that $R_\pi(\sigma)$ is a normal subgroup of $R(\sigma)$. In this section we first describe the quotient $R(\sigma)/R_\pi(\sigma)$. Then, specializing to the case where $\widetilde{G} = U_n$ and $G = SU_n$, we show that $R(\sigma)$ is a semidirect product of $R(\pi)$ and a naturally occurring group of characters.

Definition 3.1 Let $\widehat{W(\sigma)} = \{\chi \in X(\widetilde{M}) \mid \pi^w \simeq \pi\chi \text{ for some } w \in W(\sigma)\}$.

Let $w \in R(\sigma)$. Then, by Corollary 2.2, there is a $\chi \in X(\widetilde{M})$ for which $\pi^w \simeq \pi\chi$. If $\pi^w \simeq \pi\chi_1 \simeq \pi\chi_2$, then $\chi_1\chi_2^{-1} \in X(\pi)$. Thus $\varphi\colon R(\sigma) \to \widehat{W(\sigma)}/X(\pi)$ given by

$$\varphi(w) = \chi \text{ if and only if } \pi^w \simeq \pi\chi \tag{3.1}$$

is a well defined homomorphism. Let $\chi \in \widehat{W(\sigma)}/X(\pi)$. Choose $w \in W(\sigma)$ with $\pi^w \simeq \pi\chi$. Then $w = rw'$, with $r \in R(\sigma)$ and $w' \in W'(\sigma)$. By Lemma 2.3, $w' \in W'(\pi)$, and in particular, $\pi^{w'} \simeq \pi$. Therefore, $\pi^r \simeq \pi\chi$. Hence, for any $\chi \in \widehat{W(\sigma)}$

there is an element $r \in R(\sigma)$ with $\pi^r \simeq \pi\chi$. Thus, φ is surjective, with kernel $R_\pi(\sigma)$. The following result is now obvious.

Proposition 3.2 *Suppose $[\widetilde{G}, \widetilde{G}] = [G, G]$, and $ZG\backslash\widetilde{G}$ is finite abelian. For any $\pi \in \mathcal{E}_2(\widetilde{M})$ and any irreducible component σ of $\pi|_M$ we have $R_\pi(\sigma) \lhd R(\sigma)$ and $R(\sigma)/R_\pi(\sigma) \simeq \widehat{W(\sigma)}/X(\pi)$.*

We now specialize to the case where $\widetilde{G} = U_n(F)$ and $G = SU_n(F)$. For $C \in \mathcal{Z}$ there is a $J \subseteq \{1, 2, \dots, r\}$ with

$$C = C_J = \prod_{i \in J} C_i. \tag{3.2}$$

By [G2] there is a subset $J(\pi)$ so that $R(\pi) = \langle C_i | i \in J(\pi) \rangle$. By Corollary 2.10, $R(\pi) \subset R(\sigma)$. Also note that if $\pi^{w_1} \simeq \pi^{w_2} \simeq \pi\chi$, then $\pi^{w_1^{-1}w_2} \simeq \pi$, which implies $\sigma^{w_1^{-1}w_2} \simeq \sigma$. Hence if $\chi \in \widehat{W(\sigma)}$, then $\sigma^w \simeq \sigma$ for all $w \in W$ with $\pi^w \simeq \pi\chi$.

Lemma 3.3 *Suppose $\chi \in \widehat{W(\sigma)}$ and further suppose there are elements $w_1 = s_1c_1$, $w_2 = s_2c_2 \in R(\sigma)$ with $\pi^{w_1} \simeq \pi\chi \simeq \pi^{w_2}$. Here $s_i \in \mathcal{S}$, and $c_i \in \mathcal{Z}$. Then $s_1 = s_2$.*

Proof Note $w_1w_2^{-1} \in R(\sigma) \cap W(\pi) = R(\pi)$. Therefore, $w_1w_2^{-1} \in \mathcal{Z}$. Since $w_1w_2^{-1} = (s_1s_2^{-1})(s_2c_1s_2^{-1}c_1)$ is the decomposition of $w_1w_2^{-1}$ in the semidirect product $W = \mathcal{S} \ltimes \mathcal{Z}$, we have $s_1 = s_2$. ■

For $\chi \in \widehat{W(\sigma)}$, denote by s_χ the unique element in $\mathcal{S}$ so that $s_\chi c \in R(\sigma)$ for some $c \in \mathcal{Z}$. We will give an explicit description of s_χ. First, we need a lemma.

Lemma 3.4 *Suppose $\pi \simeq \pi_1 \otimes \cdots \otimes \pi_r \otimes \tau$. Let $w = sc \in W(\mathbf{G}, \mathbf{A})$ with $\pi^w \simeq \pi\chi$, for some χ. Suppose, $c(i) \neq i$. Then $\pi_i\chi \simeq \pi_i\chi^\varepsilon$.*

Proof First suppose that $s(i) = i$, so $w(i) = -i$. Then $\pi^w \simeq \pi\chi$ implies $\pi_i^\varepsilon \simeq \pi_i\chi$. Applying w again we see $\pi_i \simeq \pi_i\chi^2$, while $\pi_i \simeq (\pi_i\chi)^\varepsilon \simeq \pi_i^\varepsilon\chi^\varepsilon \simeq \pi_i\chi\chi^\varepsilon$. Thus, $\pi_i\chi \simeq \pi_i\chi^\varepsilon$.

Now suppose that $s(i) \neq i$. Assume $s = s_1s_2\cdots s_\ell$, is the disjoint cycle decomposition of s, and $s_1(i) \neq i$. Without loss of generality, suppose that $s_1 = (12\cdots j)$, and that, for some $1 \leq d \leq j$, we have $c = C_dC_{d+1}\cdots C_jc'$, with c' trivial on $\{1, \dots, j\}$. If $d \geq 2$, then $\pi^w \simeq \pi\chi$ implies $\pi_2 \simeq \pi_1\chi$. Then $\pi^{w^2} \simeq \pi\chi^2$, implies $\pi_2^\varepsilon \simeq \pi_j^\varepsilon\chi^2 \simeq \pi_1^\varepsilon\chi$. Thus $\pi_2 \simeq \pi_1\chi^\varepsilon \simeq \pi_1\chi$, proving the claim in this case. If $d = 1$, then $\pi^w \simeq \pi\chi$ implies $\pi_2^\varepsilon \simeq \pi_1\chi$, while applying w^2 gives $\pi_2 \simeq \pi_1^\varepsilon\chi \simeq \pi_1^\varepsilon\chi^\varepsilon$, proving the claim in this case as well. ■

We now describe the permutation s_χ explicitly. For $\chi \in \widehat{W(\sigma)}$, we let $\Omega(\chi, \pi) = \{i | \pi_i \simeq \pi_i\chi, \text{ or } \pi_i^\varepsilon \simeq \pi_i\chi\}$. We then let $\Omega_1(\chi, \pi) = \{1, 2, \dots, r\} \setminus \Omega(\pi, \chi)$. For $i \in \Omega(\chi, \pi)$, let $s(i) = i$. If $\Omega_1(\chi, \pi) = \varnothing$, we are done and $s = 1$. Otherwise,

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for each $i \in \Omega_1(\chi,\pi)$, we let $\Omega_1(i,\chi,\pi) = \{j \in \Omega_1(\chi,\pi) | \pi_j \simeq \pi_i\chi\}$, and let $\Omega_1^\varepsilon(i,\chi,\pi) = \{j \in \Omega_1(\chi,\pi) | \pi_j^\varepsilon \simeq \pi_i\chi\}$. Let $i_{11} = \min(\Omega_1(\chi,\pi))$. Define

$$i_{12} = \begin{cases} \min(\Omega_1(i_{11},\chi,\pi)) & \text{if } \Omega_1(i_{11},\chi,\pi) \neq \varnothing, \\ \max(\Omega_1^\varepsilon(i_{11},\chi,\pi)) & \text{otherwise.} \end{cases}$$

Suppose we have defined $i_{11}, i_{12}, \dots i_{1j}$. If $i_{1j} = i_{11}$, then we let $s_1 = (i_{11}, i_{12}, \dots, i_{1j-1})$. Otherwise, we let

$$i_{1j+1} = \begin{cases} \min(\Omega_1(i_{1j},\chi,\pi)) & \text{if } \Omega_1(i_{1j},\chi,\pi) \neq \varnothing, \\ \max(\Omega_1^\varepsilon(i_{1j},\chi,\pi)) & \text{otherwise.} \end{cases}$$

This, inductively, defines an element s_1 of $\mathcal{S}$.

Now let $\Omega_2(\chi,\pi) = \Omega_1(\chi,\pi) \setminus \{i_{11},\dots,i_{1j-1}\}$. If $\Omega_2(\chi,\pi) = \varnothing$, we are done, and $s = s_1$. Otherwise, repeat the above process by taking $i_{21} = \min(\Omega_2(\chi,\pi))$, and defining

$$i_{2j+1} = \begin{cases} \min(\Omega_1(i_{2j},\chi,\pi)) & \text{if } \Omega_1(i_{2j},\chi,\pi) \neq \varnothing, \\ \max(\Omega_1^\varepsilon(i_{2j},\chi,\pi)) & \text{otherwise,} \end{cases}$$

and let j_2 be the smallest integer greater than 1 for which $i_{2j_2} = i_{21}$. Set $s_2 = (i_{21},\dots,i_{2j_2-1})$. Proceed inductively to define $\Omega_3(\chi,\pi), \Omega_4(\chi,\pi), \dots, \Omega_k(\pi,\chi)$, with associated cycles $s_3, s_4, \dots, s_k$, and suppose that k is minimal with the property that $\Omega_{k+1}(\pi,\chi) = \varnothing$. Then let $s = s_1 s_2 \cdots s_k$. By construction, if $\pi_i \simeq \pi_j$ for $i < j$, and $s_{i'}(i) \neq i$, then $s_{j'}(j) \neq j$ for some $j' > i'$.

Lemma 3.5 *Let $\chi \in \widehat{W(\sigma)}$, and define $s = s_1 s_2 \cdots s_k$, as above. Then $s = s_\chi$.*

Proof We define $w = sc$ with the property that $\pi^w \simeq \pi\chi$, and then show $w \in R(\sigma)$. That will show that $s = s_\chi$. We let c be defined by

$$c(i) = \begin{cases} i & \text{if } \pi_{s(i)} \simeq \pi_i\chi, \\ -i & \text{if } \pi_{s(i)}^\varepsilon \simeq \pi_i\chi \text{ and } \pi_{s(i)} \not\simeq \pi_i\chi. \end{cases}$$

Then, setting $w = sc$, we have $\pi^w \simeq \pi\chi$ by construction. We show that w preserves the positivity of all elements of $\Delta'(\sigma)$. First suppose $\alpha = \alpha_{ij} = e_{b_i} - e_{b_{j-1}+1}$, where $b_i = n_1 + n_2 + \cdots + n_i$. Then, $\pi_i \simeq \pi_j$. If $w(i) = s(i)$, then $w\alpha_{ij} = e_{s(i)} \pm e_{s(j)}$. This would be a negative root if and only if $w(j) = s(j)$ and $s(j) < s(i)$. But as $w(i) = s(i)$, we know that if $w(j) = s(j)$, then $s(j) > s(i)$, by construction. On the other hand, if $w(i) = -s(i)$, then we must have $s(j) < s(i)$ and $c(j) = -j$, as well, so again $w\alpha > 0$.

Suppose that $\alpha = \alpha'_{ij} = e_{b_i} + e_{b_{j-1}+1} \in \Delta'(\sigma)$. Then $\pi_j \simeq \pi_i^\varepsilon$. If $w\alpha'_{ij} < 0$, then certainly either $w(i) = -s(i)$, or $w(j) = -s(j)$. Thus, by Lemma 3.4, $\pi_i\chi \simeq \pi_i\chi^\varepsilon$. If $w(i) = s(i)$, and $w(j) = -s(j)$, then $\pi_{s(i)} \simeq \pi_i\chi$. Note that $\pi_{s(j)} \simeq \pi_j^\varepsilon\chi^\varepsilon \simeq \pi_i\chi$, and thus, by the choice of $s(i)$, we must have $s(i) < s(j)$. Therefore $w\alpha'_{ij} > 0$, contradicting our assumption. So we must have $w(i) = -s(i)$. If $w(j) = -s(j)$, then $\pi^{\varepsilon}_{s(j)} \simeq \pi_j\chi \simeq \pi^{\varepsilon}_i\chi$ which implies $\pi_{s(j)} \simeq \pi_i\chi$. Then, by construction of s, we would have assigned $s(i) = j'$, with $j' \leq s(j)$, and $\pi_{j'} \simeq \pi_i\chi$. We would then have $w(i) = s(i) = j'$, contradicting our assumption. Thus, we must have $w(j) = s(j)$. Then $\pi_{s(j)} \simeq \pi_j\chi \simeq \pi^{\varepsilon}_i\chi \simeq (\pi_i\chi)^{\varepsilon}$, and thus by construction, $s(i) > s(j)$. Therefore, $w\alpha'_{ij} = \alpha_{s(j)s(i)} > 0$, which contradicts our assumption. Hence it is impossible for $w\alpha'_{ij} < 0$ if $\alpha'_{ij} \in \Delta'(\sigma)$.

Finally, suppose that

$$\beta_i = \begin{cases} e_{b_i} & \text{if } n \text{ is odd,} \\ 2e_{b_i} & \text{if } n \text{ is even.} \end{cases}$$

Then $\pi^{\varepsilon}_i \simeq \pi_i$. Assume $w\beta_i < 0$. Then we must have $c(i) = -i$. If $s(i) = i$, then $\pi^{\varepsilon}_i \simeq \pi\chi$, which now says $\pi_i \simeq \pi_i\chi$, and by assumption we have $c(i) = i$, contradicting our assumption. Now suppose that $s(i) \neq i$. Then as $c(i) = -i$, we have, by Lemma 3.4, $\pi_i\chi \simeq \pi_i\chi^{\varepsilon}$ and $\pi^{\varepsilon}_{s(i)} \simeq \pi_i\chi$. Since $\pi^{\varepsilon}_i \simeq \pi_i$, we have $\pi_{s(i)} \simeq \pi_i\chi$, which means that $\Omega_\ell(i, \chi, \pi) \neq \varnothing$ and therefore, we are forced to take $c(i) = i$. This also contradicts our assumption and hence $w\beta_i > 0$ for all $\beta_i \in \Delta'(\sigma)$.

In order to conclude that $s = s_\chi$, we must know that $w\sigma \simeq \sigma$. However, by assumption, $\chi \in \widehat{W(\sigma)}$, and thus $\sigma^w \simeq \sigma$. (See the remark preceding Lemma 3.3.) ■

Remark Note that it is possible that $s_\chi = s_\eta$ with $\pi\eta \not\simeq \pi\chi$. In particular, it is possible that $s_\chi = 1$, and $c \in \mathcal{Z}$ is an element of $W(\sigma) \setminus W(\pi)$.

Lemma 3.6 *Suppose $\chi \in \widehat{W(\sigma)}/X(\pi)$. Then there is a unique minimal $J_\chi \subset \{1, 2, \dots, r\}$ with $s_\chi C_{J_\chi} \in R(\sigma)$. Define $w_\chi = s_\chi C_{J_\chi}$.*

Proof Choose any element $w \in R(\sigma)$ with $\pi^w \simeq \pi\chi$. By Lemma 3.3, we have $w = s_\chi C_J$ for some J. Let $J' = \{i \in J | C_i \in R(\pi)\}$. Let $J_\chi = J \setminus J'$. Then, $C_{J'} \in R(\sigma)$, and thus $w_\chi = s_\chi C_{J_\chi} = s_\chi C_J C_{J'} \in R(\sigma)$. For any J_1, with $s_\chi C_{J_1} \in R(\sigma)$, we have $(s_\chi C_{J_\chi})^{-1} s_\chi C_{J_1} = C_{J_2} \in R(\pi)$ so $J_2 \subset J(\pi)$, and $s_\chi C_{J_1} = s_\chi C_{J_\chi} C_{J_2}$. Therefore, as $J_\chi \cap J(\pi) = \varnothing$, $J_\chi \subset J_1$. ■

Since, by Corollary 2.10, $R(\pi) = R_\pi(\sigma)$, Proposition 3.2 gives us the exact sequence

$$1 \to R(\pi) \to R(\sigma) \to \widehat{W(\sigma)}/X(\pi) \to 1. \tag{3.3}$$

The next result will show that this exact sequence splits, so $R(\sigma)$ is a semidirect product.

Theorem 3.7 *Let $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. Let $\chi_1, \chi_2 \in \widehat{W(\sigma)}/X(\pi)$, and w_{χ_i} be as in Lemma 3.6. Then $w_{\chi_1} w_{\chi_2} = w_{\chi_1\chi_2}$. Thus $\Gamma_\sigma = \{w_\chi \mid \chi \in \widehat{W(\sigma)}\}$ is a subgroup of $R(\sigma)$. Furthermore, $R(\sigma) = \Gamma_\sigma \ltimes R(\pi)$.*

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Proof Let $w_i = w_{\chi_i}$ and suppose $w_i = s_i C_{J_i}$ is the decomposition given in Lemma 3.6. Then $J_i \cap J(\pi) = \varnothing$. Note that $w_1 w_2 = s_1 s_2 C_{s_2^{-1}(J_1)} C_{J_2}$. Suppose $s_2^{-1}(J_1) \cap J(\pi) \neq \varnothing$. Let $j \in s_2^{-1}(J_1) \cap J(\pi)$. Then $C_{s_2^{-1}(J_1)} C_j = C_{J'}$, with $|J'| < |s_2^{-1}(J_1)|$, and hence $|J'| < |J_1|$. Now

$$w' = w_1 w_2 C_j w_2^{-1} = s_1 s_2 C_{s_2^{-1}(J_1)} C_j C_{J_2} C_{J_2} s_2^{-1} = s_1 s_2 C_{J'} s_2^{-1} = s_1 C_{s_2(J')},$$

and $\pi^{w'} \simeq \pi\chi$. However, since $|s_2(J')| = |J'| < |J_1|$, this contradicts our choice of J_1. Thus, $s_2^{-1}(J_1) \cap J(\pi) = \varnothing$. Since $J_2 \cap J(\pi) = \varnothing$, as well we see $C_{s_2^{-1}(J_1)} C_{J_2} = C_J$, with $J \cap J(\pi) = \varnothing$. Now $w_1 w_2 = s_1 s_2 C_J = s_{\chi_1\chi_2} C_J \in R(\sigma)$, and since $J \cap J(\pi) = \varnothing$, we know $J = J_{\chi_1\chi_2}$, by the proof of Lemma 3.6. Thus, $w_1 w_2 = w_{\chi_1\chi_2}$. This shows that Γ_σ is a subgroup of $R(\sigma)$.

In order to show that the sequence (3.3) splits, we consider the composition

$$\widehat{W(\sigma)}/X(\pi) \xrightarrow{\psi} \Gamma_\sigma \xrightarrow{\varphi} \widehat{W(\sigma)}/X(\pi),$$

where φ is given by (3.1) and $\psi(\chi) = w_\chi$. Note that Lemma 3.6 shows ψ is well defined, and the above argument shows that ψ is a homomorphism. By definition, $\varphi \circ \psi(\chi) = \varphi(w_\chi) = \chi$. Thus, $\varphi \circ \psi = 1$, which proves that (3.3) splits. Thus, $R(\sigma) = \Gamma_\sigma \ltimes R(\pi) \simeq \widehat{W(\sigma)}/X(\pi) \ltimes R(\pi)$. ∎

We address non-abelian R-groups. Suppose $w_\chi = s_\chi C_{J_\chi} \in \Gamma_\sigma$. Suppose further that $s_\chi(i) = i$ for all $i \in J(\pi)$. Then w_χ centralizes $R(\pi)$, and hence lies in the center of $R(\sigma)$. Thus, R is non-abelian if and only if there is some $w = s_\chi c \in R(\sigma)$ with $s(i) \neq i$ for some $i \in J(\pi)$.

Lemma 3.8 *Let* $R(\pi) = \langle C_i | i \in J(\pi) \rangle$.

(a) *There is no* $j \notin J(\pi)$ *for which* $\pi_j^\varepsilon \simeq \pi_j$ *and* $C_j \in R(\sigma)$.
(b) *If* $w = sc \in R(\sigma)$ *then* $s(J(\pi)) = J(\pi)$.

Proof (a) Suppose $j \notin J(\pi)$ and $\pi_j^\varepsilon \simeq \pi_j$. Then $C_j \in W(\pi)$ and $C_j \notin R(\pi)$, by assumption. Thus, $C_j\alpha < 0$ for some $\alpha \in \Delta'(\pi) = \Delta'(\sigma)$, which shows that $C_j \notin R(\sigma)$.

For (b), suppose that there is some $i \in J(\pi)$ with $s(i) \notin J(\pi)$. Then, since $C_i \in R(\pi) \subset R(\sigma)$, we see that $wC_i \in R(\sigma)$. Let $c = C_J$. If $i \in J$, then $wC_i = sC_{J\setminus\{i\}} \in R(\sigma)$, and thus we may assume that $i \notin J$. Now since $wC_i = C_{s(i)} w \in R(\sigma)$, we see that $C_{s(i)} \in R(\sigma)$, as well. Note that if $\pi^{C_{s(i)}} \simeq \pi\eta$, then $\pi_{s(i)}^\varepsilon \simeq \pi_{s(i)}\eta$, while $\pi_i \simeq \pi_i\eta$. Since $\pi_{s(i)} \simeq \pi_i\xi$, for some ξ, we have $\pi_{s(i)}\eta \simeq \pi_{s(i)}$. Thus, $\pi^{C_{s(i)}} \simeq \pi$. However, this contradicts part (a), which completes the lemma. ∎

We now wish to describe the conditions under which $i_{G,M}(\sigma)$ has elliptic constituents when σ is generic. With this assumption, the cocycle is a coboundary (see [K2, p. 62]), so $\mathcal{C}(\sigma) \simeq \mathbb{C}[R(\sigma)]$. Recall that the regular elements are given by $R(\sigma)_{\text{reg}} = \{w \in R(\sigma) \mid \mathfrak{a}_w = \{0\}\}$. We first describe all the regular elements of $R(\sigma)$, without regard to whether σ is generic. If we know that the cocycle split in general, this would then describe the elliptic spectrum in general.

Theorem 3.9 *Let $\widetilde{\mathbf{G}} = U_n$ and $\mathbf{G} = SU_n$. Let $\pi \in \mathcal{E}_2(M)$, and suppose that σ is a component of $\pi|_M$. Then $R(\sigma)_{\text{reg}} \neq \varnothing$ if and only if there is an element $w = s_\chi C_J \in R(\sigma)$ with s_χ an r-cycle and $|J|$ odd, where $J \subset \{1, 2, \dots, r\}$ is given by* (3.2)*.*

Proof If s_χ has more than one orbit, one can easily construct a non-zero element $X \in \mathfrak{a}_w$. Let $O_1 = \{1, s(1), s^2(1), \dots, s^{k-1}(1)\}$ and $O_2 = \{j, s(j), \dots, s^{\ell-1}(j)\}$ be two distinct orbits of s_χ. Let $a_i = 0$, for $i \notin O_1 \cup O_2$. We set $a_1 = a_j = 1$, and for $1 \le i \le k = 2$, let

$$a_{s^i(1)} = \begin{cases} a_{s^{i-1}(1)} & \text{if } s^{i-1}(1) \notin J, \\ -a_{s^{i-1}(1)} & \text{if } s^{i-1}(1) \in J. \end{cases}$$

similarly, for $1 \le i \le \ell - 2$ we let

$$a_{s^i(j)} = \begin{cases} a_{s^{i-1}(j)} & \text{if } s^{i-1}(j) \notin J, \\ -a_{s^{i-1}(j)} & \text{if } s^{i-1}(j) \in J. \end{cases}$$

Let $X = Y_M(a_1, a_2, \dots, a_r)$, as in Section 1. Then $wX = X$, and $\operatorname{tr} X = 0$, so $X \in \mathfrak{a}_w$. Therefore, if $w = s_\chi C_J$ is regular, then s_χ is an r-cycle.

Suppose, without loss of generality, that $s_\chi = (12 \cdots r)$, and $X = Y_M(a_1, \dots, a_r) \in \mathfrak{a}_w$. Suppose $a_1 = \lambda$. Then, for $2 \le i \le r$, we have $a_i = \pm\lambda$. Thus, we simply denote $X = X_J(\lambda)$. Now, as $\operatorname{tr} X = 0$, we have $nr(\lambda - \bar{\lambda}) = 0$, so $\lambda \in \mathbb{R}$. Note that if $a_r = \lambda$ and $r \in J$ or $a_r = -\lambda$ and $r \notin J$, then $\lambda = -\lambda$, and hence $\mathfrak{a}_w = \{0\}$. Suppose $|J|$ is even. If $r \in J$, then $a_r = -\lambda$, as w changes an odd number of signs among $1, 2, \dots, r-1$. Then $X_J(\lambda) \in \mathfrak{a}_w$ for any $\lambda \in \mathbb{R}$. Similarly, if $r \notin J$, then $a_r = \lambda$ and hence $X_J(\lambda) \in \mathfrak{a}_w$ for any $\lambda \in \mathbb{R}$. Thus, $w \notin R(\sigma)_{\text{reg}}$ if $|J|$ is even. On the other hand, if $|J|$ is odd and $r \in J$, then $a_r = \lambda$, so $X_J(\lambda) \in \mathfrak{a}_w$ only for $\lambda = 0$. Similarly, if $|J|$ is odd and $r \notin J$, then $a_r = \lambda$, so again $\mathfrak{a}_w = \{0\}$. ∎

Remark We note that Theorem 3.9 is inconsistent with the results of [G2], and this is due to an error in that work which we now correct. Note that $\mathfrak{a}_{\widetilde{G}} = \{\lambda I_n | \lambda \in i\mathbb{R}\}$. Then we easily see that $\mathfrak{a}_w = \mathfrak{a}_{\widetilde{G}}$ if and only if $m = 0$, $r = 1$, and $R \simeq \mathbb{Z}_2$. That is, only the Siegel Levi subgroup of U_n supports non-discrete elliptic representations.

We note that by Lemma 3.6(b), there are two possibilities for $R(\sigma)$ in the situation of Theorem 3.9. Namely, if $R(\pi) = \{1\}$ then s_χ an r-cycle implies $R(\sigma) \simeq \mathbb{Z}_r$. Otherwise $R(\pi) \neq \{1\}$ implies $R(\pi) = \langle C_i | \, 1 \le i \le r \rangle \simeq \mathbb{Z}_2^r$. Now if $\eta \in \widehat{W(\sigma)}$, then $s_\eta \in R(\sigma)$. If $s_\eta(1) \neq 1$, and $s_\eta(j) = j$, then $\pi_j \simeq \pi_j \eta$ implies $\pi_1 \eta \simeq \pi_1$, which contradicts our assumption that $s_\eta(1) \neq 1$. This implies that $s_\eta = s_\chi^i$ for some i, and so $R(\sigma) \simeq \mathbb{Z}_r \ltimes \mathbb{Z}_2^r$.

We now give a specific case of this second phenomenon, and note that the induced representation will contain both elliptic and non-elliptic representations. Suppose $r = 3$ and we are in this second case, namely $R(\sigma) \simeq \mathbb{Z}_3 \ltimes \mathbb{Z}_2^3$. Let $s = (123)$. Let κ be a character of $R(\pi)$, and let R_κ be the stabilizer of κ in $R(\sigma)$. By [Se, Proposition 25] we know that any irreducible representation of $R(\sigma)$ is given by $\rho = \rho_{\kappa,\lambda} = \operatorname{Ind}_{R_\kappa}^{R(\sigma)}(\kappa \otimes \lambda)$, where κ is extended to R_κ trivially, and λ is an irreducible representation of $R_\kappa \cap$

$\mathbb{Z}_3$. Note that if $w = sc$, then w acts transitively on $R(\pi)$. Thus $R_\kappa \neq R(\pi)$ if and only if $\kappa(C_i) = \kappa(C_j)$ for $i \neq j$. This implies either $\kappa = 1$, or $\kappa(C_i) = -1$ for all i, and we denote this character by sgn. So, if $\kappa \neq 1, sgn$, then $R_\kappa = R(\pi)$, and $\rho = \rho_\kappa = \mathrm{Ind}_{R(\pi)}^{R(\sigma)} \kappa$. Then, by the induced character formula, the character θ_κ of ρ_κ on an element $w = sc$ is

$$\theta_\kappa(sc) = \frac{1}{8} \sum_{\substack{x \in R(\sigma) \\ x^{-1}(sc)x \in R(\pi)}} \kappa(x^{-1}scx) = 0,$$

as the sum has no terms since $R(\pi)$ is normal. Thus, all the components $\pi(\rho_\kappa)$ with $\kappa \neq 1$, sgn are non-elliptic. There are two such representations, as there are two orbits of such κ under $\mathbb{Z}_3$. Each of these components appears in $i_{G,M}(\sigma)$ with multiplicity three. On the other hand, the six components components $\pi(\rho_{1,\lambda})$, $\pi(\rho_{sgn,\lambda})$, are elliptic, and these appear with multiplicity one. We conclude that there are representations σ for which $i_{G,M}(\sigma)$ has some elliptic components, but not all the components are elliptic. It is clear that this phenomenon will generalize to the case where r is prime.

Note that, when $P = B$ is the Borel subgroup, these are precisely the examples discussed in [K2]. The phenomenon noted above was not mentioned there merely because the results of [A2] were not yet available.

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