Rational Surfaces with Many Nodes

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Abstract. We describe smooth rational projective algebraic surfaces over an algebraically closed field of characteristic different from 2 which contain $n \geqslant b_2 - 2$ disjoint smooth rational curves with self-intersection -2, where b_2 is the second Betti number. In the last section this is applied to the study of minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 8, 9$ which admit an automorphism of order 2.

1. Introduction

Let X be a smooth rational projective algebraic surface over an algebraically closed field **k** of characteristic $\neq 2$. It is known that for any nodal curve C on X there exists a birational morphism $f\colon X \to X'$ such that the image of C is an ordinary double point (a *node*). Let $n(X)$ be the maximal number of disjoint nodal curves on X. After blowing down all of them we obtain a rational normal surface X' with $n(X)$ nodes. The Picard number $\rho(X')$ of X' is equal to the Picard number $\rho(X)$ of X minus $n(X)$. Since X' is projective, $\rho(X') = \rho(X) - n(X) \geqslant 1$. In this paper we study the limit cases, namely $\rho(X') = 1$ or 2. More precisely, we prove that $\rho(X') = 1$ is possible only if X' is isomorphic to a quadric cone and we describe all the X''s such that $\rho(X') = 2$.

The question of the number of nodes on an algebraic surface is a very old one and has a long history, but, to our knowledge, this particular problem has not been considered. Our interest in this question arose in the course of investigating complex surfaces of general type with $p_g = 0$ admitting a double plane construction, and in the last section of this paper, working over $\mathbb{C}$, we give an application to such surfaces with $K^2 = 8, 9$. More precisely, we extend some of the results of the previous sections for surfaces with $p_g = q = 0$ and nonnegative Kodaira dimension and then we

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consider surfaces S of general type with $p_g = 0$ with an involution σ. We show that if $K^2 = 9$, then S does not admit an involution σ and we list all the possibilities for the quotient surface $S/\langle\sigma\rangle$ if $K^2 = 8$.

One of our main tools is the code associated to a set of nodal curves (see Section 2), which has already been considered by A. Beauville in [B].

1.1. NOTATIONS AND CONVENTIONS

As already explained, we work over any algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$ in Sections 2 and 3, whilst in Section 4 we work over $\mathbb{C}$.

The multiplicative group of $\mathbf{k}$ is denoted by $\mathbb{G}_m$. For any Abelian group A we denote by $_2A$ the kernel of the homomorphism [2]: $A \to A$, $a \mapsto 2a$.

All varieties are projective algebraic. We do not distinguish between line bundles and divisors on a smooth variety, and use additive and multiplicative notation interchangeably. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$. The intersection product of divisors (line bundles) A and B is denoted by $A \cdot B$. We denote by $\kappa(X)$ the Kodaira dimension of a variety X and by $\rho(X)$ the Picard number of X, that is the rank of the Néron–Severi group of X. A *nodal* curve on a surface is a smooth rational curve C such that $C^2 = -2$. The remaining notation is standard in algebraic geometry.

2. Nodal Curves, Binary Codes and Covers

In this section all varieties are defined over an algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$.

Recall that a binary code is a subspace V of a k-dimensional vector space W over $\mathbb{F}_2$ equipped with a basis $(e_1, \ldots, e_k)$. We denote by $x^1 \ldots x^k$ the coordinates of $\mathbb{F}_2^k$ with respect to the chosen basis. The dimension of W (identified in the sequel with $\mathbb{F}_2^k$) is called the *length* of V. For each $v \in V$ the number of nonzero coordinates of v with respect to the basis is called the *weight* of v.

Two codes $V_1, V_2 \subset \mathbb{F}_2^k$ are *isomorphic* if there exists a permutation of the coordinates of $\mathbb{F}_2^k$ mapping V_1 onto V_2.

We say that a code V is *reduced* if there is no $1 \leqslant i \leqslant k$ such that $V \subset \{x^i = 0\}$. To every code V one can associate a reduced code V', simply by deleting the 'useless' coordinates. The dimension and the weights of V and V' are the same, while V' has (possibly) smaller length. We say that two codes $V_1 \subset \mathbb{F}_2^{k_1}$, $V_2 \subset \mathbb{F}_2^{k_2}$ are *essentially isomorphic* if the corresponding reduced codes are isomorphic.

We mention here a code that plays an important part in what follows. Given an integer n, consider the code of even vectors $V = \left\{\sum x^i = 0\right\} \subset \mathbb{F}_2^n$. V has dimension $n - 1$ and its weights are all even. We define the *code of doubly even vectors $DE(n)$* to be the image of V via the injection $\mathbb{F}_2^n \to \mathbb{F}_2^{2n}$ defined by $(x^1 \ldots x^n) \mapsto (x^1 x^1 \ldots x^n x^n)$. So $DE(n)$ has length $2n$, dimension $n - 1$ and all its weights are divisible by 4.

It is possible to associate to a linear code $V \subset \mathbb{F}_2^n$ a lattice Γ_V in the Euclidean space $\mathbb{R}^n$ (see, for example, [Eb]). One considers the canonical homomorphism $p \colon \mathbb{Z}^n \to \mathbb{F}_2^n$ and takes Γ_V to be $\frac{1}{\sqrt{2}} p^{-1}(V)$. For example, the code V of even vectors in $\mathbb{F}_2^n$ defines the root lattice of type D_n. The code of doubly even vectors $DE(n)$ defines the root lattice D_{2n} (loc. cit., p. 25).

Binary codes arise naturally in the theory of algebraic surfaces, as follows. Consider a smooth projective surface Y and k disjoint nodal curves $C_1, \ldots, C_k$ of Y. Let $\mathcal{C}$ be the subgroup of $\mathrm{Pic}(Y)$ generated by the curves C_i, which is a free Abelian group of rank k. Let $\varphi \colon \mathcal{C}/2\mathcal{C} \to \mathrm{Pic}(Y)/2\mathrm{Pic}(Y)$ be the natural homomorphism of $2-$elementary Abelian groups. We call the kernel V of φ the (*binary*) *code associated to the* C_i and denote its dimension by r. Here we take for a basis of $W := \mathcal{C}/2\mathcal{C}$ the classes of the curves C_i modulo $2\mathcal{C}$.

We say that a curve C_i *appears in* V if V is not contained in the subspace $\{x^i = 0\}$ of $\mathbb{F}_2^k$ and we denote by m the number of C_i that appear in V (so m is the length of the reduced code associated to V). The vector $v = (x^1 \ldots x^k) \in \mathbb{F}_2^k$ is in V if and only if there exists $L_v \in \mathrm{Pic}(Y)$ such that $2L_v \equiv \sum x^i C_i$ (when it is convenient, we identify $0, 1 \in \mathbb{F}_2$ with the integers $0, 1$). Notice that $K_Y \cdot L_v = 0$ and thus L_v^2 is even by the adjunction formula. Then the weight $w(v)$ of v is equal to $-2L_v^2$ and so it is divisible by 4. Notice that L_v is uniquely determined by v if and only if $_2\mathrm{Pic}(Y) = 0$.

The following result is analogous to the construction of the Galois cover of a surface Y associated to a torsion subgroup of $\mathrm{Pic}(Y)$.

PROPOSITION 2.1. *Let Y be a smooth projective surface with $_2\mathrm{Pic}(Y) = 0$, let $C_1 \ldots C_k$ be disjoint nodal curves of Y and let V, L_v be defined as above. Then there exists a unique smooth connected Galois cover $\pi \colon Z \to Y$ such that:*

(i) *the Galois group of π is $G := \mathrm{Hom}(V, \mathbb{G}_m)$;*
(ii) *the branch locus of π is the union of the C_i that appear in V;*
(iii) *$\pi_* \mathcal{O}_Z = \bigoplus_{v \in V} L_v^{-1}$, and G acts on L_v^{-1} via the character $v \in V \cong \mathrm{Hom}(G, \mathbb{G}_m)$.*

Proof. For $v \in V$ and $g \in G$, we define $\epsilon_v(g) \in \{0, 1\}$ by $(-1)^{\epsilon_v(g)} = g(v)$. We fix a basis $v_1 \ldots v_r$ of V and we write ϵ_j for ϵ_{v_j}, $j = 1 \ldots r$.

By Proposition 2.1 of [Pa], in order to determine $\pi \colon Z \to Y$ we have to assign the (reduced) *building data*, namely:

(1) for every nonzero $g \in G = \mathrm{Hom}(V, \mathbb{G}_m)$ an effective divisor D_g;
(2) for every $j = 1 \ldots r$ a line bundle M_j

in such a way that the following relations are satisfied:

$$2M_j \equiv \sum_{g \in G} \epsilon_j(g) D_g, \quad j = 1 \ldots r. \tag{2.1}$$

For $i = 1 \ldots k$ we denote by $\psi_i \colon W = \mathbb{F}_2^k \to \mathbb{G}_m$ the homomorphism defined by $(x^1 \ldots x^k) \to (-1)^{x^i}$. We define D_g to be the sum of the C_s such that $\psi_s|_V = g$. Notice that the D_g are disjoint and that $D := \sum_g D_g$ is the union of the C_i that appear in V. If we write $v_j = (x_j^1 \ldots x_j^k)$, and we identify again $0, 1 \in \mathbb{F}_2$ with the integers $0, 1$, then it is not difficult to check that relations (2.1) can be rewritten as:

$$2M_j \equiv \sum_i x_j^i C_i, \quad j = 1 \ldots r. \tag{2.2}$$

So Equation (2.1) can be solved uniquely by setting $M_j = L_{v_j}, j = 1 \ldots r$. The corresponding cover $\pi \colon Z \to Y$ satisfies conditions (i) and (ii) of the statement. In addition, Z is smooth by Proposition 3.1 of [Pa], since D is smooth, and it is connected since the set of $g \in G$ such that $D_g \neq 0$ generates G. In order to complete the proof we have to check that for every $v = (x_v^1 \ldots x_v^k) \in V$ the eigensheaf M_v^{-1} of $\pi_* \mathcal{O}_Z$ on which G acts via the character v is L_v^{-1}. By Theorem 2.1 of [Pa], we have $2M_v \equiv \sum_g \epsilon_v(g) D_g$. This equation can be rewritten as $2M_v \equiv \sum_i x_v^i C_i$, and thus $2L_v = 2M_v$ in $\mathrm{Pic}(Y)$. The equality $L_v = M_v$ follows since $_2\mathrm{Pic}(Y) = 0$. $\square$

Remark 1. Write $U := Y \setminus \cup_i C_i$. Then there is an isomorphism $\psi \colon V \to {}_2\mathrm{Pic}(U)$ and the restriction to U of the cover $\pi \colon Z \to Y$ is the G-torsor corresponding to ψ under the natural map $H^1(U, G) \to \mathrm{Hom}(V, \mathrm{Pic}(U))$.

Remark 2. The proof of Proposition 2.1 shows that if one removes the assumption $_2\mathrm{Pic}(Y) = 0$ then the cover $\pi \colon Z \to Y$ exists but it is not determined uniquely. Also, if one assumes $\mathrm{char}(\mathbf{k}) = 2$, then the proof shows the existence of a purely inseparable cover with a G-action.

Let $\eta \colon Y \to \Sigma$ be the map that contracts the curves C_i that appear in V to singular points of type A_1. The inverse image in Z of a curve C_i that appears in V is a disjoint union of 2^{r-1} (-1)-curves. Blowing down all these (-1)-curves, we obtain a smooth surface $\bar{Z}$ and a G-cover $\bar{\pi} \colon \bar{Z} \to \Sigma$ branched precisely over the singularities of Σ. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
Z & \longrightarrow & \bar{Z} \\
\pi \downarrow & & \bar{\pi} \downarrow \\
Y & \xrightarrow{\;\eta\;} & \Sigma,
\end{array}
$$

We close this section by computing the invariants of Z and $\bar{Z}$.

LEMMA 2.2. *With the same assumptions and notations as in Proposition* 2.1 *(in particular, r is the dimension of V and m is the number of the C_i that appears in V) one has*

$$c_2(Z) = 2^r c_2(Y) - m 2^r.$$

Proof. If the base field is $\mathbb{C}$, then the formula follows easily by topological considerations. We give an algebraic proof, valid for fields of characteristic $\neq 2$.

Denote by D the branch divisor of π (which is the union of m disjoint nodal curves), and by $R = \pi^{-1}D$ the ramification divisor. Consider the following exact sequence of sheaves on Z:

$$0 \to \pi^*\Omega_Y^1 \xrightarrow{j} \Omega_Z^1 \to \mathcal{K} \to 0, \tag{2.3}$$

where the cokernel $\mathcal{K}$ is a torsion sheaf supported on R. Consider a ramification point $P \in Z$ and let R' be the irreducible component of R containing P. The subgroup $H \subset G$ consisting of the elements that induce the identity on R' is isomorphic to $\mathbb{Z}_2$ (cf. [Pa], Lemma 1.1). The surface $W := Z/H$ is smooth, since the fixed locus of H is purely one-dimensional, and π factorizes as $Z \xrightarrow{\alpha} W \xrightarrow{\beta} Y$. Let $Q = \alpha(P)$ and $D' = \alpha(R')$. The map β is étale in a neighbourhood of Q, and thus $\beta^*\Omega_Y^1 \hookrightarrow \Omega_W^1$ is an isomorphism locally near Q. It follows that the inclusion $\pi^*\Omega_Y^1 \hookrightarrow \alpha^*\Omega_W^1$ is an isomorphism locally around P. There exists an open neighbourhood U of Q in W such that $Z|_U$ is defined in $U \times \mathbb{A}^1$ by the equation $z^2 = b$, where b is a local equation for D' and z is the affine coordinate in $\mathbb{A}^1$. Notice that z is a local equation for R' around P. Let x be a function on W such that x, b are local parameters on W around Q. Then the map j of sequence (2.3) can be written locally as $(dx, db) \mapsto (dx, 2zdz)$. It follows that the cokernel $\mathcal{K}$ is naturally isomorphic to the conormal sheaf of R, $\mathcal{O}_R(-R)$. A standard computation with Chern classes gives:

$$\begin{aligned} c_2(Z) &= 2^r c_2(Y) + 2R^2 + \pi^*K_Y \cdot R = 2^r c_2(Y) + 2^{r-1}D^2 + 2^{r-1}K_Y \cdot D \\ &= 2^r c_2(Y) - m2^r. \end{aligned}$$ $\qquad\square$

PROPOSITION 2.3. *Under the same assumptions and notation as above the following holds:*

$$\kappa(Z) = \kappa(\bar{Z}) = \kappa(Y);$$
$$K_Z^2 = 2^r K_Y^2 - m2^{r-1} \quad K_{\bar{Z}}^2 = 2^r K_Y^2;$$
$$\chi(Z, \mathcal{O}_Z) = \chi(\bar{Z}, \mathcal{O}_{\bar{Z}}) = 2^r \chi(\mathcal{O}_Y) - m2^{r-3}.$$

Proof. We have $K_{\bar{Z}} = \bar{\pi}^*K_\Sigma$, since $\bar{\pi}$ is unramified in codimension 1 and Σ is normal, and therefore $K_{\bar{Z}}^2 = 2^r K_\Sigma^2 = 2^r K_Y^2$. The formula for K_Z^2 follows immediately. Since χ is a birational invariant, it is enough to compute it for Z. Then the formula for $\chi(Z, \mathcal{O}_Z)$ follows from Lemma 2.2 and Noether's formula.

If $\kappa(Z) = -\infty$, then we have $\kappa(Y) = -\infty$ (π is separable since the characteristic of $\mathbf{k}$ is $\neq 2$). So assume that $\kappa(Z) \geqslant 0$ and denote by $\tilde{Z}$ the minimal model of Z and $\bar{Z}$. Then G acts biregularly on $\tilde{Z}$. We denote by $\tilde{\pi}\colon \tilde{Z} \to \tilde{\Sigma} := \tilde{Z}/G$ the quotient map. The surface $\tilde{\Sigma}$ has canonical singularities and it is birational to Y and Σ. Denote by $\epsilon\colon \tilde{Y} \to \tilde{\Sigma}$ the minimal resolution. We have $K_{\tilde{Z}} = \tilde{\pi}^*K_{\tilde{\Sigma}}$ and thus $K_{\tilde{\Sigma}}$ and

$K_{\tilde{Y}} = \epsilon^* K_{\tilde{\Sigma}}$ are nef. So $\tilde{Y}$ is minimal and, in addition, $K_{\tilde{Y}} \sim 0$ iff $K_{\tilde{Z}} \sim 0$ and $K_{\tilde{Y}}^2 = 0$ iff $K_{\tilde{Z}}^2 = 0$. This remark shows that $\kappa(\tilde{Y}) = \kappa(\tilde{Z})$. $\qquad\square$

3. Rational Surfaces with Many Nodes

Throughout this section we assume that Y is a smooth rational surface and $C_1, \ldots, C_k$ are disjoint nodal curves of Y. As before, we let V be the code associated to the C_i, r its dimension and m the number of the C_i that appear in V. The group $\mathrm{Pic}(Y)$ is free Abelian of rank $\rho(Y) = 10 - K_Y^2$ and the intersection form on $\mathrm{Pic}(Y)$ induces a nondegenerate $\mathbb{F}_2$-valued bilinear form on $\mathrm{Pic}(Y)/2\mathrm{Pic}(Y)$. Since $C_i^2 = -2$ and the C_i are disjoint, the image of $\mathcal{C}/2\mathcal{C}$ is a totally isotropic subspace of $\mathrm{Pic}(Y)/2\,\mathrm{Pic}(Y)$. Thus the dimension r of V satisfies $r \geqslant k - [\rho(Y)/2]$. As a corollary of the results in the previous section we have the following

LEMMA 3.1. *If $r \geqslant 4$, then $m \geqslant 8$.*

  Proof. Consider the cover $\pi\colon Z \to Y$ of Proposition 2.1 associated to V and the corresponding cover of Σ, $\bar{\pi}\colon \bar{Z} \to \Sigma$. By Proposition 2.3, $\bar{Z}$ is ruled and thus $\chi(\bar{Z}) \leqslant 1$. The result follows by using the formula for $\chi(Z, \mathcal{O}_Z)$ of Proposition 2.3. $\qquad\square$

THEOREM 3.2. *Let $C_1 \ldots C_k$ be disjoint nodal curves on a rational surface Y, let V be the code associated to $C_1 \ldots C_k$ and assume that the length of the reduced code V' of V is $m \geqslant 8$. Denote by $\eta\colon Y \to \Sigma$ the map that contracts to nodes the C_i that appear in V. Then there exists a fibration $\beta : \Sigma \to \mathbb{P}^1$ such that*

 (i) *the general fibre of β is $\mathbb{P}^1$;*
 (ii) *$m = 2n$ is even and β has n double fibres, each containing two nodes of Σ;*
(iii) *the code V is essentially isomorphic to $DE(n)$.*

  Proof. Let $\pi\colon Z \to Y$ be the cover of Proposition 2.1 and let $\bar{\pi}\colon \bar{Z} \to \Sigma$ be the corresponding cover of Σ. By Proposition 2.3, the surface $\bar{Z}$ is ruled and has irregularity $q(\bar{Z}) = 1 + m2^{r-3} - 2^r > 0$. Denote by $\alpha\colon \bar{Z} \to C$ the Albanese pencil. By the canonicity of the Albanese map, the group G preserves the divisor class of a fibre. Consider the canonical homomorphism $G \to \mathrm{Aut}(C)$: if it is not injective, then there exists $g \in G$ that maps each fibre of α to itself. Hence a general fibre, being isomorphic to $\mathbb{P}^1$, has 2 fixed points of g and the ramification locus for the action of G has components of dimension 1, a contradiction since the G-cover is branched precisely over the singularities of Σ. Thus we have a commutative diagram:

$$
\begin{array}{ccc}
\bar{Z} & \xrightarrow{\bar{\pi}} & \Sigma \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\
C & \xrightarrow{p} & \mathbb{P}^1,
\end{array}
$$

where $p\colon C \to \mathbb{P}^1$ is a G-cover. The general fibre of β is $\mathbb{P}^1$, since it is isomorphic to the general fibre of α. Since the genus of C is equal to $q(\bar{Z})$, by the Hurwitz formula the branch locus of p consists of $n = m/2$ points (the inverse image of a branch point consists of 2^{r-1} simple ramification points). The cover $\bar{\pi}\colon \bar{Z} \to \Sigma$ is obtained from p by base change and normalization, thus the fibres of β over the branch points $y_1 \ldots y_n$ of p are of the form $f_i = 2\delta_i$, $i = 1 \ldots n$, and $\cup_i \delta_i$ contains all the nodes of Σ. We claim that each double fibre contains at least one node. Indeed, otherwise δ_i would be contained in the smooth part of Σ and so it would be a Cartier divisor with $\delta_i^2 = 0$, $K_\Sigma \cdot \delta_i = -1$, a contradiction to the adjunction formula.

Set $\beta' = \beta \circ \eta$. Then for every i, one can write $\beta'^* y_i = 2A_i + \sum_s C_{i,s}$ and it follows that for every choice of $h \neq j$ the divisor $\sum_s C_{h,s} + \sum_t C_{j,t}$ is divisible by 2 in $\mathrm{Pic}(Y)$, namely it corresponds to a vector of V. Since the weights of V are all divisible by 4, it follows easily that each δ_i contains precisely 2 nodes of Σ. So it is possible to relabel the C_i in such a way that $\beta'(C_{2j-1}) = \beta'(C_{2j}) = y_j$ for $j = 1 \ldots n$, and that $C_{2j} + C_{2j-1} + C_{2h} + C_{2h-1}$ is divisible by 2 in $\mathrm{Pic}(Y)$ for every choice of j, h. This shows that V is essentially isomorphic to the code $DE(n)$. $\qquad\square$

Next we apply the above results to describe rational surfaces with 'many' disjoint nodal curves. We start by describing an example.

EXAMPLE 1. Consider a relatively minimal ruled rational surface $\mathbf{F}_e := \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$, $e \geqslant 0$, and a point $y \in \mathbf{F}_e$. If one blows up y, then the total transform of the ruling of $\mathbf{F}_e$ containing y is the union of two (-1)-curves E and E' that intersect transversely in a point y_1. If one blows up also y_1, then the strict transforms of E and E' are disjoint nodal curves. By repeating this procedure n times at points lying on different rulings of $\mathbf{F}_e$, one obtains a rational surface Y containing $2n$ disjoint nodal curves. One has $\rho(Y) = 2n + 2$ and it is easy to check that the code V associated to this collection of curves is $DE(n)$. We will call Y the *standard example* of a rational surface with $\rho(Y) - 2$ disjoint nodal curves.

THEOREM 3.3. *Let Y be a smooth rational surface and let $C_1 \ldots C_k$ be disjoint nodal curves of Y. Then:*

(i) $k \leqslant \rho(Y) - 1$, *and equality holds if and only if* $Y = \mathbf{F}_2$;
(ii) *if* $k = \rho(Y) - 2$ *and* $\rho(Y) \geqslant 5$, *then* Y *is the standard example. In particular* $k = 2n$ *is even and the code* V *is* $DE(n)$.

Proof. The group $\mathrm{Pic}(Y)$ is free Abelian of rank $\rho := \rho(Y)$. The intersection form on $\mathrm{Pic}(Y)$ extends to a nondegenerate bilinear form of signature $(1, \rho(Y) - 1)$ on $N^1(Y) := \mathrm{Pic}(Y) \otimes \mathbb{R}$. The subspace of $N^1(Y)$ spanned by the classes of the C_i has dimension k and the intersection form is negative definite there, thus we get $k < \rho$.

We start by proving (ii). As before, we let $m \leqslant k$ be the number of nodal curves that appear in the code V. Recall that the dimension r of V is $\geqslant \rho - 2 - [\rho/2] = [(\rho+1)/2] - 2$. So, for $\rho \geqslant 11$, we have $r \geqslant 4$ and thus $m \geqslant 8$ by Lemma 3.1.

Assuming then that $\rho \geqslant 11$, we can apply Theorem 3.2. Thus m is even, say $m = 2n$, and there exists a morphism $\beta : \Sigma \to \mathbb{P}^1$ such that the general fibre of β is $\mathbb{P}^1$ and β has n double fibres, occurring at points $y_1 \ldots y_n$ of $\mathbb{P}^1$. Each double fibre contains precisely 2 nodes of Σ, and the code V is $DE(n)$. So we have $n - 1 = \dim V \geqslant [(\rho + 1)/2] - 2$, namely $\rho - 2 \geqslant m = 2n \geqslant 2[(\rho + 1)/2] - 2$. It follows that ρ is even and $\rho - 2 = m = 2n$. In particular, $m = k$, i.e. all the C_i appear in V.

Set $\beta' = \beta \circ \eta$, denote by F the cohomology class on Y of a fibre of β' and let

$$T = \{L \in N^1(Y) : L \cdot F = 0\}.$$

A basis of T is given by F and the classes of $C_1 \ldots C_{2n}$, since these are independent classes and $\dim T = \rho - 1 = 2n + 1$. On the other hand, it is well known that, if one removes a component from each reducible fibre of β', then F and the classes of the remaining components of the reducible fibres are independent. It follows that the $F_i := \beta'^* y_i$, $i = 1, \ldots, n$ are the only reducible fibres of Y. As in the proof of Theorem 3.2, it is possible to relabel the C_i in such a way that for each i one has $F_i = \lambda_i C_{2i-1} + \mu_i C_{2i} + 2\nu_i D_i$, with D_i irreducible and such that $D_i^2 < 0$. From $K_Y \cdot F = -2$, we get $\nu_i = 1$, $K_Y \cdot D_i = -1$, and thus $D_i^2 = -1$, namely D_i is a (-1)-curve. The curve D_i has nonempty intersection with both C_{2i-1} and C_{2i}, since F_i is connected. So the equality

$$0 = D_i \cdot F = D_i \cdot (2D_i + \lambda_i C_{2i-1} + \mu_i C_{2i}) = -2 + \lambda_i D_i \cdot C_{2i-1} + \mu_i D_i \cdot C_{2i}$$

gives:

$$\lambda_i = \mu_i = D_i \cdot C_{2i-1} = D_i \cdot C_{2i} = 1.$$

Blowing down $D_1 \ldots D_n$ one obtains a smooth surface ruled over $\mathbb{P}^1$ with precisely n reducible rulings, each consisting of two (-1)-curves intersecting transversely. Blowing down a (-1)-curve of each ruling, we obtain a ruled surface $\mathbf{F}_e$. So Y is the standard example.

In order to complete the proof of (ii), we have to describe the cases $5 \leqslant \rho \leqslant 10$. In addition we may assume $m < 8$, since for $m = 8$ (and $\rho = 10$) one can apply the argument above to show that Y is the standard example. Since $m < 8$ all the elements of V have weight 4 and it is easy to check that the only (numerical) possibilities for the pair (k, r) are: $(4, 1)$, $(6, 2)$, $(7, 3)$ and $(8, 3)$. One has $m = k$ in all cases but the last one, where $m = 7$.

Consider the first three cases. Let $Z \to Y$ be the Galois cover considered in Proposition 2.1 and $\bar{Z} \to \Sigma$ the corresponding cover of Σ. By Proposition 2.3, $\bar{Z}$ is a surface satisfying $\kappa(\bar{Z}) = \kappa(Y)$, $K_{\bar{Z}}^2 = 8$, $\chi(\bar{Z}) = 1$. So $\bar{Z}$ is rational and $K_{\bar{Z}}^2 = 8$ implies that $\bar{Z} = \mathbf{F}_e$ for some $e \geqslant 0$. Denote by t the trace of $g \in G \setminus \{1\}$ on the l-adic cohomology $H^2(\bar{Z}, \mathbb{Q}_l) \cong \mathbb{Q}_l^2$. Since the class in $H^2(\bar{Z}, \mathbb{Q}_l)$ of the canonical bundle of Y is G-invariant, t is either equal to 0 or 2. Applying the (l-adic) Lefschetz fixed point formula (see [SGA5], (4.11.3), cf. the next section for the analogous statement for the complex cohomology) we see that $t = 0$ is impossible and, hence, g acts identically on $H^2(\bar{Z}, \mathbb{Q}_l)$. In particular, given the ruling (or a ruling if $e = 0$) $f : \bar{Z} \to \mathbb{P}^1$ the

action of the Galois group G of $\bar{Z} \to \Sigma$ descends to an action on $\mathbb{P}^1$ and there is an induced fibration $h \colon \Sigma \to \mathbb{P}^1/G = \mathbb{P}^1$. The same argument as in the proof of Theorem 3.2 shows that the action of G on $\mathbb{P}^1$ is faithful. Thus each element g of G fixes precisely two fibres of h, each containing two fixed points of g. Since $\mathrm{Aut}(\mathbb{P}^1)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2^3$, we can rule out immediately the case $(n, r) = (7, 3)$. In the remaining two cases, the cover $\mathbb{P}^1 \to \mathbb{P}^1/G$ is branched over $n = m/2$ points, and over each of these points h has an irreducible double fibre containing 2 nodes. It follows easily that Y is the standard example.

Finally consider the case $(8, 3)$, $m = 7$ (the code is essentially isomorphic to the Hamming code defined by the root lattice of type E_7). By Proposition 2.3 the G-cover $\bar{Z} \to \Sigma$ is a smooth ruled surface with invariants $K_{\bar{Z}}^2 = 0$, $\chi(\bar{Z}) = 1$. Thus $\bar{Z}$ is rational. The preimage of the nodal curve of Y not appearing in V is a set $D_1 \ldots D_8$ of disjoint nodal curves on which $G = \mathbb{Z}_2^3$ acts transitively. The code $\tilde{V} \subset \mathbb{F}_2^8$ associated to $D_1 \ldots D_8$ is acted on by G, and therefore all the nodes appear in $\tilde{V}$, namely $\tilde{V}$ has $m = 8$. Thus $\bar{Z}$ is a standard example with $\rho = 10$. If there is only one pencil with rational fibres $f \colon \bar{Z} \to \mathbb{P}^1$ such that the D_i are contracted by f, then one argues as in case $(7, 3)$ and obtains a contradiction by showing the existence of a $\mathbb{Z}_2^3$-cover $\mathbb{P}^1 \to \mathbb{P}^1$. So assume that there are two pencils with rational fibres $f_j \colon \bar{Z} \to \mathbb{P}^1$, $j = 1, 2$ such that the D_i are contracted both by f_1 and f_2. Denote by F_j, $j = 1, 2$, the class in $N^1(\bar{Z}) := \mathrm{Pic}(\bar{Z}) \otimes \mathbb{R}$ of a smooth fibre of f_j. Considering the intersection form, one sees immediately that the classes of $F_1, F_2, D_1 \ldots D_8$ are a basis of $N^1(\bar{Z})$. Consider a nonzero $g \in G$. The surface $Z' := \bar{Z}/\langle g \rangle$ is a rational surface with s singular points of type A_1, that are the images of the fixed points of g on $\bar{Z}$. By the standard double cover formulas:

$$1 = \chi(\bar{Z}) = 2\chi(Z') - s/4 = 2 - s/4$$

and so $s = 4$. Denote by t the trace of g on the l-adic cohomology $H^2(\bar{Z}, \mathbb{Q}_l)$. Applying again the Lefschetz fixed point formula we get $t = 2$. The action of g on $H^2(\bar{Z}, \mathbb{Q}_l)$ preserves the subspace $\langle D_1 \ldots D_8 \rangle$ generated by the fundamental classes of the divisors $D_1 \ldots D_8$, and thus it preserves also its orthogonal subspace, which is spanned by the classes of F_1, F_2. The trace of g on $\langle D_1 \ldots D_8 \rangle$ is zero. It follows that g is the identity on $\langle F_1, F_2 \rangle$, namely every $g \in G$ preserves both pencils. Thus we can apply again, the argument above to one of the pencils and the proof of (ii) is complete.

Finally we prove (i). Assume that $k = \rho(Y) - 1$. The code V has length $\rho - 1$, dimension $r \geqslant [(\rho + 1)/2] - 1$ and all the weights divisible by 4. Thus if $\rho \geqslant 9$, then $m \geqslant 8$ by Lemma 3.1 and one can argue as in case (ii) and show that Y is the surface constructed in the standard example and V is essentially isomorphic to $DE(n)$, with $n = \rho/2 - 1$. In particular, $r = n - 1 = \rho/2 - 2$, contradicting $r \geqslant [(\rho + 1)/2] - 1$. So assume $\rho \leqslant 8$. If $\rho = 2$, then $K_Y^2 = 8$ and so Y is the minimal ruled surface $\mathbf{F}_2$. If $\rho > 2$, the only numerical possibility is $\rho = 8$, $r = 3$. Let Y be a surface corresponding to this possibility. We have $K_Y^2 = 2$. Up to a permutation, we may assume that $C_1 \ldots C_4$ is an even set. The corresponding double cover $Y' \to Y$ is a smooth

rational surface (same proof as Proposition 2.3), with $K^2_{Y'} = 0$. The inverse images of $C_1 \ldots C_4$ are (-1)-curves, while the inverse images of C_5, C_6, C_7 are three pairs of disjoint nodal curves. Blowing down the (-1)-curves, one obtains a rational surface Y'' with $\rho(Y'') = 6$ and containing 6 disjoint nodal curves. This is impossible, and the proof is complete.   □

Remark 3. For a rational surface Y with $\rho(Y) \leqslant 4$ containing $k = \rho(Y) - 2$ disjoint nodal curves, the code V is zero and one cannot argue as in Theorem 3.3. On the other hand, this case can be studied directly and it is easy to check that the possibilities for (k, ρ) are:

 (i) $(0, 2)$ and Y is a surface $\mathbf{F}_e$, $e \neq 2$.
(ii) $(1, 3)$ and Y the blowup of $\mathbf{F}_2$ at a point outside the negative section (the nodal curve is the pull back of the negative section); or Y is the blowup of $\mathbf{F}_1$ at a point on the negative section (the nodal curve is the strict transform of the negative section);
(iii) $(2, 4)$ and Y is the standard example with $k = 2$; or Y is the blowup of $\mathbf{F}_2$ at points x_1, x_2, with x_1 not on the negative section and x_2 infinitely near to x_1 (the nodal curves are the pullback of the negative section and the strict transform of the exceptional curve of the first blowup).

4. An Application

Throughout all of this section the ground field is $\mathbb{C}$ and a 'surface' is a smooth projective complex surface.

We apply the previous results to study involutions (i.e. automorphisms of order 2) on minimal surfaces of general type with $p_g = 0$ and $K^2_S = 8$ or 9.

We start by extending the results of Section 3 to complex surfaces with $p_g = q = 0$ and nonnegative Kodaira dimension. The use of Miyaoka's formula is a key ingredient for the proof below and explains the assumption that the ground field is $\mathbb{C}$ in this section.

PROPOSITION 4.1. *Let Y be a surface with $p_g(Y) = q(Y) = 0$ and $\kappa(Y) \geqslant 0$, and let $C_1 \ldots C_k \subset Y$ be disjoint nodal curves. Then:*

 (i) *$k \leqslant \rho(Y) - 2$;*
(ii) *if $k = \rho(Y) - 2$, then Y is minimal.*

Proof. Assume first that Y is minimal. In this case we can apply Miyaoka's formula ([M], § 2): $3c_2(Y) - K^2_Y \geqslant (9/2)k$, and (i) follows immediately using $0 \leqslant K^2_Y \leqslant 9$ and Noether's formula.

Now assume that Y is not minimal and let $\bar{Y}$ be the minimal model of Y. We use induction on $v := \rho(Y) - \rho(\bar{Y})$. Let $E \subset Y$ be an irreducible (-1)-curve and let Y' be

the surface obtained by blowing down E. If E does not intersect any of the C_i, then Y' contains k disjoint nodal curves and induction gives: $k \leqslant \rho(Y') - 2 = \rho(Y) - 3$. So assume, say, $C_1 \cdot E = \alpha > 0$. Then the image C_1' of C_1 in Y' is an irreducible curve such that $(C_1')^2 = -2 + \alpha^2$, $C_1' \cdot K_{Y'} = -\alpha$. Now necessarily $\alpha = 1$. In fact suppose that $\alpha \geqslant 2$. Then $C_1'^2 > 0$ and, therefore, the image of C_1' in the minimal model $\bar{Y}$ of Y is a curve C_1''. Since $C_1'' \cdot K_{\bar{Y}} \leqslant C_1' \cdot K_{Y'} < 0$ and $K_{\bar{Y}}$ is nef because $\kappa(Y') \geqslant 0$, we have a contradiction. Therefore C_1' is a (-1)-curve. In addition, $E \cdot C_i = 0$ for $i > 1$, since otherwise Y' would contain a pair of irreducible (-1)-curves with non-empty intersection, which is impossible again because $\kappa(Y') \geqslant 0$. Now blowing down C_1' we obtain a surface Y'' containing a set of $k - 1$ disjoint irreducible nodal curves. Using induction again, we have $k - 1 \leqslant \rho(Y'') - 2 = \rho(Y) - 4$ and the proof is complete. $\qquad\square$

Let S be a surface admitting an involution σ. Let k be the number of isolated fixed points of σ and let D be the 1-dimensional part of the fixed-point locus. The divisor D is smooth (possibly empty). If we consider the blow-up X of the set of isolated fixed points, then the involution σ lifts to an involution on X (which we still denote by σ) and the quotient $Y := X/\langle\sigma\rangle$ has k disjoint nodal curves C_i.

We recall the following two well-known formulas:

(Holomorphic Fixed Point Formula) (see [AS], p. 566):

$$\sum_{i=0}^{2} (-1)^i \text{Trace}(\sigma | H^i(S, \mathcal{O}_S)) = \frac{k - D \cdot K_S}{4}$$

(Topological Fixed Point Formula) (see [Gr], (30.9)):

$$\sum_{i=0}^{4} (-1)^i \text{Trace}(\sigma | H^i(S, \mathbb{C})) = k + e(D),$$

where $e(D) = -D^2 - D \cdot K_S$ is the topological Euler characteristic of D.

LEMMA 4.2. *Let S be a surface with $p_g(S) = q(S) = 0$ and let σ be an automorphism of S of order 2. Let D be the divisorial part of the fixed locus of σ, let k be the number of isolated fixed points of σ and let t be the trace of $\sigma | H^2(S, \mathbb{C})$. Then: $k = K_S \cdot D + 4$; $t = 2 - D^2$.*

Furthermore, if X is the blow-up of the k isolated fixed points of σ, and $Y = X/\langle\sigma\rangle$ one has $\rho(S) + t = 2\rho(Y) - 2k$.

Proof. The first fixed point formula gives

$$k = 4 + K_S \cdot D \tag{4.1}$$

Together with the second formula we obtain

$$t := \text{Trace}(\sigma | H^2(S, \mathbb{C})) = 2 - D^2. \tag{4.2}$$

For the last part notice that we have

$$e(S) + k = e(X) = 2e(Y) - 2k - e(D).$$

Since by the topological fixed point formula $e(D) = -k + 2 + t$, one has $e(S) + t + 2 = 2e(Y) - 2k$.

Now $p_g = q = 0$ implies $e(S) = \rho(S) + 2$, $e(Y) = \rho(Y) + 2$ and we obtain

$$\rho(S) + t = 2\rho(Y) - 2k \tag{4.3}$$

$\square$

THEOREM 4.3. *A surface of general type S with $p_g(S) = 0$ and $K_S^2 = 9$ has no automorphism of order* 2.

Proof. Assume otherwise. Since $\rho(S) = 1$, we have $t = 1$. Lemma 4.2 gives $D^2 = 1$. Since the canonical class is invariant for σ, we have $K_S \sim rD$ for some $r \in \mathbb{Q}$. Then $K_S^2 = 9$ yields $K_S \sim 3D$ and $K_S \cdot D = 3$. Thus Lemma 4.2 gives $k = 7$ and $2 = 2\rho(Y) - 14$, i.e. $\rho(Y) = 8$. So Y contains $\rho(Y) - 1$ disjoint nodal curves and $K_Y^2 = 2$. This is a contradiction in view of Theorem 3.3 and Proposition 4.1. $\square$

THEOREM 4.4. *Let S be a minimal surface of general type with $p_g(S) = 0$, $K_S^2 = 8$ and let σ be an automorphism of S of order 2. Let D be the divisorial part of the fixed locus of σ, let k be the number of isolated fixed points of σ and let Y be a minimal resolution of the quotient $S/\langle\sigma\rangle$. Then: $D^2 = 0$, $K_S \cdot D = k - 4$ and one of the following cases occurs:*

(i) $k = 4$, $D = 0$ *and Y is a minimal surface of general type with $p_g(Y) = 0$ and $K_Y^2 = 4$.*

(ii) $k = 6$, *and Y is a minimal surface of general type with $p_g(Y) = 0$ and $K_Y^2 = 2$.*

(iii) $k = 8$, *Y is a minimal surface with $p_g(Y) = q(Y) = 0$, $\kappa(Y) = 1$ for which the elliptic fibration $Y \to \mathbb{P}^1$ has two reducible fibres of Kodaira type I_0^*, and as such constant moduli.*

(iv) $k = 10$, *and Y is a rational surface from Example 1 with $\rho = 12$. The fibration with connected rational fibres $f \colon Y \to \mathbb{P}^1$ pulls back on S to a pencil of hyperelliptic curves of genus 5.*

(v) $k = 12$, *and Y is a rational surface from Example 1 with $\rho = 14$. The fibration with connected rational fibres $f \colon Y \to \mathbb{P}^1$ pulls back on S to a pencil of hyperelliptic curves of genus 3.*

Proof. Since $\rho(S) = 2$, the possible values for the trace t are 0 and 2.

The case $t = 0$ does not occur. Indeed, assume otherwise. By Lemma 4.2, $D^2 = 2$ so that $D \neq 0$. Since $t = 0$, the invariant part of $H^2(S, \mathbb{Q})$ is one-dimensional and thus (because the canonical class is invariant for σ), $K_S \sim rD$ for some $r \in \mathbb{Q}$. Thus $K_S \sim 2D$ and, hence, $K_S \cdot D = 4$. Lemma 4.2 gives $k = 8$ and $\rho(Y) = 9$, and so by Noether's formula $K_Y^2 = 1$. Since Y contains 8 disjoint nodal curves, we have a contradiction to Theorem 3.3 and Proposition 4.1. So $t \neq 0$.

Now we consider the case $t = 2$, that is, the involution σ acts identically on $H^2(S, \mathbb{Q})$. In this case $D^2 = 0$.

If $D = 0$, we get $k = 4$ and the surface Y is a surface of general type with $K_Y^2 = 4$ and $\rho(Y) = 6$. It contains an even set of four disjoint nodal curves $C_1, \ldots, C_4$ and thus it is minimal by Proposition 4.1. This is case (i).

The last case to consider is $t = 2$ and $D \neq 0$. Since $D^2 = 0$, we have $K_S \cdot D = 2m$, with $m > 0$. Then Lemma 4.2 gives $k = 4 + 2m$, so that in particular k is $\geqslant 6$ and even, and $\rho(Y) = 6 + 2m = k + 2$.

Assume that $\kappa(Y) \geqslant 0$. Since Y is a minimal surface by Proposition 4.1, $K_Y^2 \geqslant 0$ and so $k = 8 - K_Y^2 \leqslant 8$. So either $k = 6$ or $k = 8$. If $k = 6$, $K_Y^2 = 2$ and so Y is of general type and we have case (ii). If $k = 8$, then $K_Y^2 = 0$ and thus Y, being minimal, is not of general type. Since $p_g(Y) = q(Y) = 0$, Y is either an Enriques surface or a surface of Kodaira dimension 1. The first case cannot occur. In fact since $K_Y \sim 0$ and $D^2 = 0$ we would have $K_S \cdot D = 0$, a contradiction. So $\kappa(Y) = 1$ and Y is a minimal properly elliptic surface. Denote by $f \colon Y \to \mathbb{P}^1$ the elliptic fibration and let F be a general fibre of f. Since K_Y is numerically a rational multiple of F, we have $F \cdot C_i = 0$ for every i, namely the C_i are mapped to points by f. Let $\bar{F}$ be a fibre containing, say, $C_1 \ldots C_s$ and let $A_1 \ldots A_p$ be the remaining irreducible components of $\bar{F}$. It is well known that the classes of $A_1 \ldots A_p, C_1 \ldots C_s$ in $H^2(Y, \mathbb{Q})$ are independent and span a subspace U_1 on which the intersection form is seminegative. The classes of $C_{s+1} \ldots C_8$ are also independent and span a subspace U_2 such that the intersection form is negative on U_2 and $U_1 \cap U_2 = \{0\}$. Since $\rho(Y) = 10$, we see that the only possibility is $p = 1$. Looking at Kodaira's list of singular elliptic fibres (see e.g. [BPV], pg.150), one sees that the possible types of singular fibres containing some of the C_i are I_2, I_0^* and III. In addition, we have $12 = e(Y) = \sum_t e(F_t)$, where F_t is the fibre of f over the point $t \in \mathbb{P}^1$ and e denotes the topological Euler–Poincaré characteristic. It is easy to check that the only numerical possibility is that f has two I_0^* fibres, each containing 4 of the C_i, and that every other singular fibre is a multiple of a smooth elliptic curve. Up to a permutation we may assume that the I_0^* fibres of f are $C_1 + \cdots + C_4 + 2D_1$ and $C_5 + \cdots + C_8 + 2D_2$. So $C_1 + \cdots + C_8 \equiv 2(F - D_1 - D_2)$ is divisible by 2 in $\mathrm{Pic}(Y)$. Let $\pi \colon Y' \to Y$ be the corresponding double cover. For a general fibre F of f, $\pi^* F$ is disconnected and the Stein factorization of $f \circ \pi$ gives rise to an elliptic fibration $f' \colon Y' \to \mathbb{P}^1$ 'with the same fibres' as f. The inverse images of D_1, D_2 are smooth elliptic curves. The inverse images of $C_1, \ldots, C_8$ are 8 (-1)-curves contained in the fibres of f'. Blowing these exceptional curves down, one obtains an elliptic fibration $f'' \colon Y'' \to \mathbb{P}^1$ whose only singular fibres are multiples of smooth elliptic fibres. Thus f'' has constant moduli, and therefore f' and f have constant moduli too. This is case (iii).

Finally, assume that Y is a rational surface. Since $k \geqslant 6$ and $\rho(Y) = k + 2$ we can apply Theorem 3.3 to obtain that Y is as in the standard example. In particular there is a fibration $f \colon Y \to \mathbb{P}^1$ with general fibre F isomorphic to $\mathbb{P}^1$. If we write $K_S \cdot D = 2m$ (hence $k = 2m + 4$), then f has precisely $m + 2$ singular fibres of the form $C_{2i-1} + C_{2i} + 2E_i$, with E_i a (-1)-curve and $E_i \cdot C_{2i-1} = E_i \cdot C_{2i} = 1$. Denote

by $\bar{D}$ the image of D on Y and by L the line bundle of Y such that $2L \equiv \bar{D} + C_1 + \cdots + C_k$. The intersection number $E_i \cdot \bar{D} = E_i \cdot 2L - E_i \cdot (C_1 + \cdots + C_k) = 2L \cdot E_i - 2$ is even. Thus we may write $\bar{D} \cdot F = \bar{D} \cdot (2E_i + C_{2i-1} + C_{2i}) = 2\bar{D} \cdot E_i = 4d$, and the pre-image in X of the ruling on Y is a pencil of hyperelliptic curves of genus $2d-1$. Blowing down the curves E_i and then the images of the C_{2i}, we obtain a birational morphism $p \colon Y \to \mathbf{F}_e$ onto a relatively minimal ruled surface. Let C be the image of $\bar{D}$ on $\mathbf{F}_e$. Let F, S be the standard generators of $\mathrm{Pic}(\mathbf{F}_e)$ with $F^2 = 0, S^2 = -e \leqslant 0, F \cdot S = 1$. We have $C \sim aF + 4dS$. The curve C has $m+2$ singular points of type $(2d, 2d)$, that are solved by the morphism p. Since $\bar{D}^2 = D^2/2 = 0$, we get

$$0 = C^2 - (m+2)8d^2 = 8d(a - 2de - d(m+2)).$$

This gives us a first equation:

$$a = d(m+2) + 2de. \tag{4.4}$$

We also know that $\bar{D} \cdot K_Y = 2m$. On the other hand,

$$\bar{D} \cdot K_Y = C \cdot K_{\mathbf{F}_e} + 2(m+2)2d,$$

and we get the second equation

$$a = 2d(m+e) - m. \tag{4.5}$$

Comparing the two equations, we get $dm = m + 2d$. This has the solutions $(m, d) = (3, 3), (4, 2)$, which yield the cases (iv) and (v), respectively. $\qquad\square$

Remark 4. We do not know whether all the possibilities in Theorem 4.4 really occur. One can check that in the case of the bicanonical involution of the surface S of example (4.2) of [MP] the quotient is as in case (v). In addition, $\mathrm{Aut}(S) = \mathbb{Z}_2^3$ and the remaining involutions are as in case (iii). Example (4.3) of [MP] has a group Γ of automorphisms isomorphic to $\mathbb{Z}_2^4$: some elements of Γ have no one-dimensional fixed part, and thus are as in case (i), while the others are as in case (iii). Both examples are Beauville-type surfaces (cf [BPV], p. 236).

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