VECTOR LAYPUNOV FUNCTIONALS AND STABILITIES AND CHAOTICITIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS OCCURRING IN CELESTIAL MECHANICS AND STELLAR DYNAMICS

D.R.K. Sastry
Research Centre for Advanced Studies, Kondapur, Opp: Central University, Hyderabad - 500133, India

ABSTRACT

Vector Laypunov functions and stabilities and chaoticities of Functional Differential Equations occurring in Celestial Mechanics and Stellar Dynamics have been discussed. Fuzzy dynamical systems are more realistic while considering problems occurring in Celestial Mechanics and Stellary Dynamics.

DISCUSSION

LaSalle used Vector Laypunov Functions and the Principle of invariance to have geometric pictures of the regions of Stabilities in [4] which in turn are extended to Difference Equations in [14] and difference equations are exploited to obtain qualitative results in Chaos, and stabilities with rich applications [5 (i), 5 (ii), 13]. All the researchers in Celestial Mechanics and Stellary Dynamics, while investigating several complicated problems, could consider differential equations, ignoring the time delay equations which are equally important because the origin and evolution of the solar system along with the other planets and the stars depend on their past history also. We feel the Vector Laypunov Functionals could be conveniently used even to deal with the Chaoticities besides Stabilities, instabilities and conditional stabilities. Computational aspects has not been considered although it is equally important. These results can easily be extended to functional differential system $z'(t) = F(z_t, \lambda t)$ where $t$ is the time, $z_t(\theta) = z(t+\theta)$, $-\tau \leq \theta \leq 0$, $\lambda t(\theta) = \lambda(t+\theta)$, $-\tau \leq \theta \leq 0$ $\lambda t$ is a fast oscillating functional parameter which is more general than the fast oscillating parameter considered by...
Richard Bellman and his collaborators in [15]. Analogous to
the system (3.01) in [1]. We consider \((m+n)\) dimensional multi-
frequency oscillatory functional differential

\[
\begin{align*}
\frac{dx}{dt} &= \lambda f(x_t, y_t) \\
\frac{dy}{dt} &= \omega(x_t) + \lambda g(x_t, y_t)
\end{align*}
\]

with \(n\) fast and \(m\) slow variables, \(x_t = x(t+\Theta), \ -r \leq \Theta < 0, \ x_t, f \in C_1([-r,0], \mathbb{R}^m), \ y_t, \omega, g \in C_2([-r,0], \mathbb{R}^n). \ C_1, C_2 \) and \(C([-r,0], \mathbb{R}^{m+n}) = C\) are spaces of continuous functions \(f\) and \(g\) are defined and \(2\pi\)-periodic with \(\Theta\) respect to the second variable \(y_t\). System (1) can be reduced to the multifrequency oscillatory system of ordinary differential equations of the

\[
\begin{align*}
\frac{d^2 x_1}{dt^2} + \frac{G(m_0 + m_1)x_1}{r_1^3} &= Gm_2 \left( \frac{x_2 - x_1}{\Delta_{12}^3} - \frac{x_1}{r_1} \right) \\
\frac{d^2 y_1}{dt^2} + \frac{G(m_0 + m_1)y_1}{r_1^3} &= Gm_2 \left( \frac{y_2 - y_1}{\Delta_{12}^3} - \frac{y_1}{r_1} \right) \\
\frac{d^2 z_1}{dt^2} + \frac{G(m_0 + m_1)z_1}{r_1^3} &= Gm_2 \left( \frac{z_2 - z_1}{\Delta_{12}^3} - \frac{z_1}{r_1} \right) \\
\frac{d^2 x_2}{dt^2} + \frac{G(m_0 + m_2)x_2}{r_2^3} &= Gm_1 \left( \frac{x_1 - x_2}{\Delta_{12}^3} - \frac{x_2}{r_2} \right) \\
\frac{d^2 y_2}{dt^2} + \frac{G(m_0 + m_2)y_2}{r_2^3} &= Gm_1 \left( \frac{y_1 - y_2}{\Delta_{12}^3} - \frac{y_2}{r_2} \right)
\end{align*}
\]

System (1) also deals with the three-body problem dealing with the study of the motion of each of
the three objects \(P_0, P_1\) and \(P_2\) having arbitrary masses \(m_0, m_1\)
and \(m_2\) respectively and attracting one another in accordance
with the Newton's Law of gravitation and the three-body problem can be interpreted as a two-planet problem where in the
bodies with the masses \(m_1\) and \(m_2\) are the two planets and the
body with the mass \(m_0\) is the SUN. And the differential equa-
tions of motion of the planets will be of the form

\[
\begin{align*}
\frac{d^2 x_1}{dt^2} + \frac{G(m_0 + m_1)x_1}{r_1^3} &= Gm_2 \left( \frac{x_2 - x_1}{\Delta_{12}^3} - \frac{x_1}{r_1} \right) \\
\frac{d^2 y_1}{dt^2} + \frac{G(m_0 + m_1)y_1}{r_1^3} &= Gm_2 \left( \frac{y_2 - y_1}{\Delta_{12}^3} - \frac{y_1}{r_1} \right) \\
\frac{d^2 z_1}{dt^2} + \frac{G(m_0 + m_1)z_1}{r_1^3} &= Gm_2 \left( \frac{z_2 - z_1}{\Delta_{12}^3} - \frac{z_1}{r_1} \right) \\
\frac{d^2 x_2}{dt^2} + \frac{G(m_0 + m_2)x_2}{r_2^3} &= Gm_1 \left( \frac{x_1 - x_2}{\Delta_{12}^3} - \frac{x_2}{r_2} \right) \\
\frac{d^2 y_2}{dt^2} + \frac{G(m_0 + m_2)y_2}{r_2^3} &= Gm_1 \left( \frac{y_1 - y_2}{\Delta_{12}^3} - \frac{y_2}{r_2} \right)
\end{align*}
\]
\[
\frac{d^2 z_2}{dt^2} + \frac{G(m_0 + m_2)z_2}{r_2^3} = Gm_1 \left( \frac{z_1 - z_2}{\Delta_{12}^3} - \frac{z_2}{r_2^3} \right)
\]  

(2)

where \((x_1, y_1, z_1)\) are the rectangular coordinates of the planet \(P_1\) and \((x_2, y_2, z_2)\) are the rectangular coordinates of the planet \(P_2\) with the \text{SUN} as the origin, \(G\) is the gravitational constant and

\[
\Delta_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2
\]

\[
r_i^2 = x_i^2 + y_i^2 + z_i^2, \quad i = 1, 2.
\]

Let \(z = (x, y)\), \(\phi = (\phi_1, \phi_2)\), \(F(z_t, \lambda) = F(x_t, y_t, \lambda) = \)

\[
= \begin{pmatrix}
\lambda f(x_t, y_t) \\
\omega(x_t) + \lambda g(x_t, y_t)
\end{pmatrix}
\]

Functional differential equation (1) can now be written as

\[
z'(t) = F(z_t, \lambda) 
\]

(3)

where \(F: C([-r, 0], \mathbb{R}^{m+n}) \times (0, \infty) \rightarrow C([-r, 0], \mathbb{R}^{m+n})\) with the initial function \(z(t) = \phi(t), -r < t < 0\). Consider Vector Laypunov functional \(V(z_t): C_3(C, \mathbb{R}^p)\) and as usual its derivative is taken along the solutions of (3)

\[
V'(z_t) = \lim \inf_{t \to 0^+} \frac{V(z_t(\phi) - V(\phi)}
\]

Define \(E = \{z_t: V'(z_t) = 0, z_t \in C([-r, 0], \mathbb{R}^{m+n})\}\) and let \(M\) be the largest invariant set in \(E, \) that is \(M\) is the union of all solutions of (3) defined on \(R = (-\infty, 0)\) that remains for all \(t\) in \(E\). Consider a set

\[
V^{-1}(c) = \{z_t: V(z_t) = c, z_t \in C\}
\]
Theorem: Let $V(z_t)$ be a Laypunov functional of (3) defined on $C$ satisfying the above conditions. If $z_t(\phi)$ is a solution of (3) that is compact for all $t \geq 0$, then there is a constant vector $c \in \mathbb{R}^{m+n}$ such that $z_t(\phi) \rightarrow M \cap V^{-1}(c)$ as $t \rightarrow \infty$.

Proof of this Theorem is analogous to that of Theorem 3.1 in [4].

By using this Theorem we can have a picture of the regions of stability and also instability for system (3).

Even for Difference Equations Laypunov Functions have been exploited to obtain similar results in [14]. These difference equations are used to get results of qualitative nature in Chaos, besides stabilities in [5(i), 5(ii),13].

Remarks: (1) Instead of (3), we may consider Functional Differential Equation

$$z'(t) = F(z_t, \lambda_t) \quad (4)$$

where

$$\lambda_t = \lambda(t+\theta), \quad -r \leq \theta \leq 0$$

$$\lambda_t \in C_4([-r, 0], \mathbb{R}^q), \quad F: C \times C_4 \rightarrow C$$

and $\lambda_t$ is a functional parameter.

The above results and the Theorem can be extended to (4).

Our Functional Differential Equation (4) is much more general than equation (1) in [15] considered by Bellman and his collaborators.

(2) Fuzzy Dynamical Systems are more realistic while considering problems occurring in Celestial Mechanics and Stellary Dynamics. The above results can also be obtained by considering Generalized Dynamical Systems and Fuzzy Dynamical Systems occurring in Celestial Mechanics and Stellar Dynamics, besides Stochastic Dynamical Systems.

REFERENCES