LINEAR ORTHOGONALITY PRESERVERS OF
HILBERT BUNDLES

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Abstract

A C-linear map $\theta$ (not necessarily bounded) between two Hilbert $C^*$-modules is said to be ‘orthogonality preserving’ if $\langle \theta(x), \theta(y) \rangle = 0$ whenever $\langle x, y \rangle = 0$. We prove that if $\theta$ is an orthogonality preserving map from a full Hilbert $C_0(\Omega)$-module $E$ into another Hilbert $C_0(\Omega)$-module $F$ that satisfies a weaker notion of $C_0(\Omega)$-linearity (called ‘localness’), then $\theta$ is bounded and there exists $\phi \in C_b(\Omega)_+$ such that $\langle \theta(x), \theta(y) \rangle = \phi \cdot \langle x, y \rangle$ for all $x, y \in E$.

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1. Introduction

It is common knowledge that the inner product of a Hilbert space determines both the norm and orthogonality; and conversely, the norm structure determines the inner product structure. It may be slightly less well known that the orthogonality structure of a Hilbert space also determines its norm structure. Indeed, if $\theta$ is a linear map between Hilbert spaces preserving orthogonality, then it is easy to see that $\theta$ is a scalar multiple of an isometry (see [5, 6]).

We are interested in the corresponding relations for Hilbert $C^*$-modules. Note that, in the case of a commutative $C^*$-algebra $C_0(\Omega)$, Hilbert $C_0(\Omega)$-modules are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces over $\Omega$. By modifying the proof of [12, Theorem 6] (see also [9, 13, 16]), one may show that any surjective isometry between two continuous fields of Hilbert spaces with nonzero fibers over each point is given by a homeomorphism and a field of unitary operators. Thus, the norm structure (and linearity) determines the unitary structure in this situation.
Our primary concern is the question of whether the orthogonality structure of a Hilbert $C^*$-module determines its unitary structure. More precisely, let $A$ be a $C^*$-algebra, and $E$ and $F$ be two Hilbert $A$-modules. If $\theta : E \to F$ is an $A$-module homomorphism, not necessarily bounded, which preserves orthogonality, that is, $\langle \theta(x), \theta(y) \rangle_A = 0$ whenever $\langle x, y \rangle_A = 0$, then we ask whether there is a central positive multiplier $u$ in $M(A)$ such that

$$\langle \theta(e), \theta(f) \rangle_A = u \langle e, f \rangle_A \quad \forall e, f \in E.$$ 

When $A = \mathbb{C}$, this reduces to the case of Hilbert spaces. Recently, Ilišević and Turnšek [10] gave a positive answer in the case where $A$ is a standard $C^*$-algebra, that is, when $K(H) \subseteq A \subseteq L(H)$.

In this paper, we will give a positive answer when $A$ is a commutative $C^*$-algebra (actually, we prove a slightly stronger result that replaces $A$-linearity with the ‘localness’ property; see Definition 2.1). On the other hand, we will also consider bijective biorthogonality preserving maps between Hilbert $C^*$-modules over different commutative $C^*$-algebras. We show that if such a map also satisfies a certain locality-type property (see Definition 3.12) but is not assumed to be bounded, then it is given by a homeomorphism (between the base spaces) and a ‘continuous field of unitary operators’. We remark that in this case of Hilbert $C^*$-modules over different commutative $C^*$-algebras, one cannot define ‘$A$-linearity’, but has to consider the localness property. This is one of the reasons for considering local maps. We remark also that this case does not cover the case of Hilbert $C^*$-modules over the same commutative $C^*$-algebra, because we need to assume that the map is both bijective and biorthogonality preserving.

Note that if $\Omega$ is a locally compact Hausdorff space and $H$ is a Hilbert space, then $C_0(\Omega, H)$ is a Hilbert $C_0(\Omega)$-module. As far as we know, even in this case our results are new, and the techniques in the proofs are nonstandard and nontrivial, compared to those in the literature [1, 4, 8, 11] on separating or zero-product preservers (although some statements look similar). In a forthcoming paper, the authors will study the case where the underlying $C^*$-algebra is not commutative.

2. Terminology and notation

Recall that a (right) Hilbert $C^*$-module $E$ over a $C^*$-algebra $A$ is a right $A$-module equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ such that the following conditions hold for all $x, y \in E$ and all $a \in A$:

(i) $\langle x, ya \rangle = \langle x, y \rangle a$;
(ii) $\langle x, y \rangle^* = \langle y, x \rangle$;
(iii) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ exactly when $x = 0$.

Moreover, $E$ is a Banach space equipped with the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. We also call $E$ a Hilbert $A$-module in this case. A complex linear map $\theta : E \to F$ between two Hilbert $A$-modules is called an $A$-module homomorphism if $\theta(xa) = \theta(x)a$.
for all \(a \in A\) and \(x \in E\). See, for example, [15] or [20] for a general introduction to the theory of Hilbert \(C^*\)-modules. In this paper, we are interested in the case where the underlying \(C^*\)-algebra \(A\) is abelian, that is, the space \(A = C_0(\Omega)\) of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space \(\Omega\).

**Definition 2.1.** Let \(A\) be a \(C^*\)-algebra. Suppose that \(E\) and \(F\) are Hilbert \(A\)-modules. A \(C\)-linear map \(\theta : E \to F\) is said to be **local** if \(\theta(e)a = 0\) whenever \(ea = 0\) for any \(e \in E\) and \(a \in A\).

The idea of local linear maps is often found in research in analysis. For example, a theorem of Peetre [19] states that local linear maps of the space of smooth functions defined on a manifold modeled on \(\mathbb{R}^n\) are exactly the linear differential operators (see [18]). This was extended to the case of vector-valued differentiable functions defined on a finite-dimensional manifold by Kantrowitz and Neumann [14] and Araujo [3], and to the Banach \(C^1[0,1]\)-module setting by Alaminos et al. [2]. Note that every \(A\)-module homomorphism is local. Conversely, every bounded local map is an \(A\)-module homomorphism (see [17, Proposition A.1]). See Remark 3.4 below for more information.

Throughout this paper, \(\Omega\) and \(\Delta\) are two locally compact Hausdorff spaces, and \(\Omega_\infty\) is the one-point compactification of \(\Omega\). Moreover, \(E\) and \(F\) are a (right) Hilbert \(C_0(\Omega)\)-module and a (right) Hilbert \(C_0(\Delta)\)-module respectively, while \(\theta : E \to F\) is a \(C\)-linear map (not assumed to be bounded). We denote by \(\mathcal{B}_{C_0(\Omega)}(E, F)\) the set of all bounded \(C_0(\Omega)\)-module homomorphisms from \(E\) into \(F\). For any \(\omega \in \Omega\), we let \(\mathcal{N}_{\Omega}(\omega)\) be the set of all compact neighborhoods of \(\omega\) in \(\Omega\). If \(S \subseteq \Omega\), we denote by \(\text{Int}_{\Omega}(S)\) the interior of \(S\) in \(\Omega\). Moreover, when \(U, V \subseteq \Omega\) and the closure of \(V\) is a compact subset of \(\text{Int}_{\Omega}(U)\), we denote by \(\mathcal{U}_{\Omega}(V, U)\) the collection of all functions \(\lambda \in C_0(\Omega)\) such that \(0 \leq \lambda \leq 1\), \(\lambda \equiv 1\) on \(V\) and \(\lambda\) vanishes outside \(U\).

Note that any Hilbert \(C_0(\Omega)\)-module \(E\) may be regarded as a Hilbert \(C(\Omega_\infty)\)-module, and the results in [7] may be applied. In particular, \(E\) is the space of \(C_0\)-sections (that is, continuous sections that vanish at infinity) of an \((F)\)-Hilbert bundle \(\mathcal{E}^E\) over \(\Omega_\infty\) (see [7, p. 49]).

We define the modulus function \(|f|(\omega) := \|f(\omega)\|\) for all \(f \in E\) and \(\omega \in \Omega\). For any closed subset \(S\) of \(\Omega_\infty\) and \(\omega \in \Omega_\infty\), we set

\[
K^E_S := \{ f \in E : f(\omega) = 0 \text{ for some } \omega \in S \} \quad \text{and} \quad I^E_\omega := \bigcup_{V \in \mathcal{N}_{\Omega_\infty}(\omega)} K^E_V
\]

(for simplicity, we also denote \(K^E_{(\omega)}\) by \(K^E_\omega\)). Note that \(K^E_\infty = E\) and the fiber \(\mathcal{E}^E_\omega\) of \(\mathcal{E}^E\) at \(\omega \in \Omega_\infty\) is \(E/K^E_\omega\). Furthermore, \(K^E_S\) is a Hilbert \(C_0(\Omega)\)-module and

\[
K^E_S = E \cdot K^C_0(\Omega).
\]

We also define

\[
\Delta^E_\theta := \{ v \in \Delta : \theta(E) \nsubseteq K^E_S \} = \{ v \in \Delta : \theta(e)(v) \neq 0 \text{ for some } e \in E \}.
\]
Then $\Delta_\theta$ is an open subset of $\Delta$ and we put

$$\Omega_E := \{\omega \in \Omega : \mathcal{E}_\omega^E \neq (0)\}.$$ 

Let $\Omega_0 \subseteq \Omega$ be an open set. As in [7, p. 10], we denote by $\mathcal{E}^E|_{\Omega_0}$ the restriction of $\mathcal{E}^E$ to $\Omega_0$ and by $E_{\Omega_0}$ the set of $C_0$-sections on $\mathcal{E}^E|_{\Omega_0}$. One may make the following identifications:

$$C_0(\Omega_0) = K_{\Omega \setminus \Omega_0}^{C_0(\Omega)} \quad \text{and} \quad E_{\Omega_0} = K_{\Omega \setminus \Omega_0}^E.$$ 

### 3. Orthogonality preserving maps between Hilbert $C_0(\Omega)$-modules

We first recall two technical lemmas from [17, Lemmas 3.1 and 3.3, and Theorem 3.7] (see also [17, Remark 3.4]), which summarize, unify, and generalize techniques used sporadically in the literature [4, 8, 11].

**Lemma 3.1.** If $\sigma : \Delta_0 \to \Omega_\infty$ is a map satisfying $\theta(I_{\sigma(v)}^E) \subseteq K_v^F$ for all $v \in \Delta_0$, then $\sigma$ is continuous.

**Lemma 3.2.** Let $\sigma : \Delta \to \Omega$ be a map (not necessarily continuous) with the property that $\theta(I_{\sigma(v)}^E) \subseteq K_v^F$ for every $v \in \Delta$.

(a) If $\Lambda_0 := \{v \in \Delta : \sup_{\|e\| \leq 1} \|\theta(e)(v)\| = \infty\}$, then $\sigma(\Lambda_0)$ is a finite set.

(b) If $\Lambda_{0,\sigma} := \{v \in \Delta : \theta(K_{\sigma(v)}^E) \subseteq K_v^F\}$, then $\Lambda_{0,\sigma} \subseteq \Lambda_0$ and $\sigma(\Lambda_{0,\sigma})$ consists of nonisolated points in $\Omega$.

(c) If $\sigma$ is injective and sends isolated points in $\Delta$ to isolated points in $\Omega$, then $\Lambda_{0,\sigma} = \emptyset$ and there exist a finite set $T$ consisting of isolated points of $\Delta$, a bounded linear map $\theta_0 : K_{E(T)}^E \to K_T^F$ as well as linear maps $\theta_v : \mathcal{E}_{\sigma(v)}^E \to \mathcal{E}_v^F$ for all $v \in T$, such that $E = K_{E(T)}^E \oplus \bigoplus_{v \in T} \mathcal{E}_{\sigma(v)}^F$,

$$F = K_T^F \oplus \bigoplus_{v \in T} \mathcal{E}_v^F \quad \text{and} \quad \theta = \theta_0 \oplus \bigoplus_{v \in T} \theta_v.$$ 

For any $v \in \Delta \setminus \Lambda_{0,\sigma}$, one may define $\theta_v : \mathcal{E}_{\sigma(v)}^E \to \mathcal{E}_v^F$ by

$$\theta_v(e + K_{\sigma(v)}^E) = \theta(e) + K_v^F \quad \forall e \in E,$$ 

or equivalently, $\theta_v(e(\sigma(v))) = (\theta(e))(v)$ for all $e \in E$.

**Lemma 3.3.** Let $\sigma$ and $\Lambda_0$ be as in Lemma 3.2. Suppose, in addition, that $\sigma$ is injective and $\theta$ is orthogonality preserving. Then there exists a bounded function $\psi : \Delta \setminus \Lambda_0 \to \mathbb{R}_+$ such that

$$\langle \theta(e), \theta(g) \rangle(v) = \psi(v)^2 \langle e, g \rangle(\sigma(v)) \quad \forall e, g \in E, \forall v \in \Delta \setminus \Lambda_0.$$ 

Moreover, for each $v \in \Delta_\theta$, there is an isometry $\iota_v : \mathcal{E}_{\sigma(v)}^E \to \mathcal{E}_v^F$ such that

$$\theta(e)(v) = \psi(v)\iota_v(e(\sigma(v))) \quad \forall e \in E, \forall v \in \Delta_\theta \setminus \Lambda_0.$$
Proving any \( \nu \in \Diamond \setminus \Upsilon \). By Lemma 3.2(b), the map \( \theta_{\nu} \), as in (3.1), is well defined. Suppose that \( \eta_1 \) and \( \eta_2 \) are orthogonal elements in \( \mathcal{E}_{\omega,\nu}^E \) and \( \eta_1 \neq 0 \) (this is possible because \( \Delta \setminus \Upsilon \subseteq \sigma^{-1}(\Omega) \)), and that \( g_1, g_2 \in E \) and \( g_i(\sigma(\nu)) = \eta_i \) when \( i = 1, 2 \). If \( \nu \in \mathcal{N}_\Omega(\sigma(\nu)) \) and \( g_1 \) does not vanish on \( \nu \), then by replacing \( g_2 \) with 

\[
\left( g_2 - \frac{\langle g_2, g_1 \rangle}{|g_1|^2} g_1 \right) \lambda,
\]

where \( \lambda \in \mathcal{U}_\Omega(\{\sigma(\nu)\}, V) \), we see that there are orthogonal elements \( e_1, e_2 \in E \) such that \( e_i(\sigma(\nu)) = \eta_i \) when \( i = 1, 2 \). Hence \( \theta_{\nu} \) is nonzero, because \( \nu \in \Delta \setminus \Upsilon \), and is an orthogonality preserving \( \mathbb{C} \)-linear map between Hilbert spaces. Consequently, there exist an isometry \( \iota_{\nu} : \mathcal{E}_{\sigma(\nu)}^E \rightarrow \mathcal{E}_{\nu}^F \) and a unique scalar \( \psi(\nu) > 0 \) such that \( \theta_{\nu} = \psi(\nu)\iota_{\nu} \). For any \( \nu \in \Delta \setminus \Upsilon \), we set \( \psi(\nu) = 0 \). Then clearly (3.2) holds. Next, we show that \( \psi \) is a bounded function on \( \Delta \setminus \Upsilon \). Suppose that this is not the case. Then there exist distinct points \( \nu_n \in \Delta \setminus \Upsilon \) such that \( \psi(\nu_n) > n^3 \). If \( e_n \in E \) such that \( \|e_n\| = 1 \) and its modulus function satisfies

\[
|e_n|(\sigma(\nu_n)) = \sqrt{\langle e_n, e_n \rangle(\sigma(\nu_n))} \geq (n - 1)/n
\]

(note that \( \nu_n \in \sigma^{-1}(\Omega_E) \)), then in light of (3.2),

\[
|\theta(e_n)(\nu_n) = \psi(\nu_n)|e_n|(\sigma(\nu_n)) > n^2(n - 1).
\]

As \( \{\sigma(\nu_n)\} \) is a set of distinct points (note that \( \sigma \) is injective), by passing to a subsequence if necessary, we may assume that there are \( U_n \in \mathcal{N}_\Omega(\sigma(\nu_n)) \) such that \( U_n \cap U_m = \emptyset \) when \( m \neq n \). Now pick any \( V_n \in \mathcal{N}_\Omega(\sigma(\nu_n)) \) such that \( V_n \subseteq \text{Int}_\Omega(U_n) \) and choose a function \( \lambda_n \in \mathcal{U}_\Omega(V_n, U_n) \) for all \( n \in \mathbb{N} \). Define \( e := \sum_{k=1}^\infty e_k \lambda_n^2/k^2 \in E \).

As \( n^2 e - e_n \lambda_n^2 \in K_{U_n}^E \) and \( e_n - e_n \lambda_n^2 = e_n(1 - \lambda_n^2) \in K_{V_n}^E \) for all \( n \in \mathbb{N} \),

\[
\|\theta(e)\| \geq \|\theta(e)(\nu_n)\| = \frac{\|\theta(e_n \lambda_n^2)(\nu_n)\|}{n^2} = \frac{\|\theta(e_n)(\nu_n)\|}{n^2} > n - 1,
\]

by the relation between \( \theta \) and \( \sigma \), which is a contradiction. \( \square \)

3.1. Hilbert bundles over the same base space.

Remark 3.4. For any \( e \in E \), we define

\[
\text{supp}_\Omega e := \{\omega \in \Omega : e(\omega) \neq 0\}.
\]

It is not hard to check that the following statements are equivalent (and this tells us that local maps are the same as support shrinking maps [8]):

(i) \( \theta \) is local (see Definition 2.1);
(ii) \( \theta(K_{U_n}^E) \subseteq K_{V_n}^E \) for all nonempty open set \( V \);
(iii) \( \text{supp}_\Omega \theta(e) \subseteq \text{supp}_\Omega e \) for all \( e \in E \);
(iv) \( \text{supp}_\Omega \theta(e) \lambda \subseteq \text{supp}_\Omega e \) for all \( e \in E \) and \( \lambda \in C_0(\Omega) \).

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**Theorem 3.5.** Let $\Omega$ be a locally compact Hausdorff space, and let $E$ and $F$ be two Hilbert $C_0(\Omega)$-modules. Suppose that $\theta : E \rightarrow F$ is an orthogonality preserving local $\mathbb{C}$-linear map. The following assertions hold.

(a) $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$.

(b) There is a bounded nonnegative function $\varphi$ on $\Omega$, continuous on $\Omega_E$, such that

$$\langle \theta(e), \theta(g) \rangle = \varphi \cdot \langle e, g \rangle \quad \forall e, g \in E.$$ 

(c) There exist a strictly positive element $\psi_0 \in C_b(\Omega_\theta)_+$ and $J \in \mathcal{B}_{C_0(\Omega)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ such that the fiber map $J_\omega$ is an isometry for all $\omega \in \Omega_\theta$ and

$$\theta(e)(\omega) = \psi_0(\omega) J(e)(\omega) \quad \forall e \in E, \forall \omega \in \Omega_\theta.$$  

**Proof.** Note that the conclusions of Lemmas 3.2 and 3.3 hold when $\Omega = \Delta$ and $\sigma = \text{Int}_\Omega$.

We prove (a). By Remark 3.4 and Lemma 3.2(c), $\theta$ is a $C_0(\Omega)$-module homomorphism. Further, as $\theta_\nu$ (as in Lemma 3.2(c)) is an orthogonality preserving, hence bounded, linear map between Hilbert spaces for all $\nu \in T$ (where $T$ is as in Lemma 3.2(c) and $\sigma = \text{Int}_\Omega$), we know from Lemma 3.2(c) that $\theta$ is bounded (note that $T$ is finite).

Now we consider (b). By part (a), $\Omega_\theta = \emptyset$. Thus, Lemma 3.3 tells us that there exists a bounded nonnegative function $\psi$ on $\Omega$ such that $\langle \theta(e), \theta(f) \rangle = |\psi|^2 \cdot \langle e, f \rangle$.

Let $\omega \in \Omega_E$ and pick any $e \in E$ for which there exists $U_\omega \in \mathcal{N}_\Omega(\omega)$ such that $e(v) \neq 0$ for all $v \in U_\omega$. Then $\psi(\omega) = |\theta(e)|^2(\omega)/|e|^2(\omega)$ for all $\omega \in U_\omega$. Hence $\psi$ is continuous on $\Omega_E$, and $\varphi = \psi^2$ is the required function.

It remains to prove (c). Note that $\Omega_\theta \subseteq \Omega_E$, by part (a). Since $\varphi(\omega) > 0$ for all $\omega \in \Omega_\theta$, we know from part (b) that $\psi = \varphi^{1/2}$ is a strictly positive element $\psi_0$ in $C_b(\Omega_\theta)_+$.

The equivalence in [7, (2.2)] (consider $E$ and $F$ as Hilbert $C(\Omega_\infty)$-bundles) tells us that the restriction of $\theta$ induces a bounded Banach bundle map, again denoted by $\theta$, from $\mathcal{S}^E|_{\Omega_\theta}$ into $\mathcal{S}^F|_{\Omega_\theta}$. For each $\eta \in \mathcal{E}^E|_{\Omega_\theta}$, we define $J(\eta) := \psi_0(\pi(\eta))^{-1} \theta(\eta)$, where $\pi : \mathcal{S}^E \rightarrow \Omega$ is the canonical projection. Then $J : \mathcal{E}^E|_{\Omega_\theta} \rightarrow \mathcal{S}^F|_{\Omega_\theta}$ is a Banach bundle map, as $\eta \mapsto \psi_0(\pi(\eta))^{-1}$ is continuous, which is an isometry on each fiber (hence $J$ is bounded) such that $\theta(\eta) = \psi(\pi(\eta)) J(\eta)$. This map $J$ induces a map, again denoted by $J$, in $\mathcal{B}_{C_0(\Omega)}(E_{\Omega_\theta}, F_{\Omega_\theta})$ that satisfies the requirement of part (c). \(\square\)

It is natural to ask if one can find $\varphi \in C_b(\Omega)$ such that the conclusion of Theorem 3.5(b) holds. Unfortunately, the following example tells us that this is not the case in general.

**Example 3.6.** Let $\Omega = \mathbb{R}_\infty$, the one-point compactification of the real line $\mathbb{R}$. Let $E$ and $F$ be the Hilbert $C(\Omega)$-module $C_0(\mathbb{R})$, and define $\theta(f)(t) = f(t) \cos t$ for all $f \in E$ and $t \in \mathbb{R}$. Then $\Omega \setminus \Omega_E = \{\infty\}$ and $\varphi(t) = \cos t$ for all $t \in \mathbb{R} = \Omega_E$. Thus $\varphi$ does not extend to a continuous function on $\Omega$.

We can now obtain the following commutative analog of [10, Proposition 2.3]. This, together with Corollary 3.11, asserts that the orthogonality structure of a Hilbert
bundle essentially determines its unitary structure, as we claimed in the introduction. Note also that a large portion of Lemma 3.2 was used to deal with the possibility of \( \theta(K^E_{\sigma(v)}) \nsubseteq K^F_v \) (this situation does not arise for \( C_0(\Omega) \)-module homomorphism), and this corollary actually has a much easier proof.

**Corollary 3.7.** Let \( \Omega \) be a locally compact Hausdorff space, and \( E \) and \( F \) be Hilbert \( C_0(\Omega) \)-modules. Suppose that \( \theta : E \rightarrow F \) is a \( C_0(\Omega) \)-module homomorphism that preserves orthogonality. Then \( \theta \) is bounded and there exists a bounded nonnegative function \( \varphi \) on \( \Omega \) that preserves orthogonality. Then this corollary actually has a much easier proof.

Recall that a Hilbert \( C_0(\Omega) \)-module \( E \) is *full* if the \( \mathbb{C} \)-linear span \( \langle E, E \rangle \) of the set \( \{ \langle e, f \rangle : e, f \in E \} \) is dense in \( C_0(\Omega) \).

**Remark 3.8.** A Hilbert \( C_0(\Omega) \)-module \( E \) is full if and only if \( E \nsubseteq K^E_\omega \) for all \( \omega \in \Omega \) (or equivalently, \( \Omega_E = \Omega \)). In fact, if \( E \subseteq K^E_\omega \), then \( f(\omega) = 0 \) for all \( f \in \langle E, E \rangle \) and \( E \) is not full. Conversely, if \( E \) is not full, then there exists \( \omega \in \Omega \) such that \( f(\omega) = 0 \) for all \( f \in \langle E, E \rangle \), because the closure of \( \langle E, E \rangle \) is an ideal of \( C_0(\Omega) \), and \( E \subseteq K^E_\omega \).

**Remark 3.9.** If \( E \) is full, then by the previous remark, the function \( \varphi \) in Theorem 3.5(b) (and Corollary 3.7) is an element of \( C_b(\Omega) \). However, there is no guarantee that this function is strictly positive.

**Remark 3.10.** Suppose that \( F \) is full and \( \theta \) is a surjective orthogonality preserving local \( \mathbb{C} \)-linear map. If there exists \( \omega \in \Omega \setminus \Omega_\theta \), then \( F = \theta(E) \subseteq K^F_\omega \), which contradicts the fullness of \( F \) (see Remark 3.8). Consequently, \( \Omega_\theta = \Omega \). As \( \theta \in \mathcal{B}_{C_0(\Omega)}(E, F) \) by Theorem 3.5(a), we see that \( \Omega = \Omega_\theta \subseteq \Omega_E \) and \( E \) is full.

**Corollary 3.11.** Let \( \Omega \) be a locally compact Hausdorff space, and let \( E \) and \( F \) be two Hilbert \( C_0(\Omega) \)-modules. Suppose that \( F \) is full and \( \theta : E \rightarrow F \) is an orthogonality preserving surjective local \( \mathbb{C} \)-linear map. Then \( \theta \in \mathcal{B}_{C_0(\Omega)}(E, F) \). Moreover, there exist a strictly positive element \( \psi \in C_b(\Omega)_+ \) and a unitary map \( U \in \mathcal{B}_{C_0(\Omega)}(E, F) \) such that \( \theta = \psi \cdot U \).

**Proof.** Remark 3.10 tells us that \( \Omega_\theta = \Omega \). By the surjectivity of \( \theta \), the bounded Banach bundle map \( J \) in Theorem 3.5 is unitary on each fiber. Therefore, the element \( U \in \mathcal{B}_{C_0(\Omega)}(E, F) \) corresponding to \( J \), as in [7, (2.2)], is unitary.

### 3.2. Hilbert bundles over different base spaces.

**Definition 3.12.** The map \( \theta \) is said to be *quasilocal* if it is bijective and, for all \( e \in E \) and \( \lambda \in C_0(\Delta) \),

\[
\text{supp}_\Omega \theta^{-1}(\theta(e)\lambda) \subseteq \text{supp}_\Omega e. \tag{3.3}
\]

Note that if \( \Delta = \Omega \) and \( \theta \) is both local and bijective (hence \( \theta^{-1} \) is also local), then \( \theta \) is quasilocal by Remark 3.4.
Lemma 3.13. Suppose that $\theta$ is bijective and quasilocal and that $\theta$ and $\theta^{-1}$ both preserve orthogonality. Then $|\theta(e)||\theta(g)| = 0$ if $e, g \in E$ and $\text{supp}_\Theta e \cap \text{supp}_\Theta g = \emptyset$.

Proof. Suppose, on the contrary, that there exist $e_1, e_2 \in E$ and $\nu \in \Delta$ such that $\text{supp}_\Theta e_1 \cap \text{supp}_\Theta e_2 = \emptyset$ but $\|\theta(e_1)(\nu)\|\|\theta(e_2)(\nu)\| \neq 0$. As $\theta$ preserves orthogonality, we may assume that $\theta(e_1)(\nu)$ and $\theta(e_2)(\nu)$ are orthogonal unit vectors in $\mathbb{Z}_\nu^F$. Take $U, W \in \mathcal{N}_\Delta(\nu)$ such that $W \subseteq \text{Int}_\Delta(U)$ and $\|\theta(e_i)(\mu)\| > 1/2$ for all $\mu \in U$. Pick any $\lambda \in \mathcal{U}_\Delta(W; U)$, and define $h_i \in F \setminus \{0\}$ (when $i = 1, 2$) by

$$h_i(\mu) := \begin{cases} \frac{\lambda(\mu)}{\|\theta(e_i)(\mu)\|} & \text{if } \mu \in \text{Int}_\Delta(U) \\ 0 & \text{if } \mu \notin \text{Int}_\Delta(U) \end{cases}$$

and set $e_i' := \theta^{-1}(h_i)$. The orthogonality of $h_1$ and $h_2$ (recall that $e_1$ and $e_2$ are orthogonal), together with that of $h_1 + h_2$ and $h_1 - h_2$ (as $|h_1| = |h_2|$), ensures the orthogonality of $e_1'$ and $e_2'$, as well as that of $e_1' + e_2'$ and $e_1' - e_2'$. It follows that $|e_1'| = |e_2'| \neq 0$, which contradicts the fact that $|e_1'||e_2'| = 0$, as $\theta$ is quasilocal. □

Theorem 3.14. Let $\Omega$ and $\Delta$ be locally compact Hausdorff spaces. Suppose that $E$ is a full Hilbert $C_0(\Omega)$-module and $F$ is a full Hilbert $C_0(\Delta)$-module. If $\theta : E \to F$ is a bijective $\mathbb{C}$-linear map such that both $\theta$ and $\theta^{-1}$ are quasilocal and orthogonality preserving, then $\theta$ is bounded and

$$\theta(e)(\nu) = \psi(\nu)J_\nu(e(\sigma(\nu))) \quad \forall e \in E, \forall \nu \in \Delta,$$

(3.4)

where $\sigma : \Delta \to \Omega$ is a homeomorphism, $\psi$ is a strictly positive element of $C_b(\Delta)_+$, and $J_\nu$ is a unitary operator from $\mathbb{Z}_\sigma(\nu)$ onto $\mathbb{Z}_\nu^F$ such that the map $\nu \mapsto J_\nu(f(\sigma(\nu)))$ is continuous for all fixed $f \in E$.

Proof. We consider $E$ as a Hilbert $C(\Omega_\infty)$-module. For each $\nu \in \Delta$, let

$$S_\nu := \{\omega \in \Omega_\infty : \theta(K_{\Omega_\infty \setminus W}^E) \nsubseteq K_v^F \forall W \in \mathcal{N}_\Omega(\omega)\}.$$

We first show that $S_\nu$ is a singleton. Indeed, assume that $S_\nu = \emptyset$. Then for all $\omega \in \Omega_\infty$, there is $W_\omega \in \mathcal{N}_\Omega(\omega)$ such that $\theta(K_{\Omega_\infty \setminus W_\omega}^E) \subseteq K_v^F$. Consider $\omega_1, \ldots, \omega_n \in \Omega_\infty$ such that

$$\bigcup_{k=1}^n \text{Int}_{\Omega_\infty}(W_{\omega_k}) = \Omega_\infty,$$

and take a partition of unity $\{\varphi_k\}_{k=1}^n$ that is subordinate to $\{\text{Int}_{\Omega_\infty}(W_{\omega_k})\}_{k=1}^n$. Then $e\varphi_k \in K_{\Omega_\infty \setminus W_{\omega_k}}^E$ for all $e \in E$, and so $\theta(e) \in K_v^F$. As $\theta$ is surjective, this shows that $F = K_v^F$, and contradicts the fullness of $F$ (see Remark 3.8). Now, assume that there are distinct elements $\omega_1, \omega_2 \in S_\nu$. Take $V_1 \in \mathcal{N}_\Omega(\omega_1)$ and $V_2 \in \mathcal{N}_\Omega(\omega_2)$ such that $V_1 \cap V_2 = \emptyset$. By the definition of $S_\nu$, there exist $e_1, e_2 \in E$ such that $\text{supp}_\Theta e_i \subseteq V_i \setminus \{\infty\}$ and $\theta(e_i)(\nu) \neq 0$ when $i = 1, 2$, which contradicts Lemma 3.13.
Thus, there is a unique element $\sigma(v) \in \Omega_{\infty}$ such that $S_v = \{\sigma(v)\}$. Next, we claim that

$$\theta(I_{E(v)}^E) \subseteq I_v^F \quad \forall v \in \Delta.$$  \hspace{1cm} (3.5)

Consider any $V \in N_{\Omega_{\infty}}(\sigma(v))$ and $e \in K_{V}^E$. Pick $U \in N_{\Omega_{\infty}}(\sigma(v))$ such that $U \subseteq \text{Int}_{\Omega_{\infty}}(V)$. By the definition of $\sigma$, there exists $g \in K_{\Omega_{\infty}}^E \setminus U$ such that $\theta(g)(v) \neq 0$. Hence, there is $W \in N_{\Delta}(v)$ such that $\theta(g)(\mu) \neq 0$ for all $\mu \in W$, and Lemma 3.13 implies that $\theta(e) \in K_{W}^E$, as claimed. If there exists $v \in \Delta \setminus \Delta_\theta$, then $f(v) = 0$ for all $f \in F$, because $\theta$ is surjective, which contradicts the fullness of $F$. Thus, $\Delta_\theta = \Delta$ and $\sigma : \Delta \to \Omega_{\infty}$ is continuous, by Lemma 3.1. As $\theta^{-1}$ is also quasilocal and orthogonality preserving, a similar argument to the above gives a continuous map $\tau : \Omega \to \Delta_{\infty}$ satisfying $\theta^{-1}(I_{\tau(\omega)}^F) \subseteq I_{\omega}^F$ for all $\omega \in \Omega$. Now, the argument of [17, Theorem 5.3] tells us that $\sigma$ is a homeomorphism from $\Delta$ to $\Omega$ such that

$$\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma \quad \forall e \in E, \forall \varphi \in C_0(\Omega),$$

and by Lemma 3.2(c), there exists a finite set $T$ consisting of isolated points of $\Delta$ such that $\theta$ restricts to a bounded map from $K_{\sigma(T)}^E$ to $K_T^F$. Since any $v \in T$ is an isolated point, $\theta$ induces an orthogonality preserving, hence bounded, map $\theta_v$ from the Hilbert space $\mathcal{E}_{\sigma(v)}^E$ onto the Hilbert space $\mathcal{E}_v^F$. This shows that $\theta$ is bounded, by Lemma 3.2(c) and the fact that $T$ is finite. By Lemma 3.3, there is a surjective isometry $J_v : \mathcal{E}_{\sigma(v)}^E \to \mathcal{E}_v^F$ such that

$$\theta(e)(v) = J_v(e(\sigma(v))) \quad \forall e \in E, \forall v \in \Delta.$$

Now the fullness of $E$ implies that $\psi(v) > 0$ for all $v \in \Delta$, and the map $v \mapsto \theta(e)(v)/\psi(v)$ is evidently continuous. \hfill $\square$

The following example shows the necessity of the assumption in Theorem 3.14 that $\theta^{-1}$ preserves orthogonality.

**Example 3.15.** Let $\Omega$ be a nonempty locally compact Hausdorff space, $\Omega_2$ be the topological disjoint sum of two copies of $\Omega$, and $j_1, j_2 : \Omega \to \Omega_2$ be the embeddings into the first and the second copies of $\Omega$ in $\Omega_2$, respectively. Let $H$ be a nonzero Hilbert space, and let $H_2$ be the Hilbert space direct sum of two copies of $H$. Then the map $\theta : C_0(\Omega_2, H) \to C_0(\Omega, H_2)$, defined by

$$\theta(f)(\omega) = (f(j_1(\omega)), f(j_2(\omega))),$$

is a bijective $\mathbb{C}$-linear map preserving orthogonality satisfying condition (3.3). However, $\theta$ is not of the expected form. Note that $\theta^{-1}$ does not preserve orthogonality.

**References**


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