SOME REMARKS ON LIMITS IN CATEGORIES

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1. Introduction. The object of this paper is to give simple criteria for the existence of direct limits in categories and for the permuting of a functor with direct limits.

The notion of direct limit of a diagram that we shall use here is essentially that of Kan (4), which is more general than the usual notion of direct limit of a directed diagram.

Our treatment is based on the fact (Lemma 2) that the usual process for constructing the direct limit of a diagram of modules, which consists in taking a direct sum of the modules in the diagram and then considering a certain homomorphic image of this direct sum (3, p. 220), is essentially, once certain notions have been properly generalized, the only process for constructing the direct limit of any diagram in any category.

It follows quite naturally from this that in a category every diagram has a direct limit if and only if every family of objects has a direct sum and every pair of maps has a cokernel (Theorem 1), and that a functor permutes with direct limits if and only if it is right exact and permutes with direct sums (Theorem 2).

The duals of these results concerning direct limits, being just the same results for the dual categories, are not mentioned.

The paper ends with a theorem on direct limits in categories of modules that is not dualizable.

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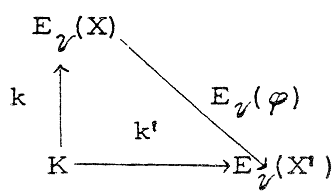
Except for trivial alterations, this material was presented by the author in the course of a series of lectures given at the Seminar of the Canadian Mathematical Congress in August 1961.

2. <u>The existence of direct limits in categories.</u> The categories considered all have the property that the class of maps from one object to another is a set. In a category $\mathcal{X}$ the objects and maps will be denoted by the letters X and x, with or without subscripts or superscripts, respectively. A category $\mathcal{V}$ will be called proper if the class of its maps is a set. The functors from such a proper category $\mathcal{V}$ to a category $\mathcal{X}$ will be called the $\mathcal{V}$-diagrams of $\mathcal{X}$. The $\mathcal{V}$-diagrams of $\mathcal{X}$ and their natural transformations obviously form a category $\mathcal{X}_{\mathcal{V}}$. The embedding functor

$$E_{\mathcal{V}}: \mathcal{X} \to \mathcal{X}_{\mathcal{V}}$$

assigns to each object X of $\mathcal{X}$ the constant $\mathcal{V}$-diagram that maps each object of $\mathcal{V}$ onto X and assigns to every map $x: X \to X'$ in $\mathcal{X}$ the natural transformation $E_{\mathcal{V}}(x) : E_{\mathcal{V}}(X) \to E_{\mathcal{V}}(X')$, where $(E_{\mathcal{V}}(x))_V = x$ for each $V \in \mathcal{V}$.

Definition 1 (Kan). If $K \in \mathcal{X}_{\mathcal{V}}$, if $X \in \mathcal{X}$ and if $k: K \to E_{\mathcal{V}}(X)$ is a natural transformation, then k is a direct limit of K if for any natural transformation $k' : K \to E_{\mathcal{V}}(X')$, $X' \in \mathcal{X}$, there exists one and only one map $\varphi : X \to X'$ such that $k' = E_{\mathcal{V}}(\varphi)k$.



We notice that a natural transformation $k: K \to E_{\mathcal{V}}(X)$ is just a family of maps $\{k_V : K(V) \to X\}_{V \in \mathcal{V}}$ such that for every map $v: V \to V'$ of $\mathcal{V}$, $k_V = k_{V'} K(v)$, and that k is a direct limit of K by Definition 1, if given any other such family $\{k'_V\}_{V \in \mathcal{V}}$, there exists one and only one map $\varphi : X \to X'$ such that

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$k'_V = \varphi k_v$ for all $V \in \mathcal{V}$.

We now make a list of different interesting special cases corresponding to different types of proper categories $\mathcal{V}$.

1) The cokernel of a family of pairs of maps. There is a particular object $\tilde{V}$ of $\mathcal{V}$ such that the set of non-identity maps of $\mathcal{V}$ consists of one pair of distinct maps from V to $\tilde{V}$ for each $V \neq \tilde{V}$. Then a $\mathcal{V}$-diagram $K: \mathcal{V} \to \mathcal{X}$ is completely determined by the family of pairs of maps $\{(f_V, g_V : X_V \to X_{\tilde{V}})\}_{V \in \mathcal{V},\, V \neq \tilde{V}}$ where f_V and g_V are the images by K of the two maps from V to $\tilde{V}$. A natural transformation $k: K \to E_{\mathcal{V}}(X)$ is completely determined by the map $k_{\tilde{V}} : X_{\tilde{V}} \to X$ which has the property $k_{\tilde{V}} f_V = k_{\tilde{V}} g_V$ for all $V \in \mathcal{V}$. It is a direct limit of K if and only if for any map $h : X_{\tilde{V}} \to X'$ with the same property, there exists a unique $\varphi : X \to X'$ such that $h = \varphi k_{\tilde{V}}$. We will then say that $k_{\tilde{V}}$ is a cokernel of the family of pairs $\{(f_V, g_V)\}_{V \in \mathcal{V},\, V \neq \tilde{V}}$. One can easily show that such a cokernel is an epimorphism.

2) Quasi-ordered diagrams. For any two objects of $\mathcal{V}$, there is at most one map from one to the other. In this case, $\mathcal{V}$ is essentially a quasi-ordered set; $V \leq V'$ meaning that there is a map from V to V'. A $\mathcal{V}$-diagram $K: \mathcal{V} \to \mathcal{X}$ is completely determined by the family $\{f_{V,V'}\}_{V \leq V' \in \mathcal{V}}$ where $f_{V,V'} = K(v)$, v being the only map from V to V'. All the following cases considered are special cases of quasi-ordered diagrams.

3) Direct sums. All the maps of $\mathcal{V}$ are identity maps so that $\mathcal{V}$ is essentially a set. Then a $\mathcal{V}$-diagram $K: \mathcal{V} \to \mathcal{X}$ is just a family $\{X_V\}_{V \in \mathcal{V}}$. In this case a direct limit of K is called a direct sum of the family $\{X_V\}_{V \in \mathcal{V}}$.

4) Direct sums with amalgamated maps. If $\mathcal{V}$ is a quasi-ordered set with the property that there exists an element V_o of $\mathcal{V}$ such that $V < V'$ implies that $V = V_o$, then a

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$\mathcal{V}$-diagram $K: \mathcal{V} \to \mathcal{X}$ is completely determined by the family $\{f_V\}_{V \in \mathcal{V}, V \neq V_o}$ where f_V is the image by K of the only map from V_o to V. A natural transformation $k:K \to E_{\mathcal{V}}(X)$ is completely determined by the family $\{k_V\}_{V \in \mathcal{V}, V \neq V_o}$. If k is a direct limit of K, then $\{k_V\}_{V \in \mathcal{V}, V \neq V_o}$ will be called the direct sum of $\{K(V)\}_{V \in \mathcal{V}, V \neq V_o}$ where the maps f_V are amalgamated.

5) Directed diagrams. This is the case where $\mathcal{V}$ is a directed set.

LEMMA 1. In a category $\mathcal{X}$, let $\{(f_i, g_i: X_i \to X)\}_{i \in I}$ be a family of pairs of maps where I is a set, let $\{k_i: X_i \to X'\}_{i \in I}$ be a direct sum of $\{X_i\}_{i \in I}$ and let $f, g: X' \to X$ be the only maps for which $f_i = fk_i$ and $g_i = gk_i$ for all $i \in I$. Then $\varphi: X \to X''$ is a cokernel of $\{(f_i, g_i)\}_{i \in I}$ if and only if it is a cokernel of the pair (f, g).

Proof. Let us assume that φ is a cokernel of (f, g). Then for each $i \in I$, $\varphi f_i = \varphi f k_i = \varphi g k_i = \varphi g_i$ and if $\psi: X \to X'''$ is such that $\psi f_i = \psi g_i$ for all $i \in I$, then $\psi f k_i = \psi g k_i$ for all $i \in I$ so that $\psi f = \psi g$ and therefore there exists a unique $\chi: X'' \to X'''$ such that $\psi = \chi \varphi$.

The converse argument is just as easy.

COROLLARY. If every family of objects of a category $\mathcal{X}$ has a direct sum, then every family of pairs of maps has a cokernel if and only if every single pair of maps has a cokernel.

LEMMA 2. Let $K: \mathcal{V} \to \mathcal{X}$ be a $\mathcal{V}$-diagram and let $\{k_V: K(V) \to X\}_{V \in \mathcal{V}}$ be a direct sum of $\{K(V)\}_{V \in \mathcal{V}}$. If $\varphi: X \to X'$ is a cokernel of the family of pairs

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$$\{(k_V, k_{V'} K(v))\}_{v:V \to V',\, v \in \mathscr{V}} \qquad (1)$$

then $\{\varphi k_V\}_{V \in \mathscr{V}}$ is a direct limit of K.

If $h: K \to E_{\mathscr{V}}(X')$ is a direct limit of K and if φ is the only map from X to X' such that $h_V = \varphi k_V$ for all $V \in \mathscr{V}$, then φ is a cokernel of the family of pairs (1).

Proof. Assume first of all that φ is a cokernel of the family of pairs (1) and let $k': K \to E_{\mathscr{V}}(X'')$ be a natural transformation. Then there exists a unique $\psi: X \to X''$ such that $k'_V = \psi k_V$ for each $V \in \mathscr{V}$. Then, for each $v: V \to V'$ in $\mathscr{V}$,

$$\psi k_V = k'_V = k'_{V'} K(v) = \psi (k_{V'} K(v))$$

so that there exists a unique $\chi: X' \to X''$ such that $\psi = \chi \varphi$. Then, for each $V \in \mathscr{V}$, $k'_V = \psi k_V = \chi(\varphi k_V)$, i.e. $\{\varphi k_V\}_{V \in \mathscr{V}}$ is a direct limit of K.

Now assume that $h: K \to E_{\mathscr{V}}(X')$ is a direct limit of K and that φ is the only map from X to X' such that $h_V = \varphi k_V$ for all $V \in \mathscr{V}$. Then, for all $v: V \to V'$ in $\mathscr{V}$,

$$\varphi k_V = h_V = h_{V'} K(v) = \varphi(k_{V'} K(v)) .$$

If $\psi: X \to X''$ is such that $\psi k_V = \psi(k_{V'} K(v)) = (\psi k_{V'}) K(v)$ for all $v: V \to V'$ in $\mathscr{V}$, then there exists a unique $\chi: X' \to X''$ such that $\psi k_V = \chi h_V = \chi \varphi k_V$ for all $V \in \mathscr{V}$, so that $\psi = \chi \varphi$ and therefore φ is a cokernel of the family (1).

THEOREM 1. In a category $\mathscr{X}$, every diagram has a direct limit if and only if every family of objects has a direct sum and every pair of maps from one object to another has a cokernel.

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Proof. Assume that every family of objects of $\mathcal{X}$ has a direct sum and that every pair of maps from one object to another in $\mathcal{X}$ has a cokernel. Let $K: \mathcal{V} \to \mathcal{X}$ be a diagram in $\mathcal{X}_{\mathcal{V}}$, let $\{k_V: K(V) \to X\}_{V \in \mathcal{V}}$ be a direct sum of $\{K(V)\}_{V \in \mathcal{V}}$ and let φ be a cokernel of

$$\{(k_V, k_{V'} K(v))\}_{v: V \to V', v \in \mathcal{V}}.$$

Then by Lemma 2, $\{\varphi k_V\}_{V \in \mathcal{V}}$ is a direct limit of K.

This Theorem may be applied for example to prove that the category of topological spaces and continuous functions has direct limits. For any family of topological spaces (which is a set) has a direct sum which is known as the topological union of the given spaces and also, any pair of continuous maps $f, g: X \to X'$ has a cokernel. To establish this last statement one considers the intersection R of all those equivalence relations on X' for which $f(a) \equiv g(a)$ for every $a \in X$. Then, the projection h of X' onto the quotient space X'/R is a cokernel of (f, g).

Similar arguments apply to the category of all groups and homomorphisms and in general to the category of all sets with a certain type of structure and all functions that preserve this type of structure (see 1).

If $\mathcal{X}$ is an additive category, then we notice that a cokernel h of a pair $(f, g: X \to X')$ is just a direct sum of X and X' with 0 and $f-g$ amalgamated (or also a cokernel of $f-g$ in the ordinary sense) so that in particular, it is a direct limit of a quasi-ordered diagram. Therefore, we conclude that in an additive category every diagram has a direct limit if and only if every quasi-ordered diagram has a direct limit.

3. Functors permuting with direct limits. If $S, T: \mathcal{X} \to \mathcal{Y}$ and $U: \mathcal{Y} \to \mathcal{Z}$ are functors and if $\alpha: S \to T$ is a natural transformation, then $\{U(\alpha_X)\}_{X \in \mathcal{X}}$ is a natural transformation from US to UT which we denote by $U(\alpha)$.

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Definition 2. If $\mathcal{C}$ is a class of proper categories and if $T: \mathcal{X} \to \mathcal{Y}$ is a functor, then we say that T is $\mathcal{C}$-admissible if given any diagram $K \in \mathcal{X}_{\mathcal{V}}$ where $\mathcal{V} \in \mathcal{C}$, if $k: K \to E_{\mathcal{V}}(X)$ is a direct limit of K, then $T(k): T(K) \to TE_{\mathcal{V}}(X) = E_{\mathcal{V}}(T(X))$ is a direct limit of TK.

A $\mathcal{C}$-admissible functor T will be said to

1) commute with direct limits, if $\mathcal{C}$ is the class of all proper categories

2) commute with cokernels, if $\mathcal{C}$ consists of those proper categories that are described in 1) of section 1.

3) be right exact, if it commutes with cokernels of single pairs of maps

4) be of type $L\Sigma$ or commute with direct sums, if $\mathcal{C}$ is the class of all trivial proper categories in which all maps are identity maps

5) be of type $L\Sigma^*$, if $\mathcal{C}$ is the class of all directed sets

6) be of type $L\Sigma^{**}$, if $\mathcal{C}$ is the class of all quasi-ordered sets.

LEMMA 3. If a functor $T: \mathcal{X} \to \mathcal{Y}$ is right exact and of type $L\Sigma$ and if in the category $\mathcal{X}$ every family of objects has a direct sum, then T permutes with cokernels.

Proof. Let $\varphi: X \to X''$ be a cokernel of $\{(f_i, g_i: X_i \to X)\}_{i \in I}$. If $\{k_i: X_i \to X'\}_{i \in I}$ is a direct sum of $\{X_i\}_{i \in I}$ and if f and g are the only maps from X' to X for which $f_i = fk_i$ and $g_i = gk_i$ for all $i \in I$, then by Lemma 1, φ is a cokernel of the pair (f, g). Since T is right exact, $T(\varphi)$ is a cokernel of $(T(f), T(g))$. But since T is of type $L\Sigma$, $\{T(k_i)\}_{i \in I}$ is a direct sum of $\{T(X_i)\}_{i \in I}$. Then, since for each $i \in I$, $T(f_i) = T(fk_i) = T(f)\,T(k_i)$ and $T(g_i) = T(g)\,T(k_i)$, again by

Lemma 1, $T(\varphi)$ is a cokernel of $\{(T(f_i), T(g_i))\}_{i \in I}$.

THEOREM 2. Given a functor $T: \mathcal{X} \to \mathcal{Y}$, where in $\mathcal{X}$ every family of objects has a direct sum, T commutes with direct limits if and only if it is right exact and of type $L\Sigma$.

Proof. Assume that T is right exact and of type $L\Sigma$, let $k:K \to E$ (X) be a direct limit of $K \in \mathcal{X}_{\mathcal{V}}$ and let $\{k'_V : K(V) \to X'\}_{V \in \mathcal{V}}$ be a direct sum of $\{K(V)\}_{V \in \mathcal{V}}$. By Lemma 2, if $\gamma : X' \to X$ is the only map for which $k_V = \gamma k'_V$ for each $V \in \mathcal{V}$, then γ is a cokernel of

$$\{(k'_V, k'_{V'} K(v))\}_{v:V \to V', \, v \in \mathcal{V}}$$

Since T is right exact, $T(\gamma)$ is a cokernel of

$$\{(T(k'_V), T(k'_{V'})K(v))\}_{v:V \to V', v \in \mathcal{V}}$$

Since T is of type $L\Sigma$, $\{T(k'_V)\}_{V \in \mathcal{V}}$ is a direct sum of $\{T(X_V)\}_{V \in \mathcal{V}}$. Therefore, again by Lemma 2,

$$\{T(\gamma) T(k'_V) = T(\gamma k'_V) = T(k_V)\}_{V \in \mathcal{V}}$$

is a direct limit of TK.

Definition 2 may be extended readily to functors of more than one variable. For example, if $T(\mathcal{X}, \mathcal{Y})$ is a functor of two variables taking its values in $\mathcal{Z}$, and if $\mathcal{E}$ is a class of proper categories, then we say that T is $\mathcal{E}$-admissible if given two direct limits $k:K \to E_{\mathcal{V}}(X)$ and $h:H \to E_{\mathcal{W}}(Y)$, where $K \in \mathcal{X}_{\mathcal{V}}$, $H \in \mathcal{Y}_{\mathcal{W}}$ and $\mathcal{V}, \mathcal{W} \in \mathcal{E}$, then $\{T(k_V, h_W)\}_{V \in \mathcal{V}, W \in \mathcal{W}}$, which is a natural transformation from $T(K,H) \in \mathcal{Z}_{\mathcal{V} \times \mathcal{W}}$ to $E_{\mathcal{V} \times \mathcal{W}}(T(X,Y))$, is a direct limit of $T(K,H)$. It should be noticed that this does not mean that T, considered as a functor from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$ is $\mathcal{E}$-admissible or even $\mathcal{E} \times \mathcal{E}$-admissible.

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However, with this definition one can show that $T(\mathcal{X}, \mathcal{Y})$ is $\mathcal{C}$-admissible if and only if for each $X \in \mathcal{X}$ and each $Y \in \mathcal{Y}$, the partial functors $T(X, \mathcal{Y})$ and $T(\mathcal{X}, Y)$ are $\mathcal{C}$-admissible. That this condition is necessary is obvious. So let us assume that $T(X, \mathcal{Y})$ and $T(\mathcal{X}, Y)$ are $\mathcal{C}$-admissible for all $X \in \mathcal{X}$ and all $Y \in \mathcal{Y}$, let $k: K \to E_{\mathcal{V}}(X)$ and $h: H \to E_{\mathcal{W}}(Y)$ be direct limits of $K \in \mathcal{X}_{\mathcal{V}}$ and $H \in \mathcal{Y}_{\mathcal{W}}$, where $\mathcal{V}$ and $\mathcal{W}$ are in $\mathcal{C}$ and consider a family

$$\{\sigma_{V, W} : T(K(V), H(W)) \to Z\}_{V \in \mathcal{V}, W \in \mathcal{W}}$$

where for each $v: V \to V'$ in $\mathcal{V}$ and each $w: W \to W'$ in $\mathcal{W}$,

$$\sigma_{V, W} = \sigma_{V', W'} T(K(v), H(w)) \tag{1}$$

For each $W \in \mathcal{W}$, $\sigma_{V, W} = \sigma_{V', W} T(K(v), H(W))$ for every $v: V \to V'$ in $\mathcal{V}$, and since $T(\mathcal{X}, H(W))$ is $\mathcal{C}$-admissible, there exists a unique $\rho_W : T(X, H(W)) \to Z$ such that $\sigma_{V, W} = \rho_W T(k_V, H(W))$ for each $v \in \mathcal{V}$. Then, for each $w: W \to W'$ in $\mathcal{W}$,

$$\rho_W T(k_V, H(W)) = \sigma_{V, W} = \sigma_{V, W'} T(K(V), H(w))$$

$$= \rho_{W'} T(k_V, H(W')) T(K(V), H(w)) = \rho_{W'} T(X, H(w)) T(k_V, H(W))$$

so that $\rho_W = \rho_{W'} T(X, H(w))$. Then, since $T(X, \mathcal{Y})$ is $\mathcal{C}$-admissible, there exists a unique $\tau : T(X, Y) \to Z$ such that $\rho_W = \tau T(X, h_W)$ for each $W \in \mathcal{W}$, so that for each $V \in \mathcal{V}$ and each $W \in \mathcal{W}$,

$$\sigma_{V, W} = \rho_W T(k_V, H(W)) = \tau T(X, h_W) T(k_V, H(W)) = \tau T(k_V, h_W)$$

With these remarks it is clear that Theorem 2 extends to any functor of two or more variables. For categories of modules, because the functor $\otimes_\Lambda$ is right exact and of type $L\Sigma$, Theorem 2 permits one to conclude immediately that it

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permutes with all direct limits. Similarly, by duality, since HOM_{Λ} is left exact and of type $R\pi$, one concludes that it permutes with all inverse limits.

THEOREM 3. Given three functors $S, T, U: \mathcal{X} \to \mathcal{Y}$, if $\alpha, \beta: S \to T$ and $\gamma: T \to U$ are natural transformations and if for each $X \in \mathcal{X}$, γ_X is a cokernel of (α_X, β_X), then if S and T are $\mathcal{C}$-admissible, so is U.

Proof. Let $k: K \to E_{\mathcal{V}}(X)$ be a direct limit of $K \in \mathcal{X}_{\mathcal{V}}$, where $\mathcal{V} \in \mathcal{C}$. We must show that $U(k): UK \to E_{\mathcal{V}}(U(X))$ is a direct limit.

$$\begin{array}{ccccc}
E(S(X)) & \overset{E(\alpha_X)}{\underset{E(\beta_X)}{\rightrightarrows}} & E(T(X)) & \xrightarrow{E(\gamma_X)} & E(U(X)) \\
S(k)\uparrow & & T(k)\uparrow & & U(k)\uparrow \\
SK & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & TK & \xrightarrow{\gamma} & UK
\end{array}$$

Let $h: UK \to E_{\mathcal{V}}(Y)$ be a natural transformation. Then, $h\gamma: TK \to E_{\mathcal{V}}(Y)$ is a natural transformation and since $T(k)$ is a direct limit, there exists a unique $\varphi: T(X) \to Y$ such that $h\gamma = E_{\mathcal{V}}(\varphi)T(k)$. Then

$$E_{\mathcal{V}}(\varphi\alpha_X)S(k) = E_{\mathcal{V}}(\varphi)E(\alpha_X)S(k) = E_{\mathcal{V}}(\varphi)T(k)\alpha = h\gamma\alpha = h\gamma\beta =$$

$$E_{\mathcal{V}}(\varphi)T(k)\beta = E_{\mathcal{V}}(\varphi)E_{\mathcal{V}}(\beta_X)S(k) = E_{\mathcal{V}}(\varphi\beta_X)S(k)$$

and since $S(k)$ is a direct limit, $E_{\mathcal{V}}(\varphi\alpha_X) = E_{\mathcal{V}}(\varphi\beta_X)$ so that $\varphi\alpha_X = \varphi\beta_X$. Then, since γ_X is a cokernel of (α_X, β_X), there exists a unique $\psi: U(X) \to Y$ such that $\varphi = \psi\gamma_X$ so that

$$h\gamma = E_{\mathcal{V}}(\varphi)T(k) = E_{\mathcal{V}}(\psi\gamma_X)T(k) = E_{\mathcal{V}}(\psi)E_{\mathcal{V}}(\gamma_X)T(k) = E_{\mathcal{V}}(\psi)U(k)\gamma$$

and since $\gamma_{X'}$ is an epimorphism for each $X' \in \mathcal{X}$, $h = E_{\mathcal{V}}(\psi)U(k)$.

LEMMA 4. If $K: \mathcal{V} \to \mathcal{X}_R$ is a directed diagram of R-modules and if $k:K \to E_{\mathcal{V}}(X)$ is a natural transformation, then k is a direct limit if and only if

1) X is the set-theoretical union of all the im k_V, $V \in \mathcal{V}$

2) If $a \in K(V)$ is such that $k_V(a) = 0$, then there exists $V' \geq V$ such that $f_{V,V'}(a)=0$, where $f_{V,V'}$ is the image by K of the only map from V to V'.

Proof. Assume first of all that k is a direct limit. That X is the module-theoretical sum of the im k_V is obvious by Lemma 2. That it is the set-theoretical sum of the im k_V follows from the fact that if $V, V' \in \mathcal{V}$, there exists $V'' \in \mathcal{V}$ such that $V \leq V''$ and $V' \leq V''$ so that $k_V = k_{V''} f_{V,V''}$ and $k_{V'} = k_{V''} f_{V',V''}$ and therefore, $\operatorname{im} k_V \subseteq \operatorname{im} k_{V''}$ and $\operatorname{im} k_{V'} \subseteq \operatorname{im} k_{V''}$.

Now assume that $a \in K(\bar{V})$ is such that $k_{\bar{V}}(a) = 0$ and let $\{k'_V : K(V) \to X'\}_{V \in \mathcal{V}}$ be a direct sum of $\{K(V)\}_{V \in \mathcal{V}}$. Then, by Lemma 2,

$$k'_{\bar{V}}(a) = (k'_{V_1} - k'_{V'_1} f_{V_1, V'_1})(a_1) + \dots + (k'_{V_n} - k'_{V'_n} f_{V_n, V'_n})(a_n) \qquad (2)$$

where for each i, $a_i \in K(V_i)$ and $V_i \leq V'_i$. But there exists V_o such that $V \leq V_o$ and $V_i \leq V_o$ for each i. Then, if $\mathcal{V}_o$ denotes the quasi-ordered subset of $\mathcal{V}$ consisting of all $V \leq V_o$, $\{f_{V,V_o}\}_{V \leq V_o}$ is a direct limit of the restriction of K to $\mathcal{V}_o$ and it is clear from (2) that $f_{\bar{V},V_o}(a) = 0$.

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Conversely, assume that conditions 1) and 2) are satisfied and let $h:K \to E_{\mathcal{V}}(X')$ be a direct limit of K. Then there exists $\gamma: X' \to X$ such that $k = E_{\mathcal{V}}(\gamma)h$. If $a \in X$, by 1) there exists $V \in \mathcal{V}$ such that $a = k_V(a_V)$, where $a_V \in K(V)$. If $a = k_{V'}(a_{V'})$, where $V' \in \mathcal{V}$ and $a_{V'} \in K(V')$, there exists $V'' \in \mathcal{V}$ such that $V \leq V''$ and $V' \leq V''$, and we have

$$k_{V''}(f_{V,V''}(a_V) - f_{V',V''}(a_{V'})) = k_V(a_V) - k_{V'}(a_{V'}) = 0$$

so that by 2), there exists $V''' \geq V''$ such that

$$0 = f_{V'',V'''}(f_{V,V''}(a_V) - f_{V',V''}(a_{V'})) = f_{V,V'''}(a_V) - f_{V',V'''}(a_{V'})$$

and therefore,

$$h_V(a_V) = h_{V'''} f_{V,V'''}(a_V) = h_{V'''} f_{V',V'''}(a_{V'}) = h_{V'}(a_{V'}) \quad .$$

It is then easy to verify that the correspondence $a \to h_V(a_V)$ is an inverse R-homomorphism of γ.

THEOREM 4. Given three functors $S, T, U: \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is a category of R-modules, if $\alpha: S \to T$ and $\beta: T \to U$ are natural transformations and if for each $X \in \mathcal{X}$, α_X is a kernel of β_X, then if T and U are $\mathcal{C}$-admissible, where the proper categories in $\mathcal{C}$ are directed sets, so is S.

Proof. Let $K \in \mathcal{X}_{\mathcal{V}}$, where $\mathcal{V} \in \mathcal{C}$, and let $k:K \to E_{\mathcal{V}}(X)$ be a direct limit of K. Using Lemma 4, we will show that S(k) is a direct limit of SK.

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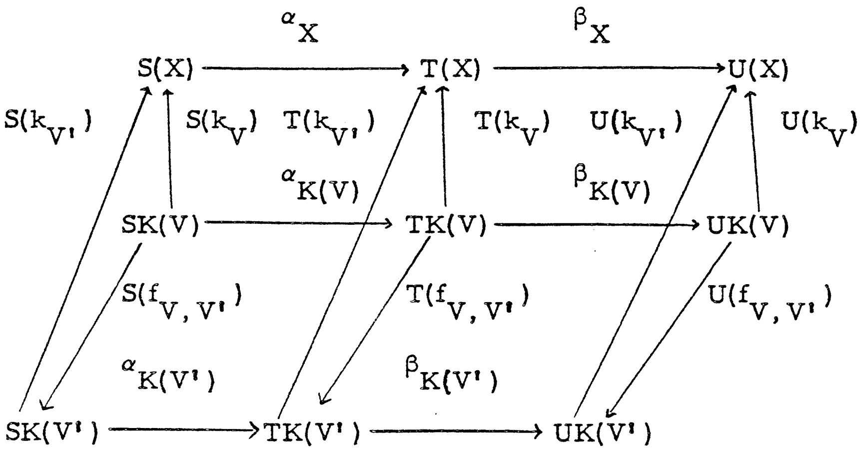

Let $a \in S(X)$. There exists $V \in \mathscr{V}$ such that $\alpha_X(a) = T(k_V)(b_V)$, where $b_V \in TK(V)$. Since

$$U(k_V)\beta_{K(V)}(b_V) = \beta_X T(k_V)(b_V) = \beta_X \alpha_X(a) = 0$$

there exists $V' \geq V$ such that $U(f_{V,V'})\beta_{K(V)}(b_V) = 0$. Then, $\beta_{K(V')} T(f_{V,V'})(b_V) = 0$ so that there exists $a_{V'} \in SK(V')$ such that $T(f_{V,V'})(b_V) = \alpha_{K(V')}(a_{V'})$ and therefore

$$\alpha_X S(k_{V'})(a_{V'}) = T(k_{V'})\alpha_{K(V')}(a_{V'}) = T(k_{V'})T(f_{V,V'})(b_V)$$

$$= T(k_V)(b_V) = \alpha_X(a).$$

Since α_X is a monomorphism, $a = S(k_{V'})(a_{V'})$.

Now assume that $S(k_V)(a_V) = 0$, where $V \in \mathscr{V}$ and $a_V \in K(V)$. Then, $T(k_V)\alpha_{K(V)}(a_V) = \alpha_X S(k_V)(a_V) = 0$ so that there exists $V' \geq V$ such that $0 = T(f_{V,V'})\alpha_{K(V)}(a_V)$ $= \alpha_{K(V')} S(f_{V,V'})(a_V)$ and therefore, since $\alpha_{K(V')}$ is a monomorphism, $S(f_{V,V'})(a_V) = 0$.

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From this theorem one may immediately deduce that the functor Z of (2) is of type $L\Sigma*$. The dual of this theorem is then certainly not valid since it would imply that the functor Z' of (2) is of type $R\Pi*$ which is not true as is noticed in (2). This means that Theorem 4 cannot be generalized to abstract categories.

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