2

MATRIX REPRESENTATIONS OF DISCRETE GROUPS

2.1 Introduction

We begin with a word of apology because this chapter is purely mathematical without any physical applications. This book is intended for students of science, but the material in this chapter is essential when applying group theory to physical problems. Most physical applications rely more on matrix representation theory than on abstract group theory; this will become apparent when we turn to applications in later chapters.

As we build up the concepts of representation theory in this chapter, important theorems are stated, their meaning is discussed, and they are illustrated by examples. Some proofs are given to show the techniques involved. Refer to more advanced texts for complete detailed proofs.

2.2 Basis Functions and Representations

A matrix representation of an abstract group is a set of matrices that is isomorphic to the abstract group. Because of the isomorphism, the matrices obey the group’s product table with matrix multiplication as the rule of combination. In this section we shall discuss the relation between the group operations and the group’s matrix representations by looking at how group operations affect functions.

To start with a familiar example, consider the basis vectors \( \hat{i}, \hat{j}, \hat{k} \) of a 3-dimensional Cartesian frame, where the hat (caret) is the symbol for unit vectors.
The basis vectors are said to span the space, because the location of a particle anywhere in 3-dimensional space can be specified by coordinates with respect to the basis vectors. If we transform to a different set of basis vectors \( \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}} \), the coordinate values change, but the particle has not moved – only our description of its location is different.

By analogy, a member of an \( n \)-dimensional group can be thought of as operating on an abstract function space spanned by a set of basis functions \( \phi_1, \phi_2, \ldots, \phi_n \). If \( T \) is a member of the group, the relation between the basis functions and the \( \alpha \) matrix representation \( D_{ij}^{(\alpha)}(T) \) of \( T \) is expressed by

\[
T \phi_j = \sum_{i=1}^{h_\alpha} D_{ij}^{(\alpha)}(T) \phi_i,
\]

where \( h_\alpha \) is the dimension of the \( \alpha \) representation. Equation (2.1) does not look like matrix multiplication where the \( ij \) subscripts are normally seen reversed. The reason for the notation is that matrices written according to Eq. (2.1) obey the group’s product table, hence are a valid faithful representation. As a proof, let \( R, S, T \) be members of a group such that \( R = ST \) according to the group’s product table. Then

\[
R \phi_j = ST \phi_j = S \sum_i D_{ij}(T) \phi_i = \sum_i D_{ij}(T) S \phi_i = \sum_k \sum_i D_{ij}(T) D_{ki}(S) \phi_k.
\]

Matrix elements are just numbers, so they can be multiplied with other matrix elements in any order, unlike the matrices themselves:

\[
R \phi_j = \sum_k \sum_i D_{ki}(S) D_{ij}(T) \phi_k = \sum_k D_{kj}(ST) \phi_k = \sum_k D_{kj}(R) \phi_k,
\]

as required. Without transposing the subscripts, the result would have been \( D(TS) \), in disagreement with the group’s product table. The matrices would not be isomorphic to the group members, hence would not be a faithful representation of the group.

Some texts omit summation symbols \( \sum \) and instead use the Einstein summation convention that a repeated index in a factor is to be summed over. This text explicitly
shows the summation symbol with a subscript for the index to be summed over, for example, \( \sum_i \).

Consider a group consisting of two members, the identity \( E \) and an operation \( T \). The group’s product table is

\[
\begin{array}{ccc}
E & T \\
E & E & T \\
T & T & E \\
\end{array}
\]

Let \( Tx = -x \). The function \( x \) spans the entire \( x \)-axis from \(-\infty \) to \(+\infty \), and \( Tx \) is also in the same space. Applying Eq. (2.1) to the single basis function \( x \), \( D_{ij}(T) \) has only a single entry, \( D_{11}(T) \),

\[
Tx = -x = D_{11}(T)x \\
D_{11}(T) = -1,
\]

so that the matrix representation of \( T \) is the 1-dimensional matrix \((-1)\). Together with the identity matrix \( D(E) = (1) \), these matrices obey the product table and are indeed a faithful matrix representation of the group.

Now perform the same calculation using the basis function \( x^2 \):

\[
Tx^2 = (-x)^2 = x^2 = D_{11}(T)x^2 \\
D_{11}(T) = 1,
\]

so that the matrix representation of \( T \) is the 1-dimensional matrix \((1)\). Together with the identity matrix \( D(E) = (1) \), these matrices obey the product table and are a homomorphic representation of the group but not a faithful representation.

A group can have more than one representation. Our examples showed that the basis function \( x \) and the basis function \( x^2 \) lead to different matrix representations of the group \( \{E, T\} \). Different sets of basis functions may give rise to different matrix representations with possibly different dimensions.

Consider the representation \( D(E) = (1) \) and \( D(T) = (1) \), in which every group member is represented by the matrix \((1)\). Every group has this homomorphic identity representation, usually given the symbol \( 1 \) in modern texts.

As a counterexample, take \( e^x \) to be a tentative basis function, and apply the same calculation.

\[
Te^x = e^{-x} = D_{11}(T)e^x \\
D_{11}(T) = e^{-2x}
\]
However, the matrix \( D(T) = (e^{-2x}) \) is not a representation of the group member \( T \). The product table is not obeyed; for instance, \( T^2 = e^{-4x} \neq E \). \( e^x \) is not a basis function for the group \( \{E, T\} \).

If a group’s representation \( \alpha \) has dimension \( h_\alpha \), there will be \( h_\alpha^2 \) matrix elements and several basis functions; so if \( h_\alpha > 1 \), the sum in Eq. (2.1) will have several terms. For \( h_\alpha = 2 \) the matrix elements for an arbitrary group member \( T \) are

\[
\begin{align*}
T\phi_1 &= D_{11}^{(\alpha)}(T)\phi_1 + D_{21}^{(\alpha)}(T)\phi_2 \\
T\phi_2 &= D_{12}^{(\alpha)}(T)\phi_1 + D_{22}^{(\alpha)}(T)\phi_2,
\end{align*}
\]

so its representation matrix is the 2 × 2 matrix

\[
D^{\alpha}(T) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.
\]

For a concrete example, suppose that operator \( T \) changes \( \phi_1 \) to \( \phi_2 \) and changes \( \phi_2 \) to \( \phi_1 \).

\[
\begin{align*}
T\phi_1 &= \phi_2 \\
T\phi_2 &= \phi_1
\end{align*}
\]

Hence \( D_{11} = 0, D_{12} = 1, D_{21} = 1, D_{22} = 0 \) so that the representation matrix for \( T \) is

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The product table says that \( T^2 = E \). The representation matrix obeys this product, so the representation is indeed faithful.

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark
\]

### 2.2.1 “Transforms Like . . .”

Lecturers on group theory often use the phrase “transforms like” when discussing the relation between basis functions and matrix representations. This phrase is a verbal description of the relation stated above in Eq. (2.1) among group operations, basis functions, and matrix representations: \( T\phi_j = \sum_i D_{ij}(T)\phi_i \), where \( T \) is any member of the group, \( \phi_i \) and \( \phi_j \) are basis functions, and \( D(T) \) is the matrix representation for \( T \) generated by the basis functions. The \( \phi_i \) transform like the group members or, in other words, according to the representation whose matrix elements are \( D_{ij} \).
2.3 Similarity Transformations

To set the stage for this concept, turn again to an example from algebraic geometry. Consider a 2-dimensional Cartesian frame \((x, y)\) and a second Cartesian frame \((x', y')\) that has the same origin as the first so that

\[
x' = a_{11}x + a_{12}y \quad y' = a_{21}x + a_{22}y,
\]

or in matrix form

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Now introduce a vector function that has components \(f_x\) and \(f_y\), respectively, in the first frame and components \(f'_x\) and \(f'_y\) in the second. Let the components be linear functions of the respective coordinates. As an explicit example, the \(s_{ij}\) would be trigonometric functions for rotation of the frames.

\[
\begin{aligned}
f_x &= s_{11}x + s_{12}y \\
f'_{x} &= s_{11}x' + s_{12}y' \\
\end{aligned}
\]

\[
\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x' \\ y' \end{pmatrix}
\]

To express the components \(f'_x\) and \(f'_y\) in terms of \(f_x\) and \(f_y\), note that

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = S^{-1} \begin{pmatrix} f_x \\ f_y \end{pmatrix},
\]

so that

\[
\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = S \begin{pmatrix} x' \\ y' \end{pmatrix} = SA \begin{pmatrix} x \\ y \end{pmatrix} = SAS^{-1} \begin{pmatrix} f_x \\ f_y \end{pmatrix}.
\]

The structure \(SAS^{-1}\) is called a similarity transformation of \(A\) with respect to \(S\). With the substitution \(P \leftrightarrow S^{-1}\) this similarity transformation can also be written \(P^{-1}AP\).

The fundamental meaning of a similarity transformation is a change to a different set of basis functions, as shown in this vector function example where the similarity transformation is from basis functions \(f_x, f_y\) to \(f'_x, f'_y\).

When all the group’s operations undergo a similarity transformation with respect to the same \(S\), the group continues to be a valid group satisfying the group axioms and with the same product table. Consider the group \(\{E, T\}\) for which \(T^2 = E\). In terms of representation matrices,

\[
STS^{-1}STS^{-1} = STTS^{-1} = SES^{-1} = E.
\]
The group’s product table is satisfied by the transformed operations $STS^{-1}$ and $SES^{-1}$.

Here is an example of a similarity transformation in quantum mechanics. The fundamental equation of quantum mechanics is $\mathcal{H}\psi_n = E_n\psi_n$, where $\mathcal{H}$ is the Hamiltonian operator for the system of interest, $\psi_n$ is the wave eigenfunction for the $n$th quantum state, and $E_n$ is the energy eigenvalue of the $n$th state. Note that here $E_n$ is a number, not an operator.

Suppose that $\psi_n$ is replaced by $\psi_n' = S\psi_n$, where $S$ is an operator. Because $S^{-1}S$ is the identity, inserting it between $\mathcal{H}$ and $\psi_n$ in the original equation causes no change: $\mathcal{H}S^{-1}S\psi_n = E_n\psi_n$. Multiply both sides of the equation from the left by $S$ to give $S\mathcal{H}S^{-1}S\psi_n = E_nS\psi_n$, again the fundamental equation with the same energy eigenvalue $E_n$, but with the original Hamiltonian replaced by its similarity transformation $S\mathcal{H}S^{-1}$ and with a new wave function $\psi_n' = S\psi_n$.

### 2.4 Equivalent Representations

A group can have many different matrix representations, each with a different set of basis functions. Let $T$ be a member of a group and let the $\phi$ be basis functions for the group with matrix representation $D(T)$.

$$T\phi_j = \sum_k D_{kj}(T)\phi_k \quad (2.2)$$

Now let the $\phi'$ be a linear combination of the $\phi$ according to matrix $C$.

$$\phi_i' = \sum_j C_{ji}\phi_j \quad (2.3)$$

Transposed subscripts are used in Eq. (2.3) because basis functions must “transform like” the representation matrices.

Any linear combination of the original basis functions is in the same function space and is therefore a basis function for the group. Applying $T$ to Eq. (2.3) and using Eq. (2.2) gives

$$T\phi_i' = \sum_j C_{ji}T\phi_j$$

$$= \sum_j \sum_k C_{ji}D_{kj}(T)\phi_k. \quad (2.4)$$

Formally solving Eq. (2.3) for the $\phi$ basis functions in terms of the $\phi'$ gives

$$\phi_k = \sum_m A_{mk}\phi'_m, \quad (2.5)$$
where the $A_{mk}$ are the elements of a coefficient matrix $A$ to be determined. Multiply Eq. (2.5) by $C_{kj}$ and sum over $k$.

$$
\sum_k C_{kj} \phi_k = \sum_k \sum_m C_{kj} A_{mk} \phi'_m = \sum_k \sum_m A_{mk} C_{kj} \phi'_m
$$

By Eq. (2.3) the sum on the left is $\phi'_j$.

$$
\phi'_j = \sum_k \sum_m C_{kj} A_{mk} \phi'_m
$$

Summing over $k$, and noting that both sides must equal $\phi'_j$:

$$
\phi'_j = \sum_m (AC)_{mj} \phi'_m
$$

$$(AC)_{mj} = \delta_{mj}$$

$$AC = E$$

$$A = C^{-1},$$

so from Eq. (2.5):

$$\phi_k = \sum_m C_{mk}^{-1} \phi'_m.$$ 

Inserting Eq. (2.4) gives

$$T \phi'_i = \sum_j \sum_k \sum_m C_{ji} D_{kj}(T) C_{mk}^{-1} \phi'_m$$

$$T \phi'_i = \sum_j \sum_k \sum_m C_{mk}^{-1} D_{kj}(T) C_{ji} \phi'_m.$$ \hfill (2.6)

Summing over $k$ and then over $j$,

$$T \phi'_i = \sum_m D_{mi}^{'}(T) \phi'_m.$$ \hfill (2.7)

where

$$D' = C^{-1} DC.$$ \hfill (2.8)

Two results from this calculation should be noted. First, Eq. (2.7) proves that the $\phi'_i$ are basis functions for a representation with new matrices $D'(T)$. Second, Eq. (2.8) expresses the transformed representation matrices $D'(T) = C^{-1} D(T) C$ in terms of
the original representation matrices. It is not surprising that Eq. (2.8) is a similarity transformation. The similarity transformation is executed by the coefficient matrix \( C \) that expresses the \( \phi' \) basis functions in terms of the \( \phi \) basis functions in Eq. (2.3). The matrices \( D \) and \( D' \) may be different, but they are both valid representations of the group. They are *equivalent matrix representations* of the group. As we shall see later, equivalent representations describe the same physics.

## 2.5 Similarity Transformations and Unitary Matrices

When converting from one representation to another, similarity transformations are always executed by unitary matrices, as we now show. Recall that a matrix \( U \) is unitary if the complex conjugate of its transpose is equal to its inverse \( U^\dagger = U^{-1} \). In terms of matrix elements, \( u^\dagger_{ij} = u_{ji}^* = u_{ij}^{-1} \). If the matrix elements are real \( u_{ij} = u_{ji} \), the matrix is real orthogonal (called orthogonal for short), as discussed in Section 1.7.5.

Consider a set of \( n \) basis functions \( \{ \phi_1, \phi_2, \ldots, \phi_n \} \). We suppose that this is an orthonormal set with some sort of scalar product denoted by angle brackets \( \langle \phi_i^*, \phi_j \rangle = \delta_{ij} \). The explicit nature of the scalar product is immaterial for our purposes here – it could be a dot product of vectors \( \mathbf{u}^* \cdot \mathbf{v} \) or it might be an integral \( \int \phi_i^* \phi_j \, d\tau \) over some space. A familiar example of an orthonormal set is the Cartesian unit basis vectors \( \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \} \) in physical space.

The set of \( n \) basis functions is considered to be *linearly independent*. By definition a set of functions is linearly independent if the equation

\[
\sum_{i=1}^{n} c_i \phi_i = 0
\]

has only the solution \( c_i = 0 \) for all \( i \), which implies that none of the functions \( \phi_i \) can be expressed in terms of the other functions of the set.

Let an operation \( \mathbf{T} \) with matrix representation \( D(\mathbf{T}) \) transform a basis set \( \phi \) to a new set \( \psi \) assumed to be orthonormal with a defined scalar product. Then, using Eq. (2.3),

\[
\psi_i = \sum_k D_{ki}(\mathbf{T}) \phi_k.
\]

\[
\delta_{ij} = \langle \psi_i^*, \psi_j \rangle
\]

\[
= \left( \sum_k D_{ki}^*(\mathbf{T}) \phi_k^* \right) \left( \sum_m D_{mj}(\mathbf{T}) \phi_m \right)
\]

\[
= \sum_k \sum_m D_{ki}^*(\mathbf{T}) D_{mj}(\mathbf{T}) \langle \phi_k^*, \phi_m \rangle = \sum_k \sum_m D_{ki}^*(\mathbf{T}) D_{mj}(\mathbf{T}) \delta_{km}
\]

\[
= \sum_k D_{ki}^* (\mathbf{T}) D_{kj} (\mathbf{T}).
\]
so that
\[ \sum_k D_{ik}^\dagger(T) D_{kj}(T) = \delta_{ij}. \]

This result implies \( D^\dagger(T) = D^{-1}(T) \), showing that \( D(T) \) is unitary.

**Theorem 1** A similarity transformation from one orthonormal set to another is always executed by a unitary matrix.

This proof has neglected to show that the \( \psi \) set is indeed orthonormal, but it can in any case be made orthonormal using Gram–Schmidt orthogonalization, a systematic procedure that converts any set of linearly independent functions to an orthonormal set. Refer to other texts for details.

### 2.6 Character and Its Invariance under Similarity Transformations

The *character*, or *trace*, of a matrix is the sum of its diagonal elements. The character of a matrix \( A \) is symbolized by \( \chi(A) \) and in terms of matrix elements as \( \chi(A) = \sum_i a_{ii} \). This would be written \( \chi(A) = a_{ii} \), using the Einstein summation convention.

Under a similarity transformation of \( A \) by some matrix \( S \), the character of \( B = SAS^{-1} \) is equal to the character of \( A \) so that \( \chi(B) = \chi(A) \). Here is a proof.

\[
\chi(A) = \sum_i a_{ii} \\
\chi(B) = \sum_i b_{ii} \\
\quad = \sum_i \sum_j \sum_k s_{ij} a_{jk} s_{ki}^{-1} = \sum_i \sum_j \sum_k s_{ki}^{-1} s_{ij} a_{jk}
\]

The sum over \( i \) is
\[
\sum_i s_{ki}^{-1} s_{ij} = \delta_{kj} \quad \text{so that}
\]
\[
\chi(B) = \sum_j \sum_k \delta_{kj} a_{jk} = \sum_j a_{jj} = \chi(A).
\]

The last step follows because \( \delta_{kj} = 1 \) for \( k = j \) and 0 for \( k \neq j \), so in the sum over \( k \) only the term \( k = j \) makes a contribution.

As an application of character invariance, consider the matrix for physical rotation by \( \theta \) about the \( z \)-axis.

\[
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Its character is $2 \cos \theta + 1$. Rotation by $\theta$ about an arbitrary axis has the same character, because the change of axis is executed by a similarity transformation.

**Theorem 2**  
*The matrix for physical rotation by $\theta$ about any axis has character $2 \cos \theta + 1$.***

Any two matrices related by a similarity transformation therefore have the same character. The German word for character is *Spur*, like the English word *spoor* for the track of an animal. A matrix cannot hide its character by undergoing a similarity transformation. If the matrices for two representations have the same characters, they are related by a similarity transformation and are equivalent.

If two sets of representation matrices for the same group do not have the same character set, they are not related by a similarity transformation and do not span the same space. Take as an example the group $\{E, T\}$ introduced earlier that has two 1-dimensional representations. In both of them $D(E) = (1)$ and the representations for $T$ were $D(T) = (1)$ and $(-1)$. The matrices are 1-dimensional, so the characters are just the matrix entries. It is convenient to display the characters in a *character table*: The column headings are the two group members. The row headings are the symbols for the representations, where $1$ always labels the identity representation and $A$ is the label for the other 1-dimensional representation. $A$ and $B$ are the standard labels for 1-dimensional representations that are not the identity representation. The characters are not the same, showing that these two representations are not equivalent; there is no similarity transformation that can change one of the representations into the other.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Put another way, the basis functions for the different representations span different spaces.

**Theorem 3**  
*If two outwardly different matrix representations of the same group have the same character tables, the representations are related by a similarity transformation. They are equivalent and not fundamentally different, because their basis functions both span the same function subspace.*

If the rows of a character table are considered to be vectors, the rows, and also the columns are orthogonal. Taking the two rows in this character table, their scalar product is $1 \times 1 + 1 \times (-1) = 0$. This behavior is generalized in Section 2.7.4.

Appendix A shows how to find characters directly from group properties and various theorems without needing to know the matrices explicitly.
The determinant of a matrix is also invariant under a similarity transformation. It is shown in problem 1.23 that for square matrices the determinant of a matrix product equals the product of the matrix determinants. Consider the similarity transformation of matrix $V$ with respect to matrix $U$ to give matrix $V'$. 

$$V' = U V U^{-1}$$

$$|V'| = |U V U^{-1}| = |U| |V| |U^{-1}|$$

$$= |V| |U U^{-1}| = |V| |E|$$

$$= |V|$$

By expanding in cofactors of the first row it is easy to show that $|E| = 1$ for $E$ of any dimension.

## 2.7 Irreducible Representations

Much of what we have discussed so far has been in preparation for this section. This is the most important section of the chapter, because irreducible representations crop up time and time again in physical applications such as molecular vibrations and quantum mechanics. This section explains the meaning of irreducible representations and why they are important. Section 2.7.3 develops the related concept of class that extends the meaning of character, and a number of theorems are stated, some with formal proofs and the others illustrated by examples.

Consider an $n$-dimensional group with basis functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ so that its representations are $n \times n$ matrices. Suppose there is a similarity transformation with some $S$ that changes the original matrix representations to matrices that have smaller nonoverlapping square “block” matrices arrayed along the diagonal and 0 elsewhere, as in this illustrative example.

$$
\begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 \\
0 & b_{11} & 0 & 0 & 0 \\
0 & 0 & c_{11} & c_{12} & 0 \\
0 & 0 & c_{21} & c_{22} & 0 \\
0 & 0 & 0 & 0 & d_{11} \\
0 & 0 & 0 & 0 & d_{21}
\end{pmatrix}
$$

This matrix is said to be in block diagonal form.

Because a similarity transformation is a transformation of the group’s basis functions to a new set, a similarity transformation by the same $S$ applies to every member of the group and puts all of the original representation matrices into block diagonal forms of the same structure. The original matrix representation is said to be reducible. If no further similarity transformation exists to break some or all of the block matrices into even smaller blocks, the representation is irreducible. Some authors use the neologism irrep to denote an irreducible representation.
The block matrices of an irreducible representation are themselves each a representation of the group member. These representations might include homomorphisms as well as faithful isomorphic representations. Also, some representations might be repeated among the blocks.

Suppose that the members of a group have representations by matrices in block diagonal form. Because all the matrices for the group have the same block diagonal structure, if two of the matrices are multiplied, the result is in the same block diagonal form, with each block just the matrix product of the corresponding blocks, as in this example.

\[
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 3 & 2
\end{pmatrix}
= 
\begin{pmatrix}
8 & 0 & 0 & 0 & 0 \\
0 & 5 & 7 & 0 & 0 \\
0 & 6 & 6 & 0 & 0 \\
0 & 0 & 0 & 5 & 4 \\
0 & 0 & 0 & 4 & 3
\end{pmatrix}
\]

The blocks retain their individuality, guaranteeing that the representation continues to satisfy the group’s product table.

The meaning of an irreducible representation can be understood in terms of basis functions. A similarity transformation with \(S\) transforms the original basis set to a new set that spans the same function space. But now the block diagonal form of the representation matrices means that the new basis set has been subdivided into subsets that each span only a portion of the function space. Put another way, any function in one of these subspaces can be expressed as a linear combination of the basis functions in the new basis subset. If a subset has dimension \(\mu\), let the new basis subset be \(\{\phi_1, \phi_2, \ldots, \phi_\mu\}\). Then any function \(\phi'\) in that subspace can be expressed as a linear combination:

\[
\phi' = c_1\phi_1 + c_2\phi_2 + \cdots + c_\mu\phi_\mu.
\]

In terms of group operations, a group member operating on an irreducible subspace can only result in a combination of the existing basis functions that span the subspace. The subspace is said to be invariant under the action of the associated group members, because no group operation can lead to a function outside the invariant subspace. We shall see in later chapters that physical meaning is attached to invariant subspaces.

### 2.7.1 The Regular Representation

One way to construct an irreducible representation is to start from a reducible representation. Suppose a group does not have a geometrical interpretation so that a reducible representation is not easy to construct. However, a representation of a group, called the regular representation, can always be constructed from the group’s product table.

The product table below is for the \(32\) group as laid out in Section 1.4.2 but with columns interchanged to put all identity operations \(E\) on the diagonal.

To form the matrices of the regular representation, use the modified product table as a template with 1 entered wherever the symbol for the operation appears and 0
otherwise. Here are the regular representations for group members A and C:

\[
D^{(\text{reg})}(A) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad D^{(\text{reg})}(C) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

In any representation the matrix for E has entry 1 everywhere on the main diagonal and 0 elsewhere because this is the only way E can commute with all the other matrices.

The matrices of the regular representation are distinct because the product table is distinct. In the regular representation, there is only a single 1 in each row or column, because in the product table a given group member occurs only once in each row and column. No matrix in the regular representation except E has 1 on the main diagonal. The matrices in the regular representation are easy to multiply together to show that the product table is satisfied; it is, therefore, a faithful isomorphic representation of the group.

### 2.7.2 Example: Reducing the Regular Representation

The regular representation of the 32 group can be transformed to irreducible representations by a similarity transformation with the following matrix S.

\[
S = \frac{1}{\sqrt{6}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & -\sqrt{\frac{3}{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\
-\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & \sqrt{\frac{3}{2}}
\end{pmatrix}
\]
2.7 Irreducible Representations

$S$ is a unitary matrix, in this case a real orthogonal matrix, so its inverse is the transpose: $S^{-1} = \tilde{S}$.

$$S^{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix}
1 & 1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{\frac{3}{2}} \\
1 & -1 & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
1 & -1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \cdot \\
1 & -1 & 0 & -\sqrt{2} & \sqrt{2} & 0 \\
1 & 1 & 0 & -\sqrt{2} & -\sqrt{2} & 0 \\
1 & 1 & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}
\end{pmatrix}$$

As an example, applying a similarity transformation with $S$ to $D^{(reg)}(C)$ given above reduces $D^{(reg)}(C)$ to irreducible representations. After lengthy matrix multiplications, the result for $SD^{(reg)}(C)S^{-1}$ is

$$SCS^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\
0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2}
\end{pmatrix}.$$  

This result has several features. There are three distinct matrix representations: two different 1-dimensional matrix representations and a repeated 2-dimensional matrix representation. The 1-dimensional matrix at the extreme upper left is the representation of $C$ in the homomorphic identity representation 1, where all group members have the matrix representation $(1)$.

$$(1) \quad (1) \quad (1) \quad (1) \quad (1) \quad (1) \quad$$

$$E \quad A \quad B \quad C \quad D \quad F$$

It is evident that these matrices will occur in the irreducible representation of every group.

The second block $(-1)$ belongs to the second 1-dimensional homomorphic representation of the $32$ group introduced in Section 1.4.

$$(1) \quad (-1) \quad (-1) \quad (-1) \quad (1) \quad (1) \quad$$

$$E \quad A \quad B \quad C \quad D \quad F$$

This representation is given the symbol $A$ for a 1-dimensional representation.
The two $2 \times 2$ repeated blocks belong to an irreducible isomorphic representation of the $32$ group. They can be obtained by applying the similarity transformation with $S$ to the regular representation of each group member.

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
$$

<table>
<thead>
<tr>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
</tbody>
</table>

The symbol for this representation is $\Gamma$.

The matrices of the irreducible representation $\Gamma$ are real orthogonal. The row vectors and column vectors of each matrix form a set of orthonormal vectors. To illustrate, consider the scalar product of the rows of matrix $B$:

$$
\left( \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( \frac{\sqrt{3}}{2} \right) \left( -\frac{1}{2} \right) = 0.
$$

The scalar product of the first row of $B$ with itself is

$$
\left( \frac{1}{2} \right) \left( \frac{1}{2} \right) + \left( \frac{\sqrt{3}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) = 1.
$$

Section 2.7.4 proves that all of a group’s irreducible representations are contained in the regular representation. Another proof will show that irreducible representations occur in the reduced regular representation a number of times equal to their dimension, as illustrated by representation $\Gamma$ of the $32$ group. $\Gamma$ has matrices of dimension 2 and occurs twice in the reduction.

Any matrix representation of a group can be expressed in terms of one or more of the irreducible representations by using a suitable similarity transformation. As an example from the $32$ group, let $D^{(\text{reg})}(C)$ be the regular representation of $C$ and let $D^{(\text{red})}(C)$ be its reduced block diagonal form. $D^{(\text{reg})}(C)$ and $D^{(\text{red})}(C)$ have the same dimensions. As our example showed,

$$
SD^{(\text{reg})}(C)S^{-1} = D^{(\text{red})}(C).
$$

Multiply both sides by $S^{-1}$ from the left and by $S$ from the right to express $D^{(\text{reg})}(C)$ in terms of irreducible representations.

$$
D^{(\text{reg})}(C) = S^{-1}D^{(\text{red})}(C)S
$$

Any representation $D^{(\alpha)}$ that is in a different reduced block form is related to the original reduced block form by a similarity transformation with some $U$.

$$
D^{(\alpha)}(C) = U^{-1}D^{(\text{red})}(C)U
$$
If all the irreducible representations of a group are known, that is all there is to know about any of its other irreducible matrix representations. They are all related by similarity transformations.

The example using the 32 group illustrates how similarity transformations can generate irreducible representations of a group. It is difficult to calculate the needed transformation matrix such as $S$ in the example. Luckily, mathematicians have worked out the irreducible representations for groups of interest.

### 2.7.3 Conjugates and Classes

The example showed that the 32 group has three distinct irreducible representations. The number of irreducible representations for any group can be predicted from its product table using the concept of *classes*.

Suppose that $A$ and $R$ are both members of the same group. The product $RAR^{-1}$ is called the *conjugate* to $A$. According to the group axioms, a conjugate must be a member of the same group. The use of conjugates "unpacks" the structure of a group, dividing it into classes.

Again, take the example of the 32 group, which has the following product table.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>F</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>E</td>
<td>F</td>
<td>D</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>F</td>
<td>D</td>
<td>E</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>D</td>
<td>D</td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>F</td>
<td>E</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>E</td>
<td>D</td>
</tr>
</tbody>
</table>

The inverse of each group member is readily found from the product table.

$E^{-1} = E$  $A^{-1} = A$  $B^{-1} = B$  $C^{-1} = C$  $D^{-1} = F$  $F^{-1} = D$

To determine the classes of this group, take each member of the group and evaluate its conjugate with every member of the group.

| $EEE^{-1}$ | E | $EDE^{-1}$ | D | $EFE^{-1}$ | F |
| $AEA^{-1}$ | E | $ADA^{-1}$ | F | $AFA^{-1}$ | D |
| $BEB^{-1}$ | E | $BDB^{-1}$ | F | $BFB^{-1}$ | D |
| $CEC^{-1}$ | E | $CDC^{-1}$ | F | $CFC^{-1}$ | D |
| $DED^{-1}$ | E | $DDD^{-1}$ | D | $DFD^{-1}$ | F |
| $FEF^{-1}$ | E | $FDF^{-1}$ | D | $FFF^{-1}$ | F |
Conjugation has resolved the group members into three distinct classes: \{E\}, \{D, F\}, and \{A, B, C\}. Note that every group must have a class \{E\} with only one member, because for any member \(R\) of the group, \(RER^{-1} = RR^{-1} = E\), as the example illustrates.

All members of the same class have the same character. As a proof, let \(A\), \(B\), and \(R\) be members of a group. If \(B = RAR^{-1}\), then \(A\) and \(B\) are members of the same class and have the same character because the similarity transformation leaves character unchanged (Section 2.6).

**Theorem 4** The number of nonequivalent irreducible representations of a group is equal to the number of classes.

Without a formal proof, Theorem 4 is plausible from the example of the 32 group, which has three classes and three irreducible representations.

For any \(n\)-dimensional representation, \(\chi(E) = n\) because the identity has 1 everywhere along the main diagonal so that \(\chi(E)\) equals the dimension of the representation. Every class in irreducible representation 1 has character 1. For group 32 matrices in irreducible representation \(A\), the classes \{E\} and \{D, F\} have character 1, and matrices in the class \{A, B, C\} have character \(-1\). In irreducible representation \(\Gamma\), the matrix in the class \{E\} has character 2, the matrices in the class \{D, F\} have character \(-1\), and in the class \{A, B, C\} the matrices have character 0.

Here is the character table for the 32 group.

<table>
<thead>
<tr>
<th>{E}</th>
<th>{A, B, C}</th>
<th>{D, F}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(A)</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>(\Gamma)</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

The column headings are the classes, and the row headings are the designations of the irreducible representations.

For another application of the invariance of character under a similarity transformation, suppose as an illustrative example that an \(8 \times 8\) reducible matrix \(U\) is transformed to irreducible block diagonal form by \(S\). The block diagonal matrix is also \(8 \times 8\), and it has irreducible representations of \(U\) strung along its main diagonal.
The process might look like this:

\[
S \begin{pmatrix}
  u_{11} & u_{12} & \cdots & u_{18} \\
  u_{21} & u_{22} & \cdots & u_{28} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{81} & u_{82} & \cdots & u_{88}
\end{pmatrix} S^{-1} = \begin{pmatrix}
  a_1 & a_2 & b_1 & b_2 \\
  b_1 & b_2 & c_1 & c_2 \\
  c_1 & c_2 & c_3 & c_4 \\
  c_1 & c_2 & c_3 & c_4
\end{pmatrix}.
\]

The 0 entries in the block diagonal matrix have been omitted to focus attention on the reduced block matrices along the main diagonal.

The similarity transformation has reduced \( D(U) \) to four 1-dimensional irreducible representations and a 2-dimensional irreducible representation that occurs twice. Because a similarity transformation does not change the character of \( D(U) \),

\[
\sum_i u_{ii} = a_1 + a_2 + b_1 + b_2 + 2(c_1 + c_4).
\]

In symbolic form,

\[
\chi(U) = \sum_\alpha c_\alpha \chi^{(\alpha)}(U),
\]

where, on the left, \( \chi(U) \) is the character of the matrix before the similarity transformation. On the right, \( \chi^{(\alpha)}(U) \) is the character of the \( \alpha \) irreducible representation in the reduced block diagonal matrix and \( c_\alpha \) is the number of times the \( \alpha \) irreducible representation appears in the reduction.

The example in Section 2.7.2 showed the reduction of the regular representation of group member \( C \) of the 32 group. The character of \( D^{(\text{reg})}(C) \) in the regular representation is 0 because in the regular representation only \( D^{(\text{reg})}(E) \) has nonzero elements on the main diagonal. Check that the character of the reduced block diagonal matrix \( C \) is also 0.

\[
\sum_\alpha c_\alpha \chi^{(\alpha)}(C) = \chi^{(1)}(C) + \chi^{(A)}(C) + 2\chi^{(T)}(C)
\]

\[
= 1 + (-1) + 2(0)
\]

\[
= 0 \quad \checkmark
\]

### 2.7.4 Orthogonality Theorems

The following theorem leads to important applications, and many authors call it the “wonderful orthogonality theorem.” Its proof is a long road and is omitted here, but the proof given in more advanced texts can be followed using the material discussed so far.

**Theorem 5**

\[
\sum_R D^{(\alpha)*}_{ij}(R) D^{(\beta)}_{i'j'}(R) = \frac{n}{h} \delta_{\alpha\beta} \delta_{ii'} \delta_{jj'}
\]
About the notation: Theorem 5 applies to a group with members $\mathbf{R}$, with the sum taken over all the group members. $D^{(\alpha)}$ and $D^{(\beta)}$ are any irreducible matrix representations of the group, either the same or different. $n$ is the order of the group, and $h$ is the dimension of the irreducible representation. The right-hand side is 0 unless the representations are the same $\alpha = \beta$, so the matrices are $h \times h$ and $h$ is uniquely defined.

Theorem 5 can be viewed as a generalized orthogonality relation analogous to the scalar product of Cartesian vectors, where the scalar product of two orthogonal vectors is 0.

To demonstrate the theorem, take both representations to be the irreducible representation $\Gamma$ of the 32 group, and take the 1,1 element of each matrix. The 32 group is of order $n = 6$, and $\Gamma$ has dimension $h = 2$.

\[
(1)^2 + (-1)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{6}{2}
\]
\[
1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3
\]
\[
3 = 3 \quad \checkmark
\]

The theorem is satisfied.

Now use Theorem 5 to find a relation involving the characters of irreducible representations. The character of a matrix is the sum of its diagonal elements, so in Theorem 5 set $j = i$ and $j' = i'$ and sum over $i$ and $i'$.

\[
\sum_i \sum_{i'} \sum_R D_{ii}^{(\alpha)} D_{i'i'}^{(\beta)} = \frac{n}{h} \sum_i \sum_{i'} \delta_{\alpha\beta} \delta_{ii'} \delta_{i'i'}
\]
\[
\sum_R \chi^{(\alpha)}(\mathbf{R})^{*} \chi^{(\beta)}(\mathbf{R}) = \frac{n}{h} \delta_{\alpha\beta} \sum_i \delta_{ii}
\]
\[
\delta_{ii} = 1, \text{ so the sum on the right is } \sum_i 1 = h. \text{ The result is worth listing as a theorem.}
\]

**Theorem 6**

\[
\sum_{\mathbf{R}} \chi^{(\alpha)}(\mathbf{R})^{*} \chi^{(\beta)}(\mathbf{R}) = n \delta_{\alpha\beta}
\]

Theorem 6 says that the character sets of two different irreducible representations are mutually orthogonal, a result illustrated earlier for the $\{\mathbf{E}, \mathbf{T}\}$ group in Section 2.2.

Now use Theorem 6 to calculate the number of times an irreducible representation occurs in the block diagonal reduction of a matrix $D(\mathbf{R})$. From Eq. (2.9),

\[
\chi(\mathbf{R}) = \sum_{\alpha} c_{\alpha} \chi^{(\alpha)}(\mathbf{R}).
\]

Multiply by $\chi^{(\beta)}(\mathbf{R})$ from the left, where $\chi^{(\beta)}(\mathbf{R})$ is the character of the $\beta$ irreducible representation of $\mathbf{R}$. Sum over the group members.

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2.7 Irreducible Representations

\[
\sum_R \chi^{(\beta)*}(R) \chi^{(R)}(R) = \sum_R \chi^{(\beta)*}(R) \sum_\alpha c_\alpha \chi^{(\alpha)}(R)
\]

\[
= \sum_\alpha c_\alpha \sum_R \chi^{(\beta)*}(R) \chi^{(\alpha)}(R)
\]

Use Theorem 6 to evaluate the sum over \( R \) on the right.

\[
\sum_R \chi^{(\beta)*}(R) \chi^{(\alpha)}(R) = n \sum_\alpha c_\alpha \delta_{\alpha\beta} = n c_\beta
\]

\[
c_\beta = \frac{1}{n} \sum_R \chi^{(\beta)*}(R) \chi^{(\alpha)}(R)
\]

(2.10)

The result implies that the number of each irreducible representation in the reduced matrices \( D(R) \) is unique – it can only happen one way.

Now apply Theorem 6 to the regular representation \( \alpha = reg \). In the regular representation, the characters of all the group members except \( E \) are 0, so the sum reduces to a single term: \( \chi^{(\beta)*}(E) \chi^{(reg)}(E) \). Here \( \chi^{(reg)}(E) = n \) is the order of the group, and \( \chi^{(\beta)}(E) = h_\beta \) is the dimension of the \( \beta \) irreducible representation. Substituting,

\[
c_\beta = \frac{1}{n} \chi^{(\beta)*}(E) \chi^{(reg)}(E)
\]

\[
= \frac{1}{n} (n h_\beta)
\]

\[= h_\beta.
\]

The regular representation contains every irreducible representation of the group a number of times equal to the dimension of the irreducible representation. For example, the irreducible representation \( \Gamma \) of the 32 group has dimension 2 and occurs twice in the reduction of the regular representation.

Applied to the same irreducible representation \( \alpha = \beta \), the left-hand side of Theorem 6 becomes the squared magnitude of \( \chi^{(\alpha)}(R) \) summed over the group members:

**Theorem 7**

\[
\sum_R \left| \chi^{(\alpha)}(R) \right|^2 = n
\]

If instead we sum over classes keeping in mind that all members of a class have the same character, Theorem 7 can be written

\[
\sum_\beta h_\beta \left| \chi^{(\beta)} \right|^2 = n,
\]

where \( h_\beta \) is the number of group members in the class \( \beta \).
Now apply Theorem 7 to the reduction of the regular representation into its constituent irreducible representations.

$$\chi^{(\text{reg})}(R) = \sum_{\alpha} c_{\alpha} \chi^{(\alpha)}(R)$$

Take $R = E$ and use $c_{\alpha} = h_{\alpha}$, $\chi^{(\text{reg})}(E) = n$, and $\chi^{(\alpha)}(E) = h_{\alpha}$ to obtain the result.

**Theorem 8**

$$n = \sum_{\alpha} h_{\alpha} \chi^{(\alpha)}(E)$$

$$= \sum_{\alpha} (h_{\alpha})^2$$

The sum of the squares of the dimensions of all the different irreducible representations is equal to the order of the group. For the 32 group, $n = 6$ and there are only two possible ways to satisfy Theorem 8:

$$6 = (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2$$

$$6 = (1)^2 + (1)^2 + (2)^2.$$  

But the 32 group has three classes, hence only three irreducible representations according to Theorem 4. Only the second choice fits: there are two distinct 1-dimensional irreducible representations, and one 2-dimensional irreducible representation.

**Theorem 9** As a corollary to Theorem 8, all the irreducible representations of an Abelian group are 1-dimensional.

To prove Theorem 9, let $A$ be a member of an Abelian group. The class of $A$ is $RAR^{-1}$, where $R$ is every member of the group. Because the group is Abelian, its members commute: $RA = AR$. Hence the class of $A$ is

$$RAR^{-1} = ARR^{-1} = AE = A.$$  

Every member of an Abelian group is therefore its own class. If an Abelian group is of order $n$, there are $n$ classes, hence $n$ irreducible representations by Theorem 3. Theorem 8 is satisfied only if all the irreducible representations are 1-dimensional.

$$\underbrace{(1)^2 + (1)^2 + \cdots + (1)^2}_{n \text{ terms}} = n$$
Tests for Reducibility

The first test for reducibility uses Theorem 6:

\[ c_\beta = \frac{1}{n} \sum_{R} \chi^{(\beta)*}(R) \chi^{(\alpha)}(R). \]

Here \( c_\beta \) is the number of times \( D(R) \) contains the \( \beta \) irreducible representation. If \( c_\beta \neq 0 \) for two or more different values of \( \beta \), it follows that the original representation must contain at least two different irreducible representations and is therefore reducible.

For a second test of reducibility, note that the character of a group member \( R \) is the sum of the characters of its irreducible representations, some possibly repeated.

\[ \chi(R) = \sum_{\alpha} c_\alpha \chi^{(\alpha)}(R) \]

Take the squared magnitude of both sides and sum over \( R \):

\[ \sum_{R} \chi^{*(R)} \chi(R) = \sum_{R} \sum_{\alpha} \sum_{\beta} c_\alpha c_\beta \chi^{(\alpha)}(R) \chi^{(\beta)*}(R) \]

\[ = \sum_{\alpha} \sum_{\beta} c_\alpha c_\beta \sum_{R} \chi^{(\alpha)}(R) \chi^{(\beta)*}(R) \]

\[ = n \sum_{\alpha} \sum_{\beta} c_\alpha c_\beta \delta_{\alpha\beta}, \]

where the last step follows from Theorem 6 for the orthogonality of the characters of irreducible representations. Hence

\[ \sum_{R} \chi^{*(R)} \chi(R) = \sum_{R} |\chi(R)|^2 = n \sum_{\alpha} c_\alpha^2. \]

If \( D(R) \) is irreducible, \( c_\alpha = 1 \) for a particular value of \( \alpha \) and \( c_\alpha = 0 \) otherwise. Hence

\[ \sum_{R} \left| \chi^{(\beta)}(R) \right|^2 = n \quad \rightarrow \quad \text{representation } \beta \text{ is irreducible.} \]

However, if \( D(R) \) is reducible, then two or more of the \( c_\alpha \neq 0 \) and

\[ \sum_{R} \left| \chi^{(\beta)}(R) \right|^2 > n \quad \rightarrow \quad \text{representation } \beta \text{ is reducible.} \]

As an illustration of this inequality, consider the regular representation of a group of order \( n \). All characters of the regular representation are 0 except for \( \chi(E) = n \).
The sum over $R$ in the inequality reduces to one term to give $|\chi^{(\beta)}(E)|^2 = n^2 > n$. The inequality is satisfied in this example because the regular representation is reducible.

**An Easy Way to Find Character Tables**

Appendix A shows how to calculate the characters of irreducible representations using algebra and the product table without knowing explicit matrices. The method is not needed for later developments in this text. The lengthy calculations needed to construct character tables can be avoided by noting that the work has already been done by others and tabulated for groups of interest. Character tables are displayed in references under the name of the group.

### 2.8 Kronecker (Direct) Product

It is sometimes useful to combine two groups into a larger group; for example, combining physical rotation with inversion. Such a combination is performed by the Kronecker (direct) product. Consider a group $\mathcal{G} = \{A_1, A_2, \ldots\}$ of order $n_a$ and a second group $\mathcal{B} = \{B_1, B_2, \ldots\}$ of order $n_b$. The direct product is symbolized $\mathcal{G} \otimes \mathcal{B}$. The members of the direct product are all possible combinations $A_i B_j$ and therefore number $n_a n_b$.

Assume that the members from the two groups commute so that $A_i B_j = B_j A_i$. This is not a serious restriction because in applications the two groups normally span different spaces and therefore do not interact – they don’t “talk” to each other. With this assumption the members of the direct product set form a group. Consider the product of two members of the direct product set.

$$(A_i B_j)(A_i' B_j') = (A_i A_i')(B_j B_j')$$

The product $(A_i A_i')$ is some member $A_i$ of the group $\mathcal{G}$, and $(B_j B_j')$ is some member $B_j$ of the group $\mathcal{B}$, and it follows that $(A_i B_j)$ is a member of the direct product set, satisfying the closure axiom for a group. The other axioms are also easily proved.

Let group $\mathcal{G}$ have a representation matrix $D^{(\mu)}(A_m)$ with elements $a_{ij}$, and let group $\mathcal{B}$ have a representation matrix $D^{(\nu)}(B_n)$ with elements $b_{k\ell}$. The matrix elements $c$ of the direct product representation are

$$D^{(\mu \nu)}_{ik,j\ell}(A_m B_n) = D^{(\mu)}_{ij}(A_m) D^{(\nu)}_{k\ell}(B_n)$$

$$c_{ik,j\ell} = a_{ij} b_{k\ell}.$$  \hfill (2.11)

Note the order in which the subscripts are written.
Let an element from each group have a $2 \times 2$ matrix representation so the direct product $\mathcal{C}$ is $4 \times 4$.

$$
\mathcal{C} = \mathcal{G} \otimes \mathcal{B} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \otimes \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
= \begin{pmatrix}
a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\
a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\
a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\
a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22}
\end{pmatrix}
= \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix}
$$

Matrices need not be square to form a direct product. For example, if $A$ is $m \times n$ and $B$ is $m' \times n'$, the direct product matrix $A \otimes B$ is $mm' \times nn'$.

We can use Eq. (2.11) to find the character of a direct product matrix.

$$
\chi(A_m B_n) = \sum_i \sum_j D_{ij,ij} (A_m B_n)
= \sum_i \sum_j D_{ii} (A_m) D_{jj} (B_n)
= \chi(A_m) \chi(B_n)
$$

The characters of direct product matrices are the products of the corresponding characters.

A lengthy proof, not given here, shows that the irreducible representations of a direct product group are the products of the irreducible representations of the corresponding constituent groups. Assuming this to be proven, we can use Theorem 8 and the definition of the direct product to show that these are all the irreducible representations of the direct product.

Let group $\mathcal{G}$ of order $n_a$ have $\alpha$ irreducible representations of dimensions $h_\alpha$, and let group $\mathcal{B}$ of order $n_b$ have $\beta$ irreducible representations of dimensions $h_\beta$. Let their direct product have irreducible representations of dimension $h_{\alpha,\beta}$:

$$
\sum_\alpha \sum_\beta (h_{\alpha,\beta})^2 = \sum_\alpha \sum_\beta (h_\alpha)^2 (h_\beta)^2
= \sum_\alpha (h_\alpha)^2 \sum_\beta (h_\beta)^2
= n_a n_b
= \text{the order of the direct product.}
$$
Hence, according to Theorem 8 the direct product has no other irreducible representations.

### 2.8.1 Example: The Klein Four-Group

There are only two groups of order 4, one of them cyclic and the other noncyclic. The noncyclic group is called the *Klein Four-group* symbolized \( K_4 \) or \( \mathbb{m}\mathbb{m}2 \). Table 2.1 shows its product table.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>E</td>
<td>C</td>
<td>B</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>C</td>
<td>E</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>B</td>
<td>A</td>
<td>E</td>
</tr>
</tbody>
</table>

The product table shows that the members of \( K_4 \) commute. It is therefore an Abelian group and each member is its own class. There are four classes, hence four irreducible representations all of dimension 1 according to Theorem 9.

\[(1)^2 + (1)^2 + (1)^2 + (1)^2 = n = 4 \quad \checkmark\]

Table 2.2 shows the characters of \( K_4 \). These are also the 1-dimensional irreducible representation matrices.

<table>
<thead>
<tr>
<th></th>
<th>{E}</th>
<th>{A}</th>
<th>{B}</th>
<th>{C}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Consider now \( C_2 \), a group of order 2. Its product table is shown in Table 2.3.

<table>
<thead>
<tr>
<th></th>
<th>{E}</th>
<th>{T}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( B )</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
This is also the character table because the representation is 1-dimensional. Form the direct product of $C_2$ with itself.

$$C_2 \otimes C_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1) & (1)(1) & (1)(1) & (1)(1) \\ (1)(1) & (1)(-1) & (1)(1) & (1)(-1) \\ (1)(1) & (1)(1) & (-1)(1) & (-1)(1) \\ (1)(1) & (1)(-1) & (-1)(1) & (-1)(-1) \end{pmatrix}$$

The product table and also the character table is shown in Table 2.4.

Table 2.4 Character table for $C_2 \otimes C_2$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Comparing the character tables for $K_4$ and for $C_2 \otimes C_2$, they are seen to be identical aside from row ordering. Two groups with the same character table are equivalent, so $K_4 = C_2 \otimes C_2$.

### 2.9 Kronecker Sum

The *Kronecker sum* is symbolized $\oplus$. Unlike the Kronecker (direct) product, the matrices in a Kronecker sum must be square. Let matrix $A$ be $m \times m$ and matrix $B$ be $n \times n$. Let $E_m$ be the $m \times m$ identity matrix, and let $E_n$ be $n \times n$. The Kronecker sum is defined as

$$A \oplus B = A \otimes E_n + E_m \otimes B.$$  

The Kronecker sum matrices $A \oplus B$ in this example are square $mn \times mn$.

### Summary of Chapter 2

Chapter 2 lays out fundamental principles of matrix representations of group operations. The format is numerous theorems, many with proofs. Chapter 2 is mathematical to provide the groundwork for physical applications in later chapters.

a) Group representations can be understood as the action of group members on basis functions that span certain abstract spaces.
b) The relation between basis functions and representations is that basis functions
“transform like” the group members and its matrix representations.

c) The similarity transformation of $T$ by $S$ is denoted $STS^{-1}$.

d) A group may have several different matrix representations; but if they are
related by similarity transformations, they are all equivalent.

e) Similarity transformations are carried out by unitary matrices.

f) The character of a matrix is the sum of its diagonal elements and is unchanged
by a similarity transformation. If two matrix representations of a group have
the same character set, they are equivalent and are related by a similarity
transformation. Their basis functions span the same abstract space.

g) A representation is irreducible, if it cannot be broken down into simpler block
diagonal form.

h) The regular representation can be constructed from the group’s product table.
Mirroring the product table, there is only one entry 1 in every row and in every
column, with 0 elsewhere. Only the identity matrix $D$ has entries on the main
diagonal.

i) When matrices of the regular representation are put into block diagonal form
by a suitable similarity transformation, the blocks are all the irreducible represen-
tations of the group, appearing a number of times equal to the representation’s
dimension.

j) The conjugate of a group member $A$ is formed by applying a similarity trans-
formation to $A$ using members $X$ of the group, $XAX^{-1}$. Conjugates divide the
group members into classes. The identity member $E$ is always a class of its own.
The number of irreducible representations equals the number of classes.

k) In a given representation, every group member in the same class has the same
character.

l) In the regular representation, the character of $E$ is $\chi^{(\text{reg})}(E) = n$, where $n$
is the order of the group. In an irreducible representation, $\chi^{(\alpha)}(E) = h_\alpha$, where $h_\alpha$
is the dimension of the $\alpha$ irreducible representation.

m) The matrices of irreducible representations and their characters obey orthogo-
nality relations.

n) A matrix representation can be tested for reducibility using its characters if the
characters of the irreducible representations are known.

o) The character table of a group’s irreducible representations can be found using
the group’s product table to calculate class sums and their products, but it is much
easier just to look in references.

**Stated Theorems in Chapter 2**

Theorem 1 The similarity transformation from one orthonormal set to another is
carried out by a unitary matrix.
Theorem 2 The matrix for physical rotation by $\theta$ has character $2 \cos \theta + 1$ for any axis of rotation.

Theorem 3 If two apparently different matrix representations of the same group have matrices with the same characters, the representations are related by a similarity transformation. They are equivalent: the basis functions for two equivalent representations both span the same space.

Theorem 4 The number of nonequivalent irreducible representations of a group is equal to the number of classes. For the $32$ group, this is $3$.

Theorem 5
\[ \sum_{\mathbf{R}} D^{(a)\ast}_{ij}(\mathbf{R}) D^{(b)}_{i'j'}(\mathbf{R}) = \frac{n}{h} \delta_{ab} \delta_{ii'} \delta_{jj'} \]

Theorem 6
\[ \sum_{\mathbf{R}} \chi^{(a)\ast}(\mathbf{R}) \chi^{(b)}(\mathbf{R}) = n \delta_{ab} \]

Theorem 7
\[ \sum_{\mathbf{R}} \left| \chi^{(a)}(\mathbf{R}) \right|^2 = n \]

If instead we sum over classes, keeping in mind that all members of a class have the same character, Theorem 7 is written as
\[ \sum_{\beta} h_{\beta} \left| \chi^{(\beta)} \right|^2 = n, \]

where $h_{\beta}$ is the number of group members in the class $\beta$.

Theorem 8
\[ n = \sum_{\beta} h_{\beta} \chi^{(\beta)}(\mathbf{E}) = \sum_{\beta} (h_{\beta})^2 \]

Theorem 9 As a corollary to Theorem 8, all the irreducible representations of an Abelian group are 1-dimensional.

Problems and Exercises

2.1 Consider the triangle rotation group $\{\mathbf{E}, \mathbf{A}, \mathbf{B}\}$. The group member $\mathbf{A}$ is a rotation by $120^\circ$. Show that $x, y$ are basis functions for $\mathbf{A}$, and develop a $2 \times 2$ matrix representation for $\mathbf{A}$. Compare with the representation given in Section 1.4.
2.2 Consider the triangle rotation group \( \{E, A, B\} \). The group member \( A \) is a rotation by 120°. Show that \( x^2 - y^2, xy \) are basis functions for \( A \), and develop a \( 2 \times 2 \) matrix representation for \( A \). Compare with the representation given in Section 1.4. Why is your result different?

2.3 For the group of an equilateral triangle, let operation \( A \) be a “flip” – a rotation by 180° about the \( y \)-axis. Show that \( \frac{1}{\sqrt{3}} x + \frac{1}{\sqrt{3}} y \) and \( \frac{1}{\sqrt{3}} y + \frac{1}{\sqrt{3}} z \) are basis functions for \( A \) by finding the corresponding matrix representation for \( A \).

2.4 For the group of an equilateral triangle, let operation \( A \) be a “flip” – a rotation by 180° about the \( y \)-axis. Show that \( \frac{1}{\sqrt{3}} x - \frac{1}{\sqrt{3}} y \) and \( \frac{1}{\sqrt{3}} y - \frac{1}{\sqrt{3}} z \) are basis functions for \( A \) by finding the corresponding matrix representation for \( A \).

2.5 A 2-dimensional representation of a group member \( T \) is generated by the \( \alpha \) set of basis functions to give \( D(\alpha)(T) = \left(\begin{array}{c} 1 \\ 0 \\
\end{array} \right) \) and by the \( \beta \) set of basis functions to give \( D(\beta)(T) = \left(\begin{array}{c} -1 \\ 0 \\
\end{array} \right) \). Show that these representations are equivalent, related by a similarity transformation with \( S \). Show that your \( S \) is unitary \( SS^\dagger = E \).

2.6 Under a similarity transformation with \( S \), group members \( A \) and \( B \) transform as \( SAS^{-1} = R \) and \( SBS^{-1} = T \). Show that \( SABS^{-1} = RT \).

2.7 For the group \( \{E, T\} \) introduced in Section 2.2, the regular representation for \( T \) is \( \left(\begin{array}{c} 0 \\ 1 \\
\end{array} \right) \). A similarity transformation with a matrix \( S \) reduces it to the block diagonal form \( \left(\begin{array}{cc} 1 & 0 \\ 0 & a \\
\end{array} \right) \). Show that \( a = \pm 1 \) and choose the correct value by calculating the matrix \( S \). Show that your matrix is unitary \( SS^\dagger = E \).

2.8 Consider the \( 2 \times 2 \) matrix

\[
S = \left(\begin{array}{cc} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{array} \right)
\]

(a) Show that \( S \) is orthogonal.

(b) Perform a similarity transformation with \( S \) on each of the following matrices:

\[
D(G1) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\
\end{array} \right) \quad D(G2) = \left(\begin{array}{cc} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \\
\end{array} \right) \quad D(G3) = \left(\begin{array}{cc} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \\
\end{array} \right).
\]

(c) Use the results from (b) to calculate the characters of the three transformed matrices.
2.9 Consider the $2 \times 2$ matrix

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Show that $T$ is orthogonal.
(b) Perform a similarity transformation with $T$ on each of the following matrices:

$$D(H1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D(H2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad D(H3) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$ 

(c) Use the results from (b) to calculate the characters of the three transformed matrices.

2.10 Find the character of each of the following matrices:

$$(-2) \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \begin{pmatrix} -1 + i & 0 & 3 \\ 2 & 2 & 4i \\ 3 - i & 3 + 2i & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) (b) (c) (d) (e)

2.11 Here is the product table for the triangle rotation group \{E, A, B\}. Write the matrices for the regular representation of group members E, A, and B.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>E</td>
<td>A</td>
</tr>
</tbody>
</table>

2.12 The noncyclic group of order 4 has members \{E, K, L, M\}, and its product table is

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>K</th>
<th>L</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>K</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>K</td>
<td>K</td>
<td>E</td>
<td>M</td>
<td>L</td>
</tr>
<tr>
<td>L</td>
<td>L</td>
<td>M</td>
<td>E</td>
<td>K</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>L</td>
<td>K</td>
<td>E</td>
</tr>
</tbody>
</table>

Write the regular representation of group member K.
2.13 The cyclic group of order 4 has members \( \{E, A, B = A^2, C = A^3\} \), and its product table is.

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>C</td>
<td>E</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

Write the regular representation of group member A.

2.14 The 422 group describes the symmetries of a square, including “flips.” It has eight members and five classes.
(a) How many irreducible representations does the 422 group have?
(b) What are the dimensions of this group’s irreducible representations?

2.15 The noncyclic group of order 4 has members \( \{E, K, L, M\} \), with product table

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>K</th>
<th>L</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>K</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>K</td>
<td>K</td>
<td>E</td>
<td>M</td>
<td>L</td>
</tr>
<tr>
<td>L</td>
<td>L</td>
<td>M</td>
<td>E</td>
<td>K</td>
</tr>
<tr>
<td>M</td>
<td>M</td>
<td>L</td>
<td>K</td>
<td>E</td>
</tr>
</tbody>
</table>

(a) What are the classes of this group?
(b) How many irreducible representations does this group have?
(c) What are the dimensions of its irreducible representations?

2.16 The cyclic group of order 4 has members \( \{E, A, B = A^2, C = A^3\} \), and its product table is

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>C</td>
<td>E</td>
<td>A</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

(a) What are the classes of this group?
(b) How many irreducible representations does this group have?
(c) What are the dimensions of its irreducible representations?
2.17 Consider a group with general members $T$. Starting from the relation

$$T\phi_j = \sum_k D_{kj}^{(\alpha)}(T)\phi_k,$$

use Theorem 5 to prove

$$\phi_i = \frac{h_\alpha}{n} \sum_{T} D_{ij}^{(\alpha)*}(T)T\phi_j,$$

where $h_\alpha$ is the dimension of the $\alpha$ irreducible representation and $n$ is the order of the group.

2.18 The three irreducible matrix representations of the $32$ group are given in Section 2.7.2. Demonstrate Theorem 5.

2.19 Here is the character table of the $32$ group.

<table>
<thead>
<tr>
<th></th>
<th>${E}$</th>
<th>${A, B, C}$</th>
<th>${D, F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>2</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Demonstrate Theorem 6 using this character table.

2.20 Here is the character table of the $32$ group.

<table>
<thead>
<tr>
<th></th>
<th>${E}$</th>
<th>${A, B, C}$</th>
<th>${D, F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>2</td>
<td>0</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Demonstrate Theorem 7 using this character table.

2.21 Consider the permutation group of 4 numbers (1234). There are two irreducible representations of dimension 1, and the other irreducible representations all have dimensions greater than 1. What are the dimensions of the other irreducible representations?

2.22 The cyclic group $\{E, A, B\}$ for the rotation of an equilateral triangle by $0^\circ$, $120^\circ$, and $240^\circ$ has a matrix representation
Show that this representation is reducible.

2.23 Show that \((A \otimes B)(A' \otimes B') = (AA') \otimes (BB')\). For simplicity let the matrices \(A, A', B, B'\) all be \(n \times n\). The general requirement on matrix dimensions for this relationship is that the number of columns in \(A\) must equal the number of rows in \(A'\) to allow matrix multiplication \(AA'\), and similarly for \(BB'\).