ON DEFECT RELATIONS OF MOVING HYPERPLANES

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§ 1. Introduction

The defect relation \( \sum_{j=1}^q \delta(f, H_j) \leq n + 1 \) gives the best-possible estimate, where \( f \) is a linearly non-degenerate holomorphic curve in \( P^n(C) \) and \( H_1, \ldots, H_q \) are hyperplanes in \( P^n(C) \) which are in general position. However, the case of moving hyperplanes has ever got only \( n(n + 1) \) instead of \( n + 1 \) (Stoll [4]) and it has not yet been known whether this bound is best-possible or not. In this paper we shall give some particular cases which have the bound \( n + 1 \).

The author thanks Professor Fujimoto for his useful advice and discussions with the author.

§ 2. Holomorphic curves and moving hyperplanes

In this paper, we fix one homogeneous coordinate system of the \( n \)-dimensional complex projective space \( P^n(C) \) and denote it by the notation \( w = (w_0 : \cdots : w_n) \).

A hyperplane \( H \) in \( P^n(C) \) is an \( (n-1) \)-dimensional projective subspace of \( P^n(C) \), i.e., it is given by \( H = \{ w \in P^n(C) | \sum_{j=0}^n a_j w_j = 0 \} \), where \( (a_0, \ldots, a_n) \in C^{n+1} - \{0\} \). We call the vector \( (a_0, \ldots, a_n) \) a representation of \( H \). Let \( H_j \) be hyperplanes in \( P^n(C) \) with representations \( a_j = (a_{j1}, \ldots, a_{jn}) \) \( (j = 1, \ldots, q) \). If any \( \min(q, n + 1) \) elements of \( a_1, \ldots, a_q \) are linearly independent over \( C \), \( H_1, \ldots, H_q \) are said to be in general position.

We call a holomorphic mapping \( f: C \to P^n(C) \) a holomorphic curve in \( P^n(C) \). A representation of \( f \) is a holomorphic mapping \( \tilde{f} = (f_0, \ldots, f_n): C \to C^{n+1} \) which satisfies \( \tilde{f}^{-1}(0) \neq C \) and \( f(z) = (f_0(z) : \cdots : f_n(z)) \) for all \( z \in C - \tilde{f}^{-1}(0) \). Then we write \( f = (f_0 : \cdots : f_n) \). If \( \tilde{f}^{-1}(0) = \emptyset \), then the representation \( \tilde{f} \) is said to be reduced.

**Definition 2.1.** A moving hyperplane \( H^x \) in \( P^n(C) \) is a mapping of...
$C$ into the set of all hyperplanes in $P^n(C)$ given by $H^M(z) = \{w \in P^n(C) \mid \sum_{j=0}^n a_j(z)w_j = 0\}$ ($z \in C$), where $(a_0, \ldots, a_n)$ is a reduced representation of some holomorphic curve $g$ in $P^n(C)$. We call a representation and a reduced representation of $g$ a representation and a reduced representation of $H^M$, respectively.

**DEFINITION 2.2.** Let $H_j^M$ be moving hyperplanes in $P^n(C)$ ($j = 1, \ldots, q$). $H_1^M, \ldots, H_q^M$ are said to be in general position if there exists a point $z_0$ of $C$ such that hyperplanes $H_1^M(z_0), \ldots, H_q^M(z_0)$ in $P^n(C)$ are in general position.

**DEFINITION 2.3.** Let $f$ be a holomorphic curve in $P^n(C)$ with a representation $(f_0, \ldots, f_n)$ and let $K$ be an extension field of $C$. We say that $f$ is non-degenerate over $K$ if $f_0, \ldots, f_n$ are linearly independent over $K$. In particular, $f$ is said to be linearly non-degenerate if it is non-degenerate over $C$.

§ 3. Characteristic functions, counting functions and defects

We define the norm $\|z\|$ of $z = (z_1, \ldots, z_m) \in C^m$ by $\|z\|^2 = \sum_{j=1}^m |z_j|^2$.

**DEFINITION 3.1.** The characteristic function of a holomorphic curve $f$ in $P^n(C)$ with a reduced representation $\bar{f}$ is defined for $0 < s < r$ by

$$T(f; r, s) = \frac{i}{2\pi} \int_{s}^{r} \frac{dt}{t} \int_{|z| \leq t} \frac{\partial^\delta}{\partial\theta} \log \|\bar{f}\|^2.$$  

This definition does not depend on the choice of $\bar{f}$. We see that $T(f; r, s)$ is non-negative and that if $f$ is non-constants, then $T(f; r, s) \to \infty$ monotonically as $r \to \infty$. Furthermore we can easily verify that

$$T(f; r, s) = \frac{1}{2\pi} \int_0^{2\pi} \log \|\bar{f}(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log \|\bar{f}(se^{i\theta})\| d\theta.$$  

**DEFINITION 3.3.** The counting function of zeros for a meromorphic function $F \neq 0$ on $C$ is defined for $0 < s < r$ by

$$N(F; r, s) = \int_s^r n(F; t) \frac{dt}{t},$$  

where $n(F; t)$ is the sum of zero orders of $F$ in $\{z \in C \mid |z| \leq t\}$.

By the definition, $N(F; r, s)$ is non-negative, and Jensen’s formula shows that
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In the situation of Definition 2.1, we define the characteristic function of \( H^u \) by \( T(H^u; r, s) := T(g; r, s) \). And we define the counting function of \( H^u \) for a holomorphic curve \( f \) by \( N(f, H^u; r, s) := N((\tilde{f}, \tilde{g}); r, s) \), where \( \tilde{f} = (f_0, \ldots, f_n) \) and \( \tilde{g} = (a_0, \ldots, a_n) \) are reduced representations of \( f \) and \( g \), respectively, and \( (\tilde{f}, \tilde{g}) := \sum_{j=0}^n a_j f_j \), if \( (\tilde{f}, \tilde{g}) \neq 0 \). This assumption holds if \( f \) is non-degenerate over a field containing all \( a_j/a_k \) with \( a_k \neq 0 \). This definition does not depend on the choice of \( \tilde{f} \) and \( \tilde{g} \). By (3.2), (3.4) and Schwarz’s inequality, we get

\[
N(f, H^u; r, s) \leq T(f; r, s) + T(H^u; r, s) + O(1), \quad r \to \infty.
\]

If either \( f \) or \( g \) is not constant, the defect of \( H^u \) for \( f \) is defined by

\[
\delta(f, H^u) = \liminf_{r \to \infty} \left( 1 - \frac{N(f, H^u; r, s)}{T(f; r, s) + T(H^u; r, s)} \right)
\]

which does not depend on \( s \). By (3.5), we see \( 0 \leq \delta(f, H^u) \leq 1 \). The moving hyperplane \( H^u \) is said to be of lower order than \( f \) if \( T(H^u; r, s) = o(T(f; r, s)) \) as \( r \to \infty \). Then

\[
\delta(f, H^u) = \liminf_{r \to \infty} \left( 1 - \frac{N(f, H^u; r, s)}{T(f; r, s)} \right).
\]

The definitions of counting functions and defects of (not-moving) hyperplanes are the same as those of moving hyperplanes. However, for convenience sake, we consider that the category of moving hyperplanes contains not-moving hyperplanes.

Let \( f \) be a holomorphic curve in \( P^n(C) \). We denote by \( K_f \) the set of all meromorphic functions \( g \) which satisfy the condition that \( T(g; r, s) = o(T(f; r, s)) \) as \( r \to \infty \). If a representation \((f_0, \ldots, f_n)\) satisfies that \( f \neq 0 \) for each \( j \) and that each \( f_j/f_k \) \( (j \neq k) \) is not constant, then we set \( \tilde{K}_f := \cap_{j \neq k} K_{f_j/f_k} \). Now, we present two lemmas without proofs.

**Lemma 3.6** ([4, Lemma 5.3]). The sets \( K_f \) and \( \tilde{K}_f \) are fields.

**Lemma 3.7** ([1, Proposition 5.9]). A holomorphic curve \( f = (f_0 : \ldots : f_n) \) in \( P^n(C) \) is rational, i.e., all \( f_j/f_k \) with \( f_k \neq 0 \) are rational if and only if

\[
T(f; r, s) = O(\log r) \quad \text{as} \quad r \to \infty.
\]
Proposition 3.8. Let \( f \) be a non-constant holomorphic curve and let \( g \) be a holomorphic curve in \( P^n(C) \) with a reduced representation \( (g_0, \ldots, g_n) \). Assume that \( g_j|g_k \in K \) if \( g_k \neq 0 \). Then, \( T(g; r, s) = o(T(f; r, s)) \) as \( r \to \infty \).

Proof. Without loss of generality, we may assume that \( g_0 \neq 0 \). Since the representation \( (g_0, \ldots, g_n) \) is reduced, for each point \( p \) where \( g_0 \) vanishes there is some \( g_j \) with \( g_j(p) \neq 0 \). Hence, we have

\[
N(g_0; r, s) \leq \sum_{j=1}^n N(g_j/g_0; \infty; r, s)
\]

\[
\leq \sum_{j=1}^n T(g_j; g_0; r, s) + O(1)
= o(T(f; r, s)) \quad (r \to \infty)
\]

and

\[
T(g; r, s) = \frac{1}{4\pi} \int_0^{2\pi} \log (1 + \sum_{j=1}^n |g_j(re^{i\theta})/g_0(re^{i\theta})|^p) d\theta
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} \log |g_0(re^{i\theta})| d\theta + O(1)
\]

\[
\leq \sum_{j=1}^n T(g_j; g_0; r, s) + N(g; r, s) + O(1)
= o(T(f; r, s)) \quad (r \to \infty) .
\]

Q.E.D.

In this paper, we treat non-rational holomorphic curves \( f \) and we use a notation \( S(f, r) \) for representing a quantity with a property that

\[
\lim_{r \to \infty, r \notin E} S(f; r)/T(f; r, s) = 0
\]

for a set \( E \subset (0, \infty) \) of finite Lebesgue measure.

§ 4. Defect relations

First, we give the known defect relations.

Theorem 4.1 (See, for example, [3, Chapter 3]). Let \( f \) be a linearly non-degenerate holomorphic curve in \( P^n(C) \) and let \( H_1, \ldots, H_q \) be hyperplanes in \( P^n(C) \) which are in general position. Then

\[
\sum_{j=1}^q \delta(f, H_j) \leq n + 1
\]

Theorem 4.2 ([4, Theorem 6.19]). Let \( f \) be a holomorphic curve in \( P^n(C) \) and let \( H_1^m, \ldots, H_q^m \) be moving hyperplanes in \( P^n(C) \) with lower orders than \( f \) which are in general position. Let \( (a_0^j, \ldots, a_n^j) \) be reduced representations of \( H_j^m \) \( (j = 1, \ldots, q) \) and \( K \) be the smallest extension field
of $C$ which contains all $a_j^k/a_m^l$ $(1 \leq j \leq q, 0 \leq k \leq n, \text{ and } m \in \{k | a_k^l \neq 0\})$. Assume that $f$ is non-degenerate over $K$. Then

$$\sum_{j=0}^{q} \delta(f, H_j^n) \leq n(n + 1).$$

**Theorem 4.3** ([2, Theorem 3.4]). Let $f$ be a holomorphic curve in $P^n(C)$ and let $H_j^n, \ldots, H_{n+1}^n$ are moving hyperplanes in $P^n(C)$ with lower orders than $f$ which are in general position. Let $(a_0^j, \ldots, a_n^j)$ be reduced representations of $H_j^n$ $(j = 0, \ldots, n + 1)$ and let $K$ be the smallest extension field of $C$ which contains all $a_j^k/a_m^l$ $(0 \leq j, k \leq n$ and $m \in \{k | a_k^l \neq 0\})$. Assume that $f$ is non-degenerate over $K$. Then

$$\sum_{j=0}^{n+1} \delta(f, H_j^n) \leq n + 1.$$ 

The main purpose of this paper is to prove the following:

**Theorem 4.4.** Let $f$ be a linearly non-degenerate holomorphic curve in $P^n(C)$ with a reduced representation $(f_0, \ldots, f_n)$ and let $H_0^n, \ldots, H_n^n$ be moving hyperplanes in general position in $P^n(C)$. Let $(a_0^j, \ldots, a_n^j)$ be reduced representations of $H_j^n$ $(1 \leq j \leq q)$. Assume that the following three conditions are satisfied:

1. **(C1)** $a_j^k/a_m^l \in \tilde{K}$; if $a_m^l \neq 0$;
2. **(C2)** $f$ is non-degenerate over $\tilde{K}$;
3. **(C3)** $N(f_j; r, s) = S(f; r)$ $(j = 0, \ldots, n)$.

Then

$$\sum_{j=1}^{q} \delta(f, H_j^n) \leq n + 1.$$ 

§ 5. Second main theorems

The next second main theorem is well-known and Theorem 4.1 is its corollary:

**Theorem 5.1** (See, for example, [3, Chapter 3]). In the same situation of Theorem 4.1, the inequality

$$(5.2) \quad (q - n - 1)T(f; r, s) \leq \sum_{j=1}^{q} N(f; H_j; r, s) + S(f; r)$$

holds for $0 < s < r$.

The next lemma will be proved by the same method of Theorem 4.3. For a proof, see [5, pp. 313–333].
**Lemma 5.3.** In the same situation of Theorem 4.3, the inequality

\[ T(f; r, s) \leq \sum_{j=0}^{q} N(f, H^j; r, s) + S(f; r) \tag{5.4} \]

holds for \( 0 < s < r \).

### § 6. Proof of Theorem 4.4

Before beginning to prove Theorem 4.4, we show the following lemma.

**Lemma 6.1.** Let \( f \) be as in Theorem 4.4. Let \( H^\mu \) be a moving hyperplane in \( P^n(C) \) with a reduced representation \((a_0, \ldots, a_n)\). Assume that \( a_j \neq 0 \) if \( a_k \neq 0 \). If \( a_{j_0} \neq 0, \ldots, a_{j_k} \neq 0 \) and \( a_j \equiv 0 \) for \( j \neq j_0, \ldots, j_k \), we give a hyperplane \( H = \{ w \in P^n(C) | w_{j_0} + \cdots + w_{j_k} = 0 \} \) in \( P^n(C) \). Then

\[ N(f, H; r, s) = N(f, H^\mu; r, s) + S(f; r) \tag{6.2} \]

**Proof.** For simplicity, we may assume that \( j_0 = 0, \ldots, j_k = k \). In the case of \( k = 0 \), the conclusion is evident since \( N(f; r, s) = o(T(f; r, s)) \) (\( r \to \infty \)) by Proposition 3.8. Hence we assume that \( k \geq 1 \).

Let \( h := (f_0 : \cdots : f_n) \) be a holomorphic curve in \( P^n(C) \) and let \( L^\mu \) be a moving hyperplane in \( P^n(C) \) with a reduced representation \((a_0, \ldots, a_n)\). Furthermore, we consider the hyperplanes \( L_j := \{ w \in P^n(C) | w_j = 0 \} \) (\( j = 0, \ldots, k \)) and \( L := \{ w \in P^n(C) | \sum_j w_j = 0 \} \) in \( P^n(C) \). Note that \( L^\mu \) has a lower order than \( h \). We get by Theorem 5.1 and Lemma 5.3.

\[ T(h; r, s) \leq N(h, L^\mu; r, s) + S(f; r) \]

and

\[ T(h; r, s) \leq N(h, L; r, s) + S(f; r) \]

Here we used the fact \( N(h, L_j; r, s) = S(f; r) \) (\( j = 0, \ldots, k \)). By (3.5) and the above inequalities, we have \( T(h; r, s) = N(h, L; r, s) + S(f; r) \) and \( T(h; r, s) = N(h, L^\mu; r, s) + S(f; r) \). Since \( N(h, L; r, s) + o(T(f; r, s)) = N(f, H; r, s) \) and \( N(h, L^\mu; r, s) = N(f, H^\mu; r, s) + o(T(f; r, s)) \) (\( r \to \infty \)), we obtain (6.2).

**Proof of Theorem 4.4.** There exists a point \( z_0 \) of \( C \) such that \( a^\mu_j(z_0) \neq 0 \) if \( a^\mu_k \neq 0 \) and that \( H^\mu_{j_0}(z_0), \ldots, H^\mu_{j_k}(z_0) \) are in general positions. Then by Lemma 6.1, we have

\[ N(f, H^\mu_{j_0}(z_0); r, s) = N(f, H^\mu_j; r, s) + S(f; r) \quad (j = 1, \ldots, q) \]

On the other hand, we have by Theorem 5.1,
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\[(q - n - 1)T(f; r, s) \leq \sum_{j=1}^{n} N(f, H^y_j(z_0); r, s) + S(f; r)\]

Hence we obtain

\[(q - n - 1)T(f; r, s) \leq \sum_{j=1}^{n} N(f, H^y_j; r, s) + S(f; r)\]

Therefore we have the defect relation

\[\sum_{j=1}^{n} \delta(f, H^y_j) \leq n + 1. \]

Q.E.D.

§ 7. Further result

In this section, we give a generalization of Theorem 4.4.

Before stating it, we show next lemmas.

Lemma 7.1. Let \(g\) be a linearly non-degenerate holomorphic curve in \(P^m(C)\) with a reduced representation \((g_0, \ldots, g_m)\). Assume that \(N(g_j; r, s) = S(g_k; r)\) for any distinct \(k\) and \(l\). Then \(g\) is non-degenerate over \(\tilde{K}_g\).

Proof. Assume that \(g_0, \ldots, g_m\) are linearly dependent over \(\tilde{K}_g\). So there exist \(a_0, \ldots, a_m \in \tilde{K}_g\) such that some \(a_j \neq 0\) and that \(a_0g_0 + \cdots + a_mg_m = 0\). Without loss of generality, we may assume that \(a_j \neq 0\) and that \(a_j = 0\) for all \(k, l\) where \(k + 1 < m\). If \(k = 0\), we can immediately lead a contradiction. So, let \(k \geq 1\).

Consider the holomorphic curve \(h = (g_0 : \ldots : g_k)\) in \(P^k(C)\) and moving hyperplanes

\[H^y_0(z) = \{w \in P^k(C) | w_j = 0\} \quad (0 \leq j \leq k)\]

and \(H^y_{k+1}\) with a representation \((a_0, \ldots, a_k)\) in \(P^k(C)\). They are in general position and of lower order than \(h\). By the assumption and the relation \(a_0g_0 + \cdots + a_kg_k = -a_{k+1}g_{k+1}\), we see that \(\delta(g, H^y_j) = 1\) \((0 \leq j \leq k + 1)\). This contradicts to Theorem 4.3. Hence we complete the proof of this lemma. Q.E.D.

Lemma 7.2. Let \(f\) be a linearly non-degenerate holomorphic curve in \(P^n(C)\) with a reduced representation \(\tilde{f} = (f_0, \ldots, f_n)\) and let \(g\) be a linearly non-degenerate holomorphic curve in \(P^m(C)\) with a reduced representation \(\tilde{g} = (g_0, \ldots, g_m)\). Assume that there are relations

\[(7.3) \quad f_j = \sum_{k=0}^{n} a^k_j g_k, \quad a^j_k \in C \quad (0 \leq j \leq n)\]

and that for each \(k = 0, \ldots, m\), there is a \(j(k)\) such that \(a^{(k)}_j \neq 0\). Moreover, if \(N(g_j; r, s) = S(g; r)\) for \(j = 0, \ldots, m\), then
\[ T(g; r, s) = T(f; r, s) + S(g; r). \]

**Proof.** By (7.3), we have the inequality \[ \|\tilde{f}\| \leq C\|\tilde{g}\| \] for some \( C > 0. \) Therefore we get

\[(7.4) \quad T(f; r, s) \leq T(g; r, s) + O(1).\]

Now, we can choose \( b_0, \ldots, b_n \in C \) such that \( c_k := \sum_{j=0}^{n} a_j b_j \neq 0. \) Consider hyperplanes

\[ H = \{ w \in P^n(C) | \sum_{j=0}^{n} b_j w_j = 0 \} \]

in \( P^n(C) \) and

\[ L_k = \{ w \in P^n(C) | w_k = 0 \} \quad (0 \leq k \leq m), \]
\[ L = \{ w \in P^n(C) | \sum_{k=0}^{m} c_k w_k = 0 \} \]

in \( P^n(C). \) Then by Theorem 5.1, we have

\[ T(g; r, s) \leq \sum_{k=0}^{m} N(g, L_k; r, s) + N(g, L; r, s) + S(g; r) \]
\[ = N(g, L; r, s) + S(g; r). \]

Since \( \sum_{k=0}^{m} c_k g_k = \sum_{j=0}^{n} b_j f_j, \) we have

\[ N(g, L; r, s) = N(f, H; r, s). \]

Hence, we get by (3.5)

\[ T(g; r, s) \leq N(f, H; r, s) + S(g; r) \]
\[ \leq T(f; r, s) + S(g; r). \]

Consequently, by (7.4), we obtain

\[ T(g; r, s) = T(f; r, s) + S(g; r). \quad \text{Q.E.D.} \]

The generalization of Theorem 4.4 is the following:

**Theorem 7.5.** Let \( f \) be a linearly non-degenerate holomorphic curve in \( P^n(C) \) with a reduced representation \( \tilde{f} = (f_0, \ldots, f_n) \) given by \( f_j = \sum_{k=1}^{m_j} f_{j}^k, \) where \( f_{j}^1, \ldots, f_{j}^{m_j} \) are entire functions which are linearly independent over \( C \) \( (j = 0, \ldots, n). \) Let \( H_j^r \) be as in Theorem 4.4. Assume that \( f \) is non-degenerate over \( \tilde{K}, \) and that \( N(f_{j}^k; r, s) = S(f_{j}^k f_{h}^{m_j}; r) \) if \( f_{j}^k f_{h}^{m_j} \) is not constant. Then

\[ \sum_{j=1}^{n} \delta(f, H_j^r) \leq n + 1. \]

**Proof.** Choose \( g_0, \ldots, g_m \) from \( f_{j}^k \) \( (1 \leq k \leq m_j, \quad 0 \leq j \leq n) \) such that
\{g_0, \cdots, g_m\} is a base of the vector space over \(C\) spanned by \(f_j^i\) \((1 \leq k \leq m_j, 0 \leq j \leq n)\). Let \(g\) be a holomorphic curve in \(P^n(C)\) with a reduced representation \(\bar{g} = (g_0/h, \cdots, g_m/h)\), where \(h\) is an entire function such that \(g_0/h, \cdots, g_m/h\) are entire functions without common zero. By Lemma 7.1, \(g\) is non-degenerate over \(\bar{K}_g\). It is easy to check that \(\bar{K}_f \subset \bar{K}_g\).

We define entire functions \(b_j^i\) \((1 \leq j \leq q, 0 \leq k \leq m)\) by the equations
\[
a_0^i f_0 + \cdots + a_n^i f_n = b_k^i g_0 + \cdots + b_m^i g_m \neq 0 \quad (1 \leq j \leq q) .
\]
Since \(b_j^i\) are linear combinations of \(a_0^i, \cdots, a_n^i\) with complex coefficients, we see that \(b_k^i/b_j^i \in \bar{K}_g\) if \(b_j^i \neq 0\). Let \(d_j\) be a common factor of \(a_0^i, \cdots, a_n^i\) and let \(L_j^i\) be a moving hyperplane in \(P^n(C)\) with a reduced representation \(b_j = (b_0^i/d_j, \cdots, b_m^i/d_j)\). Set \(a_j = (a_0^i, \cdots, a_n^i)\). Then \((\tilde{f}, a_j) = h d_j(\tilde{g}, b_j)\). Hence we have
\[
(7.6) \quad N(f, H_j^i; r, s) = N(g, L_j^i; r, s) + N(h; r, s) .
\]
We choose \(z_0\) of \(C\) such that \(b_j(z_0) \neq 0\) if \(b_j \neq 0\) and \(H_j^i(z_0), \cdots, H_n^i(z_0)\) are in general position. Then by Lemma 6.1, we get
\[
(7.7) \quad N(g, L_j^i(z_0); r, s) = N(g, L_j^i; r, s) + S(g; r) .
\]
Furthermore we have
\[
(7.8) \quad N(g, L_j^i(z_0); r, s) + N(h; r, s) = N(f, H_j^i(z_0); r, s)
\]
by \((\tilde{f}, a_j(z_0)) = h d_j(z_0)(\tilde{g}, b_j(z_0))\). Since \(N(d_j; r, s) = o(T(f; r, s))\) by Proposition 3.8, \(N(h; r, s)\) is \(S(f; r)\) and \(S(g; r)\) is \(S(f; r)\) by Lemma 7.2, we obtain
\[
N(f, H_j^i(z_0); r, s) = N(f, H_j^i; r, s) + S(f; r)
\]
by (7.6), (7.7) and (7.8). Hence using Theorem 5.1, we have
\[
(q - n - 1)T(f; r, s) \leq \sum_{j=1}^q N(f, H_j^i(z_0); r, s) + S(f; r)
\]
\[
\leq \sum_{j=1}^q N(f, H_j^i; r, s) + S(f; r) .
\]
Therefore we obtain the defect relation
\[
\sum_{j=1}^q \delta(f, H_j^i) \leq n + 1 .
\]
Q.E.D.

The most typical case of Theorem 4.4 is that \(f_j = \exp h_j\), where \(h_j\) are entire functions, and \(a_i^j\) are polynomials.
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